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Q1 Exponential Dichotomy on the real line: SVD and QR methods ☆

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ABSTRACT

In this work we show when and how techniques based on the Singular Value Decomposition (SVD) and the QR decomposition of a fundamental matrix solution can be used to infer if a system enjoys—or not—exponential dichotomy on the whole real line.

1. Introduction

Exponential dichotomy is at the heart of the fundamental perturbation results for linear systems of Coppel and Palmer ([10,11] and [24–26,28]), of the spectral theory of Sacker and Sell [31,32], of the geometric theory of Fenichel [19], of shadowing results [29,30], of perturbation results for invariant manifolds [18], of the fundamental perturbation results for connecting orbits of Beyn and Sandstede ([6–8] and [34]), and it has proven also a formidable ally to justify and gain insight into the behavior of various algorithmic approaches for solving boundary value problems, for approximating invariant surfaces and for computing traveling waves, among other uses (see [3,9,14,21]). Indeed, the property

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of exponential dichotomy is so ubiquitous and relevant that one would expect that there are several practical techniques to ascertain whether or not a given system enjoys exponential dichotomy on the real line. However, as far as we know, there are no reliable techniques allowing us to do that in general. The cases in which one is able to ascertain exponential dichotomy for a given system are perturbative in nature: slow varying systems (close to constant or periodic systems), roughness results (that is, systems close to systems having exponential dichotomy), L_1 perturbation results; e.g., see [11], [8], [26] and [33].

Analysis and implementation of methods for approximating Lyapunov exponents of differential equations, and more generally also Exponential Dichotomy spectrum, henceforth Σ_{L} and Σ_{ED} respectively, *on the half-line* $t \geqslant 0$ (or $t \leqslant 0$), are rather well established. Our goal in this work is to lay down the theoretical ground work for the development of methods to ascertain if a system has exponential dichotomy (henceforth ED) *on the entire real line*. Our techniques will rest on the same methodologies which have proven valuable to approximate spectra on the half-line. As a matter of fact, we will use the same assumptions needed in that context, ultimately some measure of integral separation in the system. As a result, our criteria to ascertain ED will not be of perturbation nature (for a system close to one which has or not have ED, such a constant coefficient problem), but will be resting on structural (and generic) properties of the underlying system.

We begin by reviewing the concepts of spectra on $\mathbb{R}^+$ (or $\mathbb{R}^-$) and lay down notation for later use.

1.1. Spectra of dynamical systems

Consider the n-dimensional dynamical system

$$\dot{y} = f(y), \qquad y(0) = y_0, \tag{1}$$

with solution $y(t, y_0)$. Consider the linearized problem

$$\dot{x} = Df\big(y(t, y_0)\big)x, \tag{2}$$

or simply

$$\dot{x} = A(t)x, \tag{3}$$

where we suppose that A is continuous and bounded. With $X(\cdot)$, we will indicate a fundamental matrix solution of the system. The principal matrix solution will be written as $\Phi(\cdot)$, so that $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$.

Spectra are defined for the linear problem (3), and in general they will depend on the solution trajectory of (1), that is on the initial condition y_0. [There are important situations when this dependency on y_0 can be removed, in a measure theoretical sense: [2,4,5,22,23].]

Two spectra associated to (3) are of interest to us: The Lyapunov spectrum, Σ_{L}, and the Exponential Dichotomy (or Sacker–Sell) spectrum, Σ_{ED}.

The Lyapunov spectrum: Σ_{L}. This is defined in terms of upper and lower Lyapunov exponents (LEs for short). Given a fundamental matrix X for (3), define μ_j, $j = 1, \ldots, n$, as

$$\mu_j = \limsup_{t \to \infty} \frac{1}{t} \log \big\| X(t) e_j \big\|, \tag{4}$$

where the e_j's are the standard unit vectors. When the sum of the numbers μ_j is minimized as we vary over all possible ICs (initial conditions) $X(0)$, the numbers are called (upper) Lyapunov exponents of the system, and the ICs are said to form a *normal basis*. We will write λ_j^s, $j = 1, \ldots, n$, for the ordered upper LEs of (3). By working with the adjoint system, $\dot{z} = -A^T(t)z$, one analogously defines its upper LEs which are called lower LEs of the original system (3), call them λ_j^i, $j = 1, \dots, n$, which again are considered ordered. The Lyapunov spectral intervals can now be defined:

$$\Sigma_{\mathrm{L}} := \bigcup_{j=1}^{n} [\lambda_j^i, \lambda_j^s]. \tag{5}$$

In case in which $\lambda_j^i = \lambda_j^s = \lambda_j$, for all $j = 1, \dots, n$, the system is called *regular*.

The Exponential Dichotomy, or Sacker–Sell, spectrum: Σ_{ED}. This is defined in terms of exponential dichotomy (see [11]). Recall that (3) has Exponential Dichotomy on the half-line (i.e., $t, s \geqslant 0$ below) if there exist constants $K \geqslant 1$, $\alpha > 0$, and a projection P such that

$$\begin{aligned} \left\| X(t) P X^{-1}(s) \right\| &\leqslant K e^{-\alpha(t-s)}, \quad t \geqslant s, \\ \left\| X(t)(I - P) X^{-1}(s) \right\| &\leqslant K e^{\alpha(t-s)}, \quad t \leqslant s, \end{aligned} \tag{6}$$

where X is a fundamental matrix solution. Σ_{ED} is defined to be the set of values $\lambda \in \mathbb{R}$ for which the shifted systems $\dot{y} = [A(t) - \lambda I]y$ do not have exponential dichotomy. For some $k \leqslant n$, Σ_{ED} is given by (see [31]) a collection of disjoint subintervals

$$\Sigma_{\mathrm{ED}} := [a_1, b_1] \cup \dots \cup [a_k, b_k]. \tag{7}$$

We make the simple, but fundamental, observation that (3) has ED on the half-line $t \geqslant 0$, iff $0 \notin \Sigma_{\mathrm{ED}}$.

Remark 1. By virtue of the roughness theorem for exponential dichotomies (see [10]), Σ_{ED} is stable, that is it is continuous with respect to perturbation in the coefficients $A(\cdot)$ of (3). Stability theory for Σ_{L} is more complicated (and restrictive), and one needs that the system enjoys the property of integral separation in some form (see [1,17]).

Directions: Growth subspaces. Associated to the spectral intervals, both Σ_{L} and Σ_{ED}, there is very important geometric information on the subspaces of solutions (i.e., of initial conditions) which achieve a certain asymptotic exponential growth; e.g., whose asymptotic growth is in the intervals of Σ_{ED}. As it turns out, this geometric information will be key for establishing ED on the real line, but it is typically neglected in algorithmic studies on approximation of spectra of dynamical systems.

Let us define the forward stable subspace $\mathcal{S}^+$ to be the set of initial conditions leading to decreasing solutions in forward time:

$$\mathcal{S}^+ = \left\{ x \in \mathbb{R}^n \colon \left\| \Phi(t)x \right\| \to 0 \text{ as } t \to +\infty \right\}. \tag{8}$$

Let us also define the backward stable subspace $\mathcal{S}^-$ (this is called unstable subspace by Sacker and Sell [31]) be the set of initial conditions leading to decreasing solutions in backward time:

$$\mathcal{S}^- = \left\{ x \in \mathbb{R}^n \colon \left\| \Phi(t)x \right\| \to 0 \text{ as } t \to -\infty \right\}. \tag{9}$$

ED on the real line. The problem (3) enjoys ED on the entire real line, when (6) holds with $t, s \in \mathbb{R}$. The following characterization of ED on the entire real line (see [10,11]) is the one which we will adopt in order to device techniques to verify ED:

System (3) *has ED in* $\mathbb{R}$ *if and only if it has ED in* $\mathbb{R}^+$ *and* $\mathbb{R}^-$, *and* $\mathbb{R}^n = \mathcal{S}^+ \oplus \mathcal{S}^-$.

As a consequence, to check if system (3) has ED in $\mathbb{R}$, we will check that:

(ED-1) It has ED in $\mathbb{R}^+$ and $\mathbb{R}^-$.
(ED-2) The stable and the unstable subspaces of (8) and (9) are complementary. [That is, they form a basis for $\mathbb{R}^n$.]

[Notice that if there is ED in $\mathbb{R}^+$ and $\mathbb{R}^-$, then convergence to 0 as $t \to \pm\infty$ in (8) and (9) is exponentially fast.]

We are now ready to look at the two techniques which we propose to use to detect ED on the real line. As it turns out, these are the same techniques which have been used to approximate Σ_{L} and Σ_{ED} on the half-line. The first technique is based on the SVD of a fundamental matrix X: $X = U\Sigma V^T$. The second technique, which rests on the most popular approach to approximate Σ_{L} and Σ_{ED} on the half-line, is based on the QR factorization of a (normal) fundamental matrix solution X, that is $X = QR$.

Notation. For a positive, non-vanishing function f, we will use the shorthand notation $\chi^s(f)$, $\chi^i(f)$ and $\chi(f)$ to mean respectively $\limsup_{t\to+\infty} \frac{1}{t}\log(f)$, $\liminf_{t\to+\infty} \frac{1}{t}\log(f)$ and $\lim_{t\to\infty} \frac{1}{t}\log|f(t)|$. Similar notation will be used for the limit as $t \to -\infty$.

2. SVD method

In order to approximate Σ_{L} and Σ_{ED} on the half-line, SVD techniques use the information emerging from the smooth SVD (if this is possible) of a fundamental solution X. That is, for $t \geqslant 0$, one seeks the decomposition $X(t) = U(t)\Sigma(t)V^T(t)$, with all factors being as smooth as X, $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$, and U and V orthogonal. To find this SVD, one can integrate the differential equations governing the evolution of the U, V, and Σ factors in the SVD of X. In [12], see also [16], we have shown that if the system (3) has stable and distinct Lyapunov exponents (a generic situation), then the decomposition $X = U\Sigma V^T$ with smooth factors is valid, differential equations for U and V can be given, and the evolution of the singular values is given by

$$\dot{\sigma}_j = C_{jj}(t)\sigma_j, \quad j = 1, \dots, n, \; C(t) := U^T(t)A(t)U(t). \tag{10}$$

Moreover, we have shown how, given *any* fundamental matrix solution X (i.e., any initial condition $X(0)$ for it), both Σ_{L} and Σ_{ED} can be recovered from knowledge of the diagonal of C. Finally, we have also shown, and this will be important soon, that—for systems with stable and distinct exponents—the orthogonal function V in the SVD converges exponentially fast to a constant matrix $\bar{V}$ and that the columns of $X(0)\bar{V}$ span the growth subspaces associated to (Σ_{L} and) Σ_{ED}. It must be stressed that, with SVD techniques, we are able to obtain Σ_{L} and Σ_{ED} without the need to require the fundamental matrix solution X to be normal, any initial condition $X(0)$ will do (as long as the system has stable and distinct Lyapunov exponents). It comes natural then, to use the principal matrix solution Φ to characterize the desired quantities. Clearly in this case the growth subspaces (which are those in (8) and (9)) are given by the columns of $\bar{V}$. We refer to [13] for computational details on SVD techniques.

Perhaps surprisingly, we already have all the necessary ingredients in order to ascertain ED on $\mathbb{R}$ by using the SVD method. In fact, it is sufficient to proceed as follows (see (ED-1)–(ED-2)):

(a) Relatively to (3) in forward and backward time, apply SVD techniques to the principal matrix solution Φ, to verify ED on the two half-lines $\mathbb{R}^+$ and $\mathbb{R}^-$. This requires verifying that $0 \notin \Sigma_{\mathrm{ED}}$

for both forward and backward problems. If $0 \in \Sigma_{ED}$ for either forward or backward problem, then there is no ED. Otherwise, there is ED on the two half-lines. Also, recall that we expect (for systems with stable and distinct Lyapunov exponents) that the factors V of the SVDs converge (exponentially fast) to constant orthogonal matrices, in forward and backward time, call these $\bar{V}^+$ and $\bar{V}^-$ respectively.

(b) Take the columns of $\bar{V}^+$ and $\bar{V}^-$ relative to the stable modes (those leading to intervals of Σ_{ED} to the left, respectively right, of 0): $\bar{V}_s^+$, $\bar{V}_s^-$. The key observation here (see [12]) is that $\bar{V}_s^+$ spans $\mathcal{S}^+$ and $\bar{V}_s^-$ spans $\mathcal{S}^-$.

(c) Verify if $[\bar{V}_s^- \ \bar{V}_s^+]$ is a basis for $\mathbb{R}^n$. If it is, then the system has ED on $\mathbb{R}$, otherwise it does not.

To sum up, SVD techniques based on (a), (b) and (c) above can be used to verify ED on $\mathbb{R}$. But, our analytical results justifying this approach require the assumption of distinct and stable Lyapunov exponents. Although the assumption of stable Lyapunov exponents is natural, the need to have distinct Lyapunov exponents is violated in cases of practical interest (such as when the coefficient matrix A assumes a constant limiting value $\bar{A}$, and $\bar{A}$ has complex conjugate eigenvalues). In our practical experience, the SVD method seems to work reliably also in these cases, but the lack of theoretical justification in the case of stable and not distinct LEs is bothersome, and we cannot guarantee that this technique will not encounter difficulties in the case of stable, but not distinct, Lyapunov exponents (either forward or backward in time). To avoid these difficulties, we then turn our attention to the next technique.

3. QR method

QR techniques are based on the (unique) decomposition of a *normal* fundamental matrix solution $X = QR$, where Q is orthogonal and R is upper triangular with positive diagonal. In practice, one only finds Q and (the logarithm of the) diagonal of R. For example, in a continuous realization of the QR method, we integrate differential equations for Q, so that $R = Q^T X$ would satisfy the triangular system $\dot{R} = B(t)R$ with $B(t) := Q^T AQ - Q^T \dot{Q}$. As it is well understood, see [5,15,16] for details on $\mathbb{R}^+$, one can approximate Σ_L and Σ_{ED} by using only the diagonal of B; in other words, only Q is needed and R is really never formed, since $\operatorname{diag}(B) = \operatorname{diag}(Q^T AQ)$.

So, to check (ED-1), one can use QR techniques to find Σ_{ED} for both forward and backward problems and verify that there is ED on the two half-lines. The issue is how to verify (ED-2), that is how to obtain information on the growth subspaces by the QR technique. It is our goal in this chapter to show that—under certain assumptions—one can obtain the needed directional information from the rescaled function $Y = R^{-1}\operatorname{diag}(R)$. We will prove that Y “converges” to a matrix $\bar{Y}$ for both forward and backward problems, and that these limiting values can be used to verify (ED-2).

Remark 2. As we previously observed, when using the QR technique we need to require that the fundamental matrix solution X be normal. But, we actually will need more: We will need that the initial condition $X(0)$ be chosen in such a way that conditions (12) and (13) below be verified in forward time, with the similar conditions (14)–(15) to be verified in backward time. In general, this request will imply that the initial conditions for the forward and backward integration will not be the same. An easy example of this is the diagonal system $\dot{x} = Ax$, with $A = \operatorname{diag}(1, -2)$. Obviously, this system enjoys Exponential Dichotomy, and the principal matrix solution Φ is a normal fundamental matrix solution for this problem. However, by using Φ, condition (12) is verified, but (14) is not. If for the backward time integration we choose $X(0) = [e_2, e_1]$, then we will get $Q = [e_2, e_1]$, and then for $t \leqslant 0$ $B = \operatorname{diag}(-2, 1)$ and (14) is verified. In other words, with respect to the notation of (8) and (9), we have $\mathcal{S}^+ = e_2$ and $\mathcal{S}^- = e_1$.

With the above remark in mind, our results can be summarized as follows: Given suitable initial conditions, $X^+(0)$ and $X^-(0)$, for the forward and backward problems respectively, the corresponding Y^+ and Y^- will “converge” (for $t \to +\infty$ and $t \to -\infty$, respectively) and the required growth subspaces can be extracted from these limiting Y's, for problems with *stable* Lyapunov exponents, though not necessarily distinct. [To be precise, for the case of non-distinct Lyapunov exponents, we show convergence of Y to subspaces associated to the equal exponents, though we will not have that each column of Y converges.] We will also show that convergence is taking place exponentially fast. Once these theoretical results will be at hand, the basic technique to verify (ED-2) will be similar to what we did for the SVD:

Use the columns of $X^+(0)\bar{Y}^+$ and $X^-(0)\bar{Y}^-$ relative to the forward and backward stable modes, $X^+(0)\bar{Y}_s^+$ and $X^-(0)\bar{Y}_s^-$, and verify if $[X^-(0)\bar{Y}_s^-\ X^+(0)\bar{Y}_s^+]$ is a basis for $\mathbb{R}^n$.

Remark 3. In our analysis below, we will focus only on the case $t \geqslant 0$, since, after rewriting assumptions (12) and (13) in backward time, see (14) and (15), the results for $t \leqslant 0$ are identical.

Assume now that a normal fundamental matrix solution $X(t)$ has been chosen (that is, the initial conditions $X(0)$ have been chosen) so that for its unique QR factorization $X(t) = Q(t)R(t)$, with Q orthogonal and R with positive diagonal entries, we have

$$\dot{R} = B(t)R,$$

and the Lyapunov exponents can be computed from the diagonal of B (that is, of R) and are stable. Our purpose in this chapter is to show that—under these mild conditions—certain subspaces associated to R converge and can be used to detect exponential dichotomy on the real line. The result is important, since we will only need to have stable Lyapunov exponents, though not necessarily stable and distinct, as it was the case for the SVD.

So, we will consider the system

$$\dot{x} = B(t)x, \tag{11}$$

with B continuous, bounded and upper triangular. In what follows, we will assume that the diagonal elements of B satisfy one of the following assumptions:

(i) B_{ii} and B_{jj} are integrally separated, i.e. there exist $a, d > 0$ such that

$$\int_s^t \big(B_{ii}(\tau) - B_{jj}(\tau)\big)\, d\tau \geqslant a(t-s) - d, \quad t \geqslant s \geqslant 0,\ i < j, \tag{12}$$

(ii) B_{ii} and B_{jj} are not integrally separated, but $\forall \epsilon > 0$ there exists $M_{ij}(\epsilon) > 0$ such that

$$\left| \int_s^t \big(B_{ii}(\tau) - B_{jj}(\tau)\big)\, d\tau \right| \leqslant M_{ij}(\epsilon) + \epsilon(t-s), \quad t \geqslant s \geqslant 0,\ i < j. \tag{13}$$

Remark 4. For the sake of completeness, we remark that for the backward problem $t \leqslant 0$, one should assume to have taken an initial condition $X(0)$ so that for the QR factorization $X(t) = Q(t)R(t)$, $t \leqslant 0$, one obtains a triangular system $\dot{R} = B(t)R$ satisfying the following analogs of (12) and (13):

(i) B_{ii} and B_{jj} are integrally separated, i.e.

$$\int_s^t \big(B_{ii}(\tau) - B_{jj}(\tau)\big)\, d\tau \geqslant -a(t-s) - d, \quad t \leqslant s \leqslant 0,\ i < j, \tag{14}$$

(ii) B_{ii} and B_{jj} are not integrally separated, but $\forall \epsilon > 0$ there exists $M_{ij}(\epsilon) > 0$ such that

$$\left| \int_s^t \bigl(B_{ii}(\tau) - B_{jj}(\tau)\bigr)\, d\tau \right| \leqslant M_{ij}(\epsilon) - \epsilon(t-s), \quad t \leqslant s \leqslant 0,\ i < j. \tag{15}$$

Considering again only the case of $t \geqslant 0$, assume moreover that the B_{ii}'s are arranged in p, $p \leqslant n$, subsets as follows. Let $n_1, \dots, n_p$ be such that $\sum_{i=1}^n n_i = n$, $m_i = \sum_{k=1}^i n_k$ and denote with $B_{jj}^{(i)}$ the $(m_{i-1} + j)$-th diagonal element of B. The set $\{B_{jj}^{(i)}, j = 1, \dots, n_i\}$ will be the i-th subset. Elements belonging to the same subset satisfy assumption (13) while elements belonging to different subsets satisfy (12) as follows

$$\int_s^t \bigl(B_{jj}^{(i)}(\tau) - B_{kk}^{(i+1)}(\tau)\bigr)\, d\tau \geqslant a(t-s) - d, \quad j = 1, \dots, n_i,\ k = 1, \dots, n_{i+1}.$$

In what follows we will use the notation B_{jj} where there is no ambiguity, otherwise we will use the subset-notation $B_{jj}^{(i)}$. The same criteria will apply also to other quantities.

In [17], it was shown that the assumptions (12) and (13) guarantee that the Lyapunov spectrum can be computed from the diagonal of B and that it is stable. In fact, in both cases (12) and (13), for all $j = 1, \dots, n$, the LEs are given by

$$\limsup_{t\to+\infty} \frac{1}{t} \log \int_0^t B_{jj}(\tau)\, d\tau = \chi^s(R_{jj}) = \lambda_j^s, \qquad \liminf_{t\to+\infty} \frac{1}{t} \log \int_0^t B_{jj}(\tau)\, d\tau = \chi^i(R_{jj}) = \lambda_j^i.$$

Remark 5. Denote with R the principal matrix solution of (11) and let

$$W_j = \left\{ w \in \mathbb{R}^n \;\middle|\; \limsup_{t\to+\infty} \frac{1}{t} \log \| R(t) w \| \leqslant \lambda_j^s \right\},$$

i.e. W_j is the subspace of initial conditions leading to upper LEs lesser or equal to λ_j^s. (The W_j's are called lineals in the classical theory of LEs.) Then conditions (12) and (13) guarantee that W_j is also the space of initial conditions leading to lower LEs lesser or equal to λ_j^i. Indeed, by [17, Theorem 4.2] there exists a Lyapunov transformation T which reduces system (11) into block diagonal form. The lineals for the new block system are the subspaces $\{e_{m_{j-1}+1}, \dots, e_{m_j}, \dots, e_n\}$ so that the lineals for system (11) are the spaces $W_j = \operatorname{span}\{T_{m_{j-1}+1}(0), \dots, T_{m_j}(0), \dots, T_n(0)\}$, where with e_j and T_j we denoted respectively the j-th canonical vector of $\mathbb{R}^n$ and the j-th column of T. By [17, Theorem 5.1], W_j is also the set of initial conditions leading to lower LEs lesser than λ_j^i.

Now, let $Y = R^{-1} \operatorname{diag}(R)$, so that Y satisfies the following ODE

$$\dot{Y} = -Y C(t), \tag{16}$$

where C is strictly upper triangular with entries

$$C_{ij}(t) = B_{ij}(t) \frac{R_{jj}(t)}{R_{ii}(t)} = B_{ij}(t) e^{\int_0^t B_{jj}(\tau) - B_{ii}(\tau)\, d\tau}, \quad j > i. \tag{17}$$

Observe that Y is upper triangular with 1's on the diagonal.

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Notation. We found it useful to adopt two different notations for the columns of Y. The standard Y_j for the j-th column, or the one inherited from the B_{jj}'s: $Y_j^{(i)}(t)$ for the $(m_{i-1}+j)$-th column of $Y(t)$.

Lemma 6. *The elements of C satisfy the following*

(i) $\chi^s(C_{ij}) < -a$, *if B_{ii} and B_{jj} are integrally separated*;
(ii) *for all $\epsilon > 0$, $\chi^{s,i}(B_{ij}) - \epsilon \leqslant \chi^{s,i}(C_{ij}) \leqslant \epsilon$ if B_{ii} and B_{jj} satisfy* (13).

Proof. Let $M = \sup_{t\geqslant 0}\|B(t)\|$. If B_{ii} and B_{jj} belong to different subsets, say the k-th and the l-th respectively, $k < l$, then $|C_{ij}(t)| \leqslant (\sup_{t\geqslant 0}\|B(t)\|)e^{-a(l-k)t+d} = Me^{-a(l-k)t+d}$. It follows easily that $\chi^s(C_{ij}) < -(l-k)a < -a$.

If instead B_{ii} and B_{jj} belong to the same subset then $|C_{ij}(t)| \leqslant Me^{M_{ij}(\epsilon)+\epsilon t}$ and $|C_{ij}(t)| \geqslant |B_{ij}(t)|e^{-M_{ij}(\epsilon)-\epsilon t}$ for arbitrary ϵ. It follows that

$$\chi^{s,i}(B_{ij}) - \epsilon \leqslant \chi^{s,i}(C_{ij}) \leqslant \epsilon. \qquad \square$$

Corollary 7. *Let Y be the matrix in* (16). *If the system is integrally separated, i.e. if the B_{ii}'s are integrally separated for all $i = 1,\dots,n$, then $Y \to \bar{Y}$, with $\bar{Y}$ constant, upper triangular matrix with 1's on the diagonal.*

Proof. By Lemma 6 and integral separation of the B_{ii}'s it follows that $\chi^s(C_{ij}) < 0$ for $i \neq j$, while $C_{ii}(t) = 0$, for all t and for all $i = 1,\dots,n$. Then $C \to_{t\to\infty} 0$ exponentially fast so that $Y \to_{t\to\infty} \bar{Y}$. $\square$

Lemma 8. *Let Y be the matrix in* (16). *Then for all $\epsilon > 0$ sufficiently small, and any $j = 1,\dots,n_i$, $i = 1,\dots,p$, we have*

$$\chi^s\big(Y_j^{(i)}\big) \leqslant (j-1)\epsilon \leqslant (n_i-1)\epsilon.$$

Proof. Eq. (16) can be solved explicitly and the elements of Y are given by the following expression

$$\begin{gathered} Y_{ij}(t) = Y_{ij}(0) - \int_0^t \sum_{k=i}^{j-1} Y_{ik}(\tau)C_{kj}(\tau)\,d\tau, \quad j > i, \\ Y_{ii}(t) = 1, \quad \forall t \geqslant 0. \end{gathered} \tag{18}$$

The $Y_j^{(i)}$ column of Y satisfies the following

$$Y_j^{(i)}(t) = Y_j^{(i)}(0) - \sum_{k=1}^{m_{i-1}} \int_0^t C_{k,m_{i-1}+j}(t_1)Y_k(t_1)\,dt_1 - \sum_{k=m_{i-1}+1}^{m_{i-1}+j-1} \int_0^t C_{k,m_{i-1}+j}(t_1)Y_k(t_1)\,dt_1. \tag{19}$$

Then

$$\big\|Y_j^{(i)}(t)\big\| \leqslant \big\|Y_j^{(i)}(0)\big\| + \sum_{k=1}^{m_{i-1}} \int_0^t \big\|C_{k,m_{i-1}+j}(t_1)Y_k(t_1)\big\|\,dt_1 + \sum_{k=m_{i-1}+1}^{m_{i-1}+j-1} \int_0^t \big\|C_{k,m_{i-1}(t_1)+j}Y_k(t_1)\big\|\,dt_1. \tag{20}$$

Take first $i = 1$. We show through inductive reasoning that $\chi^s(Y_j^{(1)}) \leqslant (j-1)\epsilon$. Clearly $\chi^i(Y_1^{(1)}) = \chi^s(Y_1^{(1)}) = 0$. Notice that for $i = 1$, the first sum in (19) and in (20) is zero. As far as the second sum is concerned, we have $\chi^s(C_{kj}Y_k) \leqslant \chi^s(C_{kj}) + \chi^s(Y_k) \leqslant \epsilon + (k-1)\epsilon$ for $k = 1, \dots, j-1$. Then $\chi^s(\int C_{kj}Y_k) \leqslant k\epsilon$. It follows that $\chi^s(Y_j^{(1)}) \leqslant \max_{k=1,\dots,j-1}(\chi^s(\int C_{kj}Y_k)) \leqslant (j-1)\epsilon$. The proof for $i > 1$ is analogous and it follows upon noticing that the first sum goes to zero exponentially fast since for $k = 1, \dots, m_{i-1}$ we have $\chi^s(C_{k,m_{i-1}+j}Y_k) \leqslant \chi^s(C_{k,m_{i-1}+j}) + \chi^s(Y_k) \leqslant -a + \max_{l=1,\dots,i-1}(n_l - 1)\epsilon < 0$, where in the last inequality we picked $\epsilon < \max_{l=1,\dots,i-1} \frac{a}{(n_l-1)}$. □

Remark 9. Lemma 8 implies $\chi^s(Y_1^{(i)}) = \chi(Y_1^{(i)}) = 0$ for all $i = 1, \dots, p$.

In the two limiting cases of Lemma 8 we have:

- $p = n$ so that $n_i = 1$ for all $i = 1, \dots, n$ and $\chi^i(Y_j) = \chi^s(Y_j) = 0$, $j = 1, \dots, n$ (this follows from Corollary 7 as well);
- $p = 1$ so that $n_1 = n$ and $0 \leqslant \chi^i(Y_j) \leqslant \chi^s(Y_j) \leqslant (j-1)\epsilon$, $j = 1, \dots, n$.

Next, let $S_i(t) = \operatorname{span}(Y_1^{(i)}(t), \dots, Y_{n_i}^{(i)}(t))$, $i = 1, \dots, p$.

Theorem 10. *The spaces S_i converge for $t \to +\infty$: $\lim_{t\to+\infty} S_i(t) = \bar{S}_i$, $i = 1, \dots, p$.*

Proof. Clearly the first n_1 columns of Y always span the same space so that $S_1(t) = \bar{S}_1 = \operatorname{span}\{e_1, \dots, e_{n_1}\}$ for all t. Consider now, for all t, $S_2(t) = \operatorname{span}\{Y_1^{(2)}(t), \dots, Y_{n_2}^{(2)}(t)\}$. Let $T > 0$ and fix $0 < \tau \leqslant 1$. We will prove the statement for S_2, the proof for the other subspaces being analogous. Using (19) we have

$$Y_1^{(2)}(T+\tau) - Y_1^{(2)}(T) = -\sum_{k=1}^{n_1} \int_T^{T+\tau} C_{k,n_1+1}(t_1) Y_k^{(1)}(t_1)\, dt_1. \tag{21}$$

Lemmas 6 and 8 imply $\chi^s(C_{k,n_1+1}Y_k^{(1)}) \leqslant \chi^s(C_{k,n_1+1}) + \chi^s(Y_k^{(1)}) < -a + (n_1 - 1)\epsilon < 0$, if we choose $\epsilon < \frac{a}{(n_1-1)}$. It follows that the integrals in (21) tend to zero for $T \to +\infty$, uniformly in τ. Hence

$$\lim_{T\to\infty} \left(Y_1^{(2)}(T+\tau) - Y_1^{(2)}(T)\right) = 0,$$

and $Y_1^{(2)}(t)$ tends to a constant vector: $\bar{Y}_1^{(2)}$. We do two more steps. First we show that for $T \to \infty$, $\operatorname{span}(\bar{Y}_1^{(2)}, Y_2^{(2)}(T), Y_2^{(2)}(T+\tau))$ is a two-dimensional subspace of $\mathbb{R}^n$. We have

$$\begin{aligned} &Y_2^{(2)}(T+\tau) - Y_2^{(2)}(T) \\ &\quad = -\sum_{k=1}^{n_1} \int_T^{T+\tau} C_{k,n_1+2}(t_1) Y_k^{(1)}(t_1)\, dt_1 - \int_T^{T+\tau} C_{n_1+1,n_1+2}(t_1) Y_1^{(2)}(t_1)\, dt_1. \end{aligned} \tag{22}$$

When $T \to +\infty$ the first sum in the right-hand side goes to zero. Moreover, $Y_1^{(2)} \to \bar{Y}_1^{(2)}$, i.e. for all $\delta > 0$, there exists $T_\delta > 0$ such that for all $t > T_\delta$, and for $j = 1, \dots, n_1 + 1$,

$$\bar{Y}_{1j}^{(2)} - \delta \leqslant Y_{1j}^{(2)}(t) \leqslant \bar{Y}_{1j}^{(2)} + \delta.$$

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It follows that for $j = 1, \dots, n_1 + 1$,

$$\lim_{T\to\infty} \int_T^{T+\tau} C_{n_1+1,n_1+2}(t_1) Y_{1j}^{(2)}(t_1)\,dt_1 = \lim_{T\to\infty} \int_T^{T+\tau} C_{n_1+1,n_1+2}(t_1)\,dt_1\ \bar{Y}_{1j}^{(2)},$$

and therefore $\lim_{T\to\infty}(Y_2^{(2)}(T+\tau) - Y_2^{(2)}(T)) \in \operatorname{span}(\bar{Y}_1^{(2)})$, since $Y_1^{(2)}(t)$ converges to $\bar{Y}_1^{(2)}$ exponentially fast.

Given the triangular structure of Y, this implies that these vectors belong to the same two-dimensional subspace. Denote with $\bar{Y}_1^{(2)}$ and $\bar{Y}_2^{(2)}$ a basis for this subspace. For the next step we have

$$Y_3^{(2)}(T+\tau) - Y_3^{(2)}(T) = -\sum_{k=1}^{n_1} \int_T^{T+\tau} C_{k,n_1+3}(t_1) Y_k^{(1)}(t_1)\,dt_1 - \int_T^{T+\tau} C_{n_1+1,n_1+3}(t_1) Y_1^{(2)}(t_1)\,dt_1$$
$$- \int_T^{T+\tau} C_{n_1+2,n_1+3}(t_1) Y_2^{(2)}(t_1)\,dt_1. \tag{23}$$

As before, when $T \to \infty$, the first integral goes to 0. Moreover, reasoning as above, we obtain that

$$\lim_{T\to\infty}\bigl(Y_3^{(2)}(T+\tau) - Y_3^{(2)}(T)\bigr) \in \operatorname{span}\bigl(\bar{Y}_1^{(2)}, \bar{Y}_2^{(2)}\bigr),$$

so that again, given the triangular structure of Y, these vectors belong to the same three-dimensional subspace. The proof for the other $Y_j^{(i)}$ is analogous. □

Remark 11. From the proof of Theorem 10, we notice that the (vector) functions $Y_1^{(i)}$ converge, as $t \to \infty$, for all $i = 1, \dots, p$. Instead, the remaining $Y_j^{(i)}$ ($j \geqslant 2$) do not necessarily converge, though they keep belonging to the same subspaces.

Our next goal is to show that by taking initial conditions in a direction identified by the subspaces $\bar{S}_i$ we converge to solutions having growth given by the i-th Lyapunov exponents. The following lemma will be useful.

Lemma 12. *Let $w \neq 0$ be a constant vector in $\bar{S}_i$. Write w in the basis $Y_1(t), \dots, Y_n(t)$, as $w = \sum_{j=1}^n c_j(t) Y_j(t) = \sum_{k=1}^p \sum_{j=1}^{n_p} c_j^{(k)}(t) Y_j^{(k)}(t)$. Let $c^{(i)} = [c_1^{(i)}, \dots, c_{n_i}^{(i)}]^T$. Then*

$$\chi^s\bigl(c^{(i)}\bigr) = \chi^i\bigl(c^{(i)}\bigr) = 0.$$

Proof. By Theorem 10, $c_j^{(k)} \to 0$ for $k \neq i$ and $\sum_{j=1}^{n_i} c_j^{(i)}(t) Y_j^{(i)}(t) \to w$. Assume by contradiction that $\chi^s(c^{(i)}) = \alpha > 0$. Notice $\chi^s(c^{(i)}) \leqslant \max_{j=1,\dots,n_i} \chi^s(c_j^{(i)})$. Adopt for w the same block notation we adopted for c. If $\chi^s(c_{n_i}^{(i)}) = \alpha$ then $c_{n_i}^{(i)}(t) Y_{n_i n_i}(t) = c_{n_i}^{(i)}(t) \to w_{n_i}^{(i)}$ and this is a constant quantity. So it must be $\chi^s(c_{n_i}^{(i)}) = 0$. Similarly $c_{n_i-1}^{(i)}(t) + c_{n_i}^{(i)}(t) Y_{n_i,n_i-1}^{(i)}(t) \to w_{n_i-1}^{(i)}$ and this implies $\chi^s(c_{n_i-1}^{(i)}) \leqslant 0$. Reasoning in this way, it must be $\chi^s(c_j^{(i)}) = 0$ for all $j = 1, \dots, n_1$ and this proves $\chi^s(c^{(i)}) = 0$. We still need to show $\chi^i(c^{(i)}) = 0$. Clearly $\chi^i(c^{(i)}) \leqslant 0$. Assume by contradiction that $\chi^i(c^{(i)}) = -\alpha < 0$. Then there exists an increasing sequence $(t_k)_{k\in\mathbb{N}}$, converging to ∞, such that

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$$\lim_{t_k \to +\infty} \frac{c_1^{(i)}(t_k)^2 + \cdots + c_{n_i}^{(i)}(t_k)^2}{e^{(-\alpha+\epsilon)t_k}} = 0,$$

for all $\epsilon > 0$. This means that $\lim_{t_k \to \infty} \frac{c_j^{(i)}(t_k)^2}{e^{(-\alpha+\epsilon)t_k}} = 0$, for all $j = 1, \ldots, n_i$ so that $\chi^i(c_j^{(i)}) \leqslant -\alpha$. Moreover, we have that $\limsup_{t_k \to \infty} \frac{1}{t_k} \log(|c_j^{(i)}(t_k)| \|Y_j^{(i)}(t_k)\|) < 0$ for all $j = 1, \ldots, n_i$, if we choose $\epsilon = \frac{\alpha}{n}$ in Lemma 8. But then, along this sequence, we would have

$$\chi(w) = \chi^s(w) = \limsup_{t_k \to \infty} \frac{1}{t_k} \log\left(\left\| \sum_{j=1}^{n_i} c_j^{(i)}(t_k) Y_j^{(i)}(t_k) \right\|\right) < 0,$$

and this is a contradiction since w is a nonzero constant vector, and thus $\chi(w) = 0$. □

Let $Z(t) = Y(t)^{-1}$. Then $Z(t)$ satisfies the following ODE

$$\dot{Z} = ZC, \tag{24}$$

where C is the matrix whose elements are defined in (17). The elements of Z are given by

$$Z_{ij}(t) = Z_{ij}(0) + \sum_{k=i+1}^{j} \int_0^t C_{ik}(s) Z_{kj}(s)\, ds, \quad i < j,$$
$$Z_{ii}(t) = 1, \quad \forall t \geqslant 0.$$

Denote with $z_i(t)^T$ the i-th row of Z so that $Z = \begin{pmatrix} z_1(t)^T \\ \vdots \\ z_n(t)^T \end{pmatrix}$. Lemma 8 applies to the elements of Z as well so that

$$\chi^s\big(z_j^{(i)}\big) \leqslant (j-1)\epsilon \leqslant (n_i - 1)\epsilon. \tag{25}$$

Remark 13. If the system is integrally separated then $\chi^s(z_i) = \chi(z_i) = 0$, for all $i = 1, \ldots, n$.

Remark 14. Eq. (25) implies $\chi^s(z_1^{(i)}) = \chi(z_1^{(i)}) = 0$, for $i = 1, \ldots, p$.

Denote with $z_j^{(i)}(t)$ the $(m_i + j)$-th row vector of $Z(t)$. Then, being Z the inverse of Y, we have that $z_i^T Y_i = 1$ so z_i is aligned with Y_i and perpendicular to all other Y_j's. Then $\mathrm{span}(z_1^{(i)}(t), \ldots, z_{n_i}^{(i)}(t)) = \mathrm{span}(Y_1^{(i)}(t), \ldots, Y_{n_i}^{(i)}(t)) \to \bar{S}_i$.

Theorem 15. *For all $w \neq 0$, $w \in \bar{S}_j$, and all $j = 1, \ldots, p$:*

$$\limsup_{t \to +\infty} \frac{1}{t} \log\|R(t)w\| = \lambda_j^s, \qquad \liminf_{t \to +\infty} \frac{1}{t} \log\|R(t)w\| = \lambda_j^i,$$

with $R(t)$ fundamental matrix solution of (11).

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Proof. We will first prove that $\chi^s(R(t)w) \geqslant \lambda_j^s$ (similarly, $\chi^i(R(t)w) \geqslant \lambda_j^i$). Then the statement will follow from some geometrical considerations.

Rewrite w in the basis $\{Y_1(t), \dots, Y_n(t)\}$ as

$$w = \sum_{i=1}^{n} c_i(t)Y_i(t) = \sum_{k=1}^{p}\sum_{i=1}^{n_p} c_i^{(k)}(t)Y_i^{(k)}(t) = w_1 + \dots + w_p, \quad w_k = \sum_{i=1}^{n_k} c_i^{(k)}Y_i^{(k)}(t).$$

Notice that, by Lemma 12, we have $\chi(c_i^{(j)}) = 0$, and also $c_i^{(k)}(t) \to_{t\to\infty} 0$, $k \neq j$.

We now look at the characteristic exponent of $R(t)w$. We have

$$\begin{aligned} \|R(t)w\| = \|\operatorname{diag}(R(t))Z(t)w\| &= \left\| \begin{matrix} R_{11}(t)z_1(t)^T w \\ \vdots \\ R_{nn}(t)z_n(t)^T w \end{matrix} \right\| = \left\| \begin{matrix} R_{11}(t)c_1(t) \\ \vdots \\ R_{nn}(t)c_n(t) \end{matrix} \right\| \\ &\geqslant \left\| \begin{matrix} R_{11}^{(j)}(t)c_1^{(j)}(t) \\ \vdots \\ R_{n_i n_i}^{(j)}(t)c_{n_j}^{(j)}(t) \end{matrix} \right\| \geqslant R_{n_j n_j}^{(j)}(t) \left\| \begin{matrix} c_1^{(j)}(t) \\ \vdots \\ c_{n_j}^{(j)}(t) \end{matrix} \right\|. \end{aligned} \tag{26}$$

This implies $\chi^s(R(t)w) \geqslant \lambda_j^s$, and $\chi^i(R(t)w) \geqslant \lambda_j^i$.

For $w \in \bar{S}_1$ we clearly have

$$\limsup_{t\to+\infty} \frac{1}{t}\log\|R(t)w\| = \lambda_1^s \quad \text{and} \quad \liminf_{t\to+\infty} \frac{1}{t}\log\|R(t)w\| = \lambda_1^i.$$

Next, denote with W_2 the space of initial conditions leading to exponential growth less than or equal to λ_2^s, i.e. $W_2 = \{v \in \mathbb{R}^n \text{ s.t. } \limsup_{t\to+\infty} \frac{1}{t}\log\|R(t)v\| \leqslant \lambda_2^s\}$. Then $\dim(W_2) = n - n_1$ while $\dim(\bar{S}_1 \oplus \bar{S}_2) = n_1 + n_2$ so that $\dim((\bar{S}_1 \oplus \bar{S}_2) \cap W_2) = n_2$. Let $w \in \bar{S}_1 \oplus \bar{S}_2$. If $w_1 \neq 0$ then by simply following the same reasoning as before we obtain $\chi(c^{(1)}) = 0$ and $\limsup_{t\to+\infty} \frac{1}{t}\log\|R(t)w\| = \lambda_1^s$. But then this leaves the only possibility that $\bar{S}_2 \subset W_2$ and $\limsup_{t\to+\infty} \frac{1}{t}\log\|R(t)w\| = \lambda_2^s$ for $w \in \bar{S}_2$. The proof for the lower exponent is analogous. Notice that the results for the lower LEs follow from Remark 5 as well. □

3.1. QR convergence speed

Our next goal is to show that there is exponential convergence to the subspaces $\bar{S}_i$, $i = 1, \dots, p$. We will do this following the approach used in [12] for the SVD, with appropriate modifications to account for non-orthogonal factors and of dimension greater than 1.

Consider the unique smooth QL decomposition of Y: $Y(t) = U(t)L(t)$ for all t, with $U \in \mathbb{R}^{n\times n}$ orthogonal and $L \in \mathbb{R}^{n\times n}$ lower triangular with $L_{ii} > 0$.

Notice that $\operatorname{span}\{y_i, \dots, y_n\} = \operatorname{span}\{u_i, \dots, u_n\}$ for all $i = 1, \dots, n$.

Let $M(t) = L(t)^{-1}$ and notice that $Z(t) = Y(t)^{-1} = M(t)U^T(t)$, so that $M(t) = Z(t)U(t)$ and UM^T is the unique QR decomposition of Z^T with positive diagonal elements.

Lemma 16. *Use for the diagonal elements of M the same block notation as for B. Then*

$$0 \leqslant \chi^i(M_{jj}^{(i)}) \leqslant \chi^s(M_{jj}^{(i)}) < (j-1)\epsilon \leqslant (n_i - 1)\epsilon, \qquad \chi^s(M_{ij}) \leqslant \max_{k=1,\dots,p}(n_k - 1)\epsilon.$$

Proof. Obviously $M_{jj}^{(i)}(t) \neq 0$. Moreover $|M_{jj}^{(i)}(t)| = |z_j^{(i)}(t)^T u_j(t)| \leqslant \|z_j^{(i)}(t)\|$, where with u_j we denote the j-th column of U. By (25) it follows $\chi^s(M_{jj}^{(i)}) \leqslant (j-1)\epsilon \leqslant (n_i-1)\epsilon$. Similarly for $M_{ij}(t) = z_i(t)^T u_j(t)$. □

Remark 17. If system (11) is integrally separated, then Lemma 16 can be restated as follows: $\chi^s(M_{ij}) = \chi(M_{ij}) = 0$, for all $i, j = 1, \ldots, n$.

Remark 18. Lemma 16 implies in particular that $\chi^s(M_{n_i+1,j}) = \chi(M_{n_i+1,j}) = 0$, for all $i = 1, \ldots, p$, and $j = 1, \ldots, n_i + 1$.

Remark 19. Assumptions (12) and (13) imply that there exists $T > 0$ so that $R_{jj}^{(i)}(t) > R_{kk}^{(i+1)}(t)$, for $t > T$, $i = 1, \ldots, p-1$, $j = 1, \ldots, n_i$, and $k = 1, \ldots, n_{i-1}$.

Define $U_i(t) = \operatorname{span}(u_1^{(i)}(t), \ldots, u_{n_i}^{(i)}(t))$ for $i = 1, \ldots, p$. Theorem 10 insures $S_i(t) = \operatorname{span}(y_1^{(i)}(t), \ldots, y_{n_i}^{(i)}(t)) \to \bar{S}_i$. Then clearly $U_p(t) = S_p(t) \to \bar{S}_p = \bar{U}_p$. Moreover $U_{p-1}(t) \oplus U_p(t) = S_{p-1}(t) \oplus S_p(t) \to \bar{S}_{p-1} \oplus \bar{S}_p$ and from the uniqueness of the orthogonal complement it follows $U_{p-1}(t) \to \bar{U}_{p-1}$. Reasoning in the same way, we can prove that $U_i \to \bar{U}_i$ for $i = p-2, \ldots, 1$. To determine speed of convergence of U_i to $\bar{U}_i$, define $W_i(t) = U_i(t) \oplus \cdots \oplus U_p(t)$, $\bar{W}_i = \bar{U}_i \oplus \cdots \oplus \bar{U}_p$. Let $e = (n_1, \ldots, n_p)$ and $F_e(n)$ be the space of flags in $\mathbb{R}^n$ of type e, i.e., the set of all filtrations $\mathcal{V} = (V_i)_{i=1}^p$ such that $\mathbb{R}^n = V_1 \supset \cdots \supset V_p$. Given a filtration $\mathcal{V}^{(j)} = (V_i^{(j)})_{i=1}^p$, define the sets $K_i^{(j)}$ such that $K_p^{(j)} = V_p^{(j)}$, $V_i^{(j)} = K_i^{(j)} \oplus V_{i+1}^{(j)}$, and $K_i^{(j)} \perp V_{i+1}^{(j)}$, for $i = p-1, \ldots, 1$. The following quantity defines a metric in $F_e(n)$ (see [20])

$$d\big(\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\big) = \max_{\substack{i \neq j \\ \|x\| = \|y\| = 1 \\ x \in K_i^{(1)},\, y \in K_j^{(2)}}} \big|(x, y)\big|^{\Delta / |\lambda_i^s - \lambda_j^s|}, \tag{27}$$

with $\Delta = \min_{i \neq j} |\lambda_j^s - \lambda_i^s| / (n-1)$. Let $P_i^{(j)}$ be the orthogonal projection into $K_i^{(j)}$. Then as in [12]

$$d\big(\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\big) = \max_{i \neq j} \big\| P_i^{(1)} P_j^{(2)} \big\|^{\Delta / |\lambda_j^s - \lambda_i^s|}. \tag{28}$$

Notice that, for all t, $\mathcal{W}(t) = (W_i(t))_{i=1}^p \in F_e(n)$, and that for this filtration, $U_i(t) = K_i$, $i = 1, \ldots, p$, where the K_i's are the subspaces defined above. In the same way $\bar{\mathcal{W}} = (\bar{W}_i)_{i=1}^p \in F_e(n)$ and $\bar{U}_i = K_i$. Let $P_i(t)$ be the orthogonal projection into $U_i(t)$. Then $P_i(t) \to \bar{P}_i$, with $\bar{P}_i$ orthogonal projection into $\bar{U}_i$. Our aim is to prove that the convergence of $P_i(t)$ to $\bar{P}_i$ is exponential. To do so, we will first prove Theorem 20, and the sought result will follow from (30) together with (28).

Theorem 20. *Assume the diagonal elements of B satisfy either* (12) *or* (13)*. Let $0 \leqslant \tau < 1$ be fixed and define $\beta_{ij}(t) = \|P_i(t+\tau) P_j(t)\|$, $i, j = 1, \ldots, p$, where*

$$P_i(t) = \big[u_1^{(i)}(t), \ldots, u_{n_i}^{(i)}(t)\big]\big[u_1^{(i)}(t), \ldots, u_{n_i}^{(i)}(t)\big]^T \tag{29}$$

is the orthogonal projection onto $U_i(t)$. Then, for arbitrary $\epsilon_0 > 0$ sufficiently small we have

$$\chi^s(\beta_{ij}) \leqslant \min\big(\lambda_j^s - \lambda_i^i, -a(j-i)\big) + \epsilon_0, \quad j > i,$$
$$\chi^s(\beta_{ij}) \leqslant \min\big(\lambda_i^s - \lambda_j^i, -a(i-j)\big) + \epsilon_0, \quad i > j. \tag{30}$$

Proof. To prove (30) it suffices to show that for all $x \in U_j(t)$,

$$\chi^s\big(\big\|P_i(t+\tau)P_j(t)x\big\|\big) \leqslant \min\big(\lambda_j^s - \lambda_i^s, -a(j-i)\big) + \epsilon_0.$$

Let us consider the case of $j > i$.

Take $x \in U_j(t)$, $x = \sum_{k=1}^{n_j} c_k^{(j)}(t)u_k^{(j)}(t)$. Let $c^{(j)}(t) = [0, \dots, 0, c_1^{(j)}(t), \dots, c_{n_j}^{(j)}(t), 0, \dots, 0]^T$ and $D(t) = \operatorname{diag}(R(t))$. Notice that the vector $D(t)M(t)c^{(j)}(t)$ has first m_{j-1} components equal to zero. Then by Remark 19, for $t > T$ we have

$$\big\|R(t)x\big\| = \left\|D(t)M(t)U^T(t)\sum_{k=1}^{n_j} c_k^{(j)}(t)u_k^{(j)}(t)\right\| = \big\|D(t)M(t)c^{(j)}(t)\big\| \leqslant \tilde{R}_{jj}(t)\big\|M(t)\big\|\|x\|, \quad (31)$$

with $\tilde{R}_{jj}(t) = \max_{k=1,\dots,n_j} R_{kk}^{(j)}$.

Rewrite x in the basis $u_1(t+\tau), \dots, u_n(t+\tau)$, as $x = \sum_{k=1}^{p}\sum_{l=1}^{n_k} b_l^{(k)}(t)u_l^{(k)}(t+\tau)$. In particular let $y = \sum_{l=1}^{n_i} b_l^{(i)}(t)u_l^{(i)}(t+\tau)$ be the orthogonal projection of x onto $U_i(t+\tau)$ so that $\|y\| = \|P_i(t+\tau)P_j(t)x\|$. Let $N^{-1} = \sup_{t\geqslant 0}\|R(t+\tau,t)\|$, with $R(t+\tau,t)$ being the solution at time $t+\tau$ of $\dot{R} = B(t)R$, $R(t,t) = I$. Then $R(t) = R(t+\tau,t)^{-1}R(t+\tau)$ and

$$\big\|R(t)x\big\| \geqslant N\big\|R(t+\tau)x\big\| = N\left\|D(t+\tau)M(t+\tau)\begin{bmatrix} b_1(t+\tau) \\ \vdots \\ b_n(t+\tau)\end{bmatrix}\right\|. \quad (32)$$

Take first $i = 1$, $l = 1$. We want to estimate the exponential behavior of $b_l^{(i)} = b_1^{(1)}$. Then (31) and (32) together with M being lower triangular imply

$$\big|M_{11}(t)b_1^{(1)}(t)\big| \leqslant \frac{\tilde{R}_{jj}(t)}{R_{11}(t)}\big\|M(t)\big\|\|x\|,$$

and this together with (13) and Remark 18 imply

$$\chi^s\big(b_1^{(1)}\big) = \chi^s\big(M_{11}b_1^{(1)}\big) \leqslant \min\big(\lambda_j^s - \lambda_1^i, -a(j-1)\big).$$

Consider now $i = 1$, $l = 2$. Then as above

$$\big|M_{21}(t)b_1^{(1)}(t) + M_{22}(t)b_2^{(1)}(t)\big| \leqslant \frac{\tilde{R}_{jj}(t)}{R_{22}(t)}\big\|M(t)\big\|\|x\|.$$

$\chi^s(|M_{21}b_1^{(1)} + M_{22}b_2^{(1)}|) \leqslant \max(\chi^s(M_{21}b_1^{(1)}), \chi^s(M_{22}b_2^{(1)}))$ and it is equal to the maximum if it is unique. So we have the following possibilities:

(i) $\chi^s(M_{22}b_2^{(1)}) > \chi^s(M_{21}b_1^{(1)})$,
(ii) $\chi^s(M_{22}b_2^{(1)}) < \chi^s(M_{21}b_1^{(1)})$,
(iii) $\chi^s(M_{22}b_2^{(1)}) = \chi^s(M_{21}b_1^{(1)})$.

Cases (i) and (ii) both imply

$$\chi^s\big(b_2^{(1)}\big) \leqslant \min\big(\lambda_j^s - \lambda_1^i, -a(j-1)\big).$$

For case (iii) we get

$$\chi^s\big(b_2^{(1)}\big) \leqslant \chi^i(M_{22}) + \chi^s\big(b_2^{(1)}\big) \leqslant \chi^s\big(M_{22}b_2^{(1)}\big) \leqslant \chi^s(M_{21}) + \chi^s\big(b_1^{(1)}\big) < \epsilon + \min\big(\lambda_j^s - \lambda_1^i, -a(j-1)\big),$$

where the last inequality follows from Lemma 16. The proof for $i=1$ and $l=3$ is analogous. In this case we have again three different possibilities:

(i) $\chi^s(M_{33}b_3^{(1)}) > \chi^s(M_{31}b_1^{(1)}), \chi^s(M_{32}b_2^{(1)})$,
(ii) $\chi^s(M_{33}b_3^{(1)}) < \chi^s(M_{31}b_1^{(1)})$, or $\chi^s(M_{33}b_3^{(1)}) < \chi^s(M_{32}b_2^{(1)})$,
(iii) $\chi^s(M_{33}b_3^{(1)}) = \chi^s(M_{3j}b_j^{(1)}) > \chi^s(M_{3k}b_k^{(1)})$, with $j=1,2$, $k=2,1$.

Again cases (i) and (ii) are easy. For case (iii) (when $\chi^s(M_{33}b_3^{(1)})$ is not the only maximum), we examine the equality with $j=2$, $k=1$, which gives rise to the bound $\chi^s(M_{33}b_3^{(1)}) = \chi^s(M_{32}b_2^{(1)}) \leqslant \chi^s(M_{32}) + \chi^s(b_2^{(1)}) < 2\epsilon + \min(\lambda_j^s - \lambda_1^i, -a(j-1))$, where the 2ϵ term comes from the estimate in Lemma 16 together with the bound for $\chi^s(b_2^{(1)})$. The proof for $i=1$ and $l=4,\dots,n_1$ is analogous, and we will get the following estimate: $\chi^s(b_l^{(1)}) < (l-1)\epsilon + \min(\lambda_j^s - \lambda_1^i, -a(j-1))$.

Finally, we have $\chi^s(\|y\|) = \frac{1}{2}\chi^s(\sum_{k=1}^{n_1}(b_k^{(1)})^2) \leqslant \max_{k=1,\dots,n_1}\chi^s(b_k^{(1)}) \leqslant \min(\lambda_j^s - \lambda_1^i, -a(j-1)) + (n_1-1)\epsilon$ and (30) is proved for $i=1$ and $j>1$. Here $\epsilon_0 = (n_1-1)\epsilon$. Notice that if we choose $\epsilon < \frac{a}{(n_1-1)}$, then $\min(\lambda_j^s - \lambda_1^i, -a(j-1)) + (n_1-1)\epsilon \leqslant \min(\lambda_j^s - \lambda_2^i, -a(j-2))$. This follows upon noticing that $\min(\lambda_j^s - \lambda_1^i, -a(j-1)) + a \leqslant \min(\lambda_j^s - \lambda_2^i, -a(j-2))$.

Take now $i=2$, $l=1$. Then (32) and (31) imply

$$\tilde{R}_{jj}(t)\big\|M(t)\big\|\|x\| \geqslant \big\|R(t)x\big\| \geqslant R_{11}^{(2)}(t)\left|\sum_{k=1}^{n_1} M_{n_1+1,k}(t)b_k^{(1)}(t) + M_{n_1+1,n_1+1}(t)b_1^{(2)}(t)\right|,$$

so that

$$\chi^s\left(\sum_{k=1}^{n_1} M_{n_1+1,k}b_k^{(1)} + M_{n_1+1,n_1+1}b_1^{(2)}\right) \leqslant \min\big(\lambda_j^s - \lambda_2^i, -a(j-2)\big).$$

And again we have three possibilities:

(i) $\chi^s(M_{n_1+1,n_1+1}b_1^{(2)}) > \chi^s(\sum_{k=1}^{n_1} M_{n_1+1,k}b_k^{(1)})$,
(ii) $\chi^s(M_{n_1+1,n_1+1}b_1^{(2)}) < \chi^s(\sum_{k=1}^{n_1} M_{n_1+1,k}b_k^{(1)})$,
(iii) $\chi^s(M_{n_1+1,n_1+1}b_1^{(2)}) = \chi^s(\sum_{k=1}^{n_1} M_{n_1+1,k}b_k^{(1)})$.

Cases (i) and (ii) imply $\chi^s(b_1^{(2)}) \leqslant \min(\lambda_j^s - \lambda_2^i, -a(j-2))$. Case (iii) implies $\chi^s(b_1^{(2)}) \leqslant \max_{k=1,\dots,n_1}\chi^s(M_{n_1+1,k}b_k^{(1)}) \leqslant \min(-a(j-1), \lambda_j^s - \lambda_1^i) + (n_1-1)\epsilon \leqslant \min(\lambda_j^s - \lambda_2^i, -a(j-2))$.

For $i=2$, $l=2,\dots,n_2$ the proof is analogous to the one for $i=1$. Reasoning in the same way we will obtain $\chi^s(b_l^{(2)}) \leqslant (n_2-1)\epsilon + \min(\lambda_j^s - \lambda_2^i, -a(j-2))$.

The statement of the theorem follows by choosing $\epsilon_0 = (n_i-1)\epsilon < a$, for $i=1,\dots,j-1$.

The proof for $i>j$ is analogous. □

Corollary 21. *Assume system* (11) *to be integrally separated and let* $\beta_{ij}(t) = \|P_i(t+\tau)P_j(t)\|$, *where* $P_i(t)$ *is the orthogonal projection onto* $U_i(t)$ *defined in* (29)*. Then*

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$$\chi^s(\beta_{ij}) \leqslant \min\bigl(\lambda_j^s - \lambda_i^i, -a(j-i)\bigr), \quad j > i,$$
$$\chi^s(\beta_{ij}) \leqslant \min\bigl(\lambda_i^s - \lambda_j^i, -a(i-j)\bigr), \quad i > j.$$

Finally, we are ready to estimate the rate of convergence of the orthogonal projections $P_i(t)$ to $\bar{P}_i$, $i = 1, \dots, p$. Let $\alpha_{ij}(t) = \|P_i(t)\bar{P}_j\|$, $i, j = 1, \dots, p$.

Theorem 22. *Let the diagonal elements of B satisfy assumption* (13) *or* (12)*. Then for all* $i, j = 1, \dots, p$,

$$\chi^s(\alpha_{ij}) \leqslant A\bigl|\lambda_i^s - \lambda_j^s\bigr|, \quad i \neq j, \qquad \chi^s(1 - \alpha_{ii}) \leqslant \max_{j \neq i} \chi^s(\alpha_{ij}), \tag{33}$$

where $A = \max_{k \neq l}(\chi^s(\beta_{kl})/|\lambda_k^s - \lambda_l^s|)$.

Proof. The proof for $i \neq j$ is much the same as that of [12, Theorem 5.4]. We omit the details here, and we just point out that the result follows from (30) together with (28).

In what follows, we prove the result for $i = j$. Since $P_1(t) + P_2(t) + \dots + P_p(t) = I$, with I the identity matrix in $\mathbb{R}^{n \times n}$, we can rewrite $\bar{P}_i = \sum_{j=1}^p P_j(t)\bar{P}_i$. Then $1 = \|\bar{P}_i\| \leqslant \sum_{j=1}^p \|P_j(t)\bar{P}_i\|$ so that $(1 - \|\bar{P}_i P_i(t)\|) \leqslant \sum_{j \neq i} \|P_j(t)\bar{P}_i\|$ and this rewrites as $(1 - \alpha_{ii}(t)) \leqslant \sum_{j \neq i} \alpha_{ij}(t)$. It follows that $\chi^s(1 - \alpha_{ii}) \leqslant \chi^s(\sum_{j \neq i} \alpha_{ij}) \leqslant \max_{j \neq i} \chi^s(\alpha_{ij})$. □

Remark 23. The importance of exponential convergence to the relevant subspaces cannot be stressed enough. For example, in a numerical implementation of methods to ascertain ED, it implies that one can truncate the real line to a finite (large) interval, ascertain ED on this finite interval, and then invoke perturbation results to conclude that the original problem had ED on the infinite interval. It is also important to stress that the assumptions we have needed in order to infer exponential convergence of the subspaces are *structural assumptions* on the system, namely some form of integral separation on the two half-lines (a generic assumption, see [24]), or more precisely we assumed that the system has stable Lyapunov exponents in forward and backward time. This is in contrast to existing results to infer ED, which require strong assumptions on the type of system at hand, e.g. slow varying systems, or systems for which roughness and L_1 perturbation results can be applied; again, see [8,11,19,21,26,33,34].

4. Exponential dichotomy from *QR* on $\mathbb{R}$

In Sections 2 and 3, we have used the equivalence to having ED given by the conditions (ED-1) and (ED-2), to see how the forward and backward stable subspaces can be obtained from the SVD or QR of appropriate fundamental matrix solutions. In this section we consider an alternative strategy, namely we give sufficient conditions that guarantee that the Sacker–Sell spectrum (on the whole line) is bounded away from the origin.

Consider the linear inhomogeneous problem

$$\dot{y} = A(t)y + g(t), \tag{34}$$

where A and g are uniformly bounded and A is piecewise continuous. Recall (see [11, p. 67]) that the homogeneous system (that is (3)) has exponential dichotomy on $\mathbb{R}$, that (6) holds with $t, s \in \mathbb{R}$, if and only if (34) has a unique bounded solution for any bounded and continuous function g.

First of all, we envision performing a global time dependent orthogonal change of variables ($x(t) = Q^T(t)y(t)$ and $f(t) = Q^T(t)g(t)$) which brings the system in upper triangular form B. We write this system in block form

$$\dot{x} = B(t)x + f(t), \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix}. \tag{35}$$

Below, let $\|x\| = \sup_t \|x(t)\|_2$, $\|f\| = \sup_t \|f(t)\|_2$, and $\|A\| = \sup_t \|A(t)\|_F$.

Next, we make the following assumptions on a fundamental matrix solution R associated to B: $\dot{R} = B(t)R$ (these are effectively assumptions on the existence of a transformation Q leading to a triangular problem for which these assumptions are satisfied). There exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ and $K_1 \geqslant 1$, $K_2 \geqslant 1$ such that

$$\left\| R_{11}(t)R_{11}^{-1}(s) \right\| \leqslant K_1 e^{-\alpha_1(s-t)}, \quad s \geqslant t, \tag{36}$$

$$\left\| R_{22}(t)R_{22}^{-1}(s) \right\| \leqslant K_2 e^{-\alpha_2(t-s)}, \quad t \geqslant s. \tag{37}$$

Theorem 24. *If* (36) *and* (37) *hold, then* (11) *has ED on* $\mathbb{R}$; *that is,* (6) *holds with* $t, s \in \mathbb{R}$, $X(t) = R(t) = \begin{pmatrix} R_{11}(t) & R_{12}(t) \\ 0 & R_{22}(t) \end{pmatrix}$, *with*

$$R_{12}(t) = -R_{11}(t) \int_t^{\infty} R_{11}^{-1}(\tau) B_{12}(\tau) R_{22}(\tau)\, d\tau \tag{38}$$

and $P = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. *Indeed, with this* R, *there exists a bounded solution* x *of* (35) *such that*

$$\|x\| \leqslant \|f\| \cdot \left\{ \frac{K_2}{\alpha_2} \sqrt{1 + \kappa^2 \frac{K_1^2}{(\alpha_1 + \alpha_2)^2}} + \frac{K_1}{\alpha_1} \sqrt{1 + \kappa^2 \frac{K_2^2}{(\alpha_1 + \alpha_2)^2}} \right\}, \tag{39}$$

where $\kappa = \|B_{12}\|$.

Proof. We write by the variation of parameters formula

$$x(t) = \int_{-\infty}^{t} R(t) P R^{-1}(s) f(s)\, ds - \int_t^{\infty} R(t)(I - P) R^{-1}(s) f(s)\, ds. \tag{40}$$

We employ the upper triangular fundamental matrix solution $R(t)$ (see [16, Lemma 7.4]) where $R_{12}(t)$ is given by (38). Then

$$R(t) P R^{-1}(s) = \begin{pmatrix} 0 & R_{12}(t) R_{22}^{-1}(s) \\ 0 & R_{22}(t) R_{22}^{-1}(s) \end{pmatrix},$$

$$R(t)(I - P) R^{-1}(s) = \begin{pmatrix} R_{11}(t) R_{11}^{-1}(s) & R_{11}(t)(R^{-1}(s))_{12} \\ 0 & 0 \end{pmatrix},$$

where

$$\left(R^{-1}(s)\right)_{12} = \left[\int_s^{\infty} R_{11}^{-1}(\tau) B_{12}(\tau) R_{22}(\tau)\, d\tau \right] R_{22}^{-1}(s).$$

Because of (36) and (37), to verify (6) we need to get (exponentially decreasing) bounds for $\|R_{12}(t)R_{22}^{-1}(s)\|$ when $t \geqslant s$ and for $\|R_{11}(t)(R^{-1}(s))_{12}\|$ when $t \leqslant s$. We have

$$\int_{-\infty}^{t} \left\| R_{22}(t) R_{22}^{-1}(s) \right\| ds \leqslant \int_{-\infty}^{t} K_2 e^{-\alpha_2(t-s)}\, ds \leqslant \frac{K_2}{\alpha_2}$$

and

$$\int_t^\infty \left\| R_{11}(t) R_{11}^{-1}(s) \right\| ds \leqslant \int_t^\infty K_1 e^{-\alpha_1 (s-t)} \, ds \leqslant \frac{K_1}{\alpha_1}.$$

Next

$$\int_{-\infty}^t R_{12}(t) R_{22}^{-1}(s) \, ds = - \int_{-\infty}^t R_{11}(t) \int_t^\infty R_{11}^{-1}(\tau) B_{12}(\tau) R_{22}(\tau) \, d\tau \, R_{22}^{-1}(s) \, ds$$

and for $t \geqslant s$,

$$\left\| R_{12}(t) R_{22}^{-1}(s) \right\| \leqslant \kappa \int_t^\infty K_1 e^{-\alpha_1(\tau - t)} K_2 e^{-\alpha_2(\tau - s)} \, d\tau \leqslant \kappa \frac{K_1 K_2}{(\alpha_1 + \alpha_2)} e^{-\alpha_2(t-s)}. \tag{41}$$

Moreover,

$$\begin{aligned} \int_{-\infty}^t \left\| R_{12}(t) R_{22}^{-1}(s) \right\| ds &\leqslant \kappa \frac{K_1 K_2}{(\alpha_1 + \alpha_2)} \int_{-\infty}^t e^{-\alpha_2(t-s)} \, ds \\ &\leqslant \kappa \frac{K_1 K_2}{\alpha_2(\alpha_1 + \alpha_2)}. \end{aligned} \tag{42}$$

Similarly,

$$\int_t^\infty R_{11}(t) \big(R^{-1}(s)\big)_{12} \, ds = \int_t^\infty \left[\int_s^\infty R_{11}^{-1}(\tau) B_{12}(\tau) R_{22}(\tau) \, d\tau \right] R_{22}^{-1}(s) \, ds$$

and for $s \geqslant t$,

$$\left\| R_{11}(t) R_{12}^{-1}(s) \right\| \leqslant \kappa \int_s^\infty K_1 e^{-\alpha_1(\tau - t)} K_2 e^{-\alpha_2(\tau - s)} \, d\tau \leqslant \kappa \frac{K_1 K_2}{(\alpha_1 + \alpha_2)} e^{-\alpha_1(s-t)}. \tag{43}$$

Moreover,

$$\begin{aligned} \int_t^\infty \left\| R_{11}(t) \big(R^{-1}(s)\big)_{12} \right\| ds &\leqslant \kappa \frac{K_1 K_2}{(\alpha_1 + \alpha_2)} \int_t^\infty e^{-\alpha_1(s-t)} \, ds \\ &\leqslant \kappa \frac{K_1 K_2}{\alpha_1(\alpha_1 + \alpha_2)}. \end{aligned}$$

Thus,

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$$\int_{-\infty}^{t} \left\| R(t)PR^{-1}(s) \right\| ds \leqslant \frac{K_2}{\alpha_2}\sqrt{1+\kappa^2\frac{K_1^2}{(\alpha_1+\alpha_2)^2}},$$

and

$$\int_{t}^{\infty} \left\| R(t)(I-P)R^{-1}(s) \right\| ds \leqslant \frac{K_1}{\alpha_1}\sqrt{1+\kappa^2\frac{K_2^2}{(\alpha_1+\alpha_2)^2}}. \tag{44}$$

Now, from (41) and (43), we obtain (6) on $\mathbb{R}$. Moreover, using (42) and (44) in (40), we obtain (39). □

Remark 25. If there exist $\alpha > 0$, $K \geqslant 1$ such that

$$\left\| R_{22}(t)R_{22}^{-1}(s) \right\| \cdot \left\| R_{11}(s)R_{11}^{-1}(t) \right\| \leqslant Ke^{-\alpha(t-s)}, \quad t \geqslant s, \tag{45}$$

where $R_{11}(t_0) = I$ and $R_{22}(t_0) = I$ for some t_0, then the system is exponentially separated [25]. The exponential separation condition (45) implies having integral separation between arbitrary diagonal elements of R_{11} and R_{22}. Clearly, (36) and (37) imply (45). In addition, (45) implies

$$\left\| R_{12}(t) \right\| \leqslant \kappa \int_{t}^{\infty} Ke^{-\alpha(\tau-t)}\, d\tau \cdot \left\| R_{22}(t) \right\| \leqslant \kappa\frac{K}{\alpha} \cdot \left\| R_{22}(t)R_{22}^{-1}(t_0) \right\| \to 0, \quad \text{as } t \to +\infty \tag{46}$$

using (37). This means that we are employing a fundamental matrix solution $R(t)$ such that $R_{11}(t_0) = I$, $R_{22}(t_0) = I$, and $\lim_{t\to\infty} R_{12}(t) = 0$. If (45) holds and there exist $L_1 > 0$ and $\beta_1 > 0$ such that $\alpha > \beta_1$ and

$$\left\| R_{11}(s)R_{11}^{-1}(t) \right\| \geqslant L_1 e^{-\beta_1(t-s)}, \quad t \geqslant s,$$

then (37) holds. Similarly, if (45) holds and there exist $L_2 > 0$ and $\beta_2 > 0$ such that $\alpha > \beta_2$ and

$$\left\| R_{22}(t)R_{22}^{-1}(s) \right\| \geqslant L_2 e^{-\beta_2(t-s)}, \quad t \geqslant s,$$

then (36) holds.

Q4 **Remark 26.** For the triangular problem of which in Theorem 24, we can explicitly give the forward and backward stable subspaces. In fact, from the proof of Theorem 24, we have that

$$\mathcal{S}^+ = \operatorname{span}\begin{pmatrix} R_{12}(0) \\ I_2 \end{pmatrix}, \qquad \mathcal{S}^- = \operatorname{span}\begin{pmatrix} I_1 \\ 0 \end{pmatrix},$$

where I_1 and I_2 are identity matrices of the same size as the blocks R_{11} and R_{22}, respectively, and $R_{12}(0)$ is defined from (38).

5. Examples

In this section we elucidate the previous theoretical results with examples of two-dimensional upper triangular systems.

1. First, consider the following problem

$$\dot{R} = B(t)R = \begin{pmatrix} \arctan(t) + \frac{t}{1+t^2} & \epsilon f(t) \\ 0 & -\frac{1}{2}(\arctan(t) + \frac{t}{1+t^2}) \end{pmatrix} R, \tag{47}$$

with $\epsilon \geqslant 0$ and f bounded and continuous in $\mathbb{R}$. Notice that the diagonal elements B_{11} and B_{22} of the coefficient matrix in (47) satisfy conditions (12) and (14) so that the scalar problems $\dot{R}_{ii} = B_{ii}(t)R_{ii}$, for $i = 1, 2$, do not admit ED on $\mathbb{R}$. This implies that, for $\epsilon = 0$, the system (47) does not admit ED on $\mathbb{R}$ either. Then, we consider the case of $\epsilon > 0$ and try to establish if (47) has or not ED on $\mathbb{R}$. Observe that (36)–(37) do not hold (regardless of ϵ) and Theorem 24 cannot be used to establish ED on $\mathbb{R}$.

The principal matrix solution Φ of (47) is given by

$$\Phi_{11}(t) = e^{t\arctan(t)};$$

$$\Phi_{12}(t) = \epsilon e^{t\arctan(t)} \int_0^t \left(e^{-3/2 s\arctan(s)} f(s)\right) ds;$$

$$\Phi_{21} = 0, \qquad \Phi_{22}(t) = e^{-\frac{t\arctan(t)}{2}}.$$

Hence, (ED-1) is satisfied, and in order to verify whether or not there is ED on the whole line we need to check whether or not (ED-2) is satisfied. The forward and backward stable subspaces are one-dimensional and can be computed explicitly. We are looking for two vectors $v = (v_1, v_2)^T$ and $w = (w_1, w_2)^T$ such that

$$\lim_{t\to+\infty} \left\|\Phi(t)v\right\| = 0, \qquad \lim_{t\to-\infty} \left\|\Phi(t)w\right\| = 0. \tag{48}$$

The first of (48) rewrites as

$$\Phi_{11}(t)v_1 + \Phi_{12}(t)v_2 \to 0, \qquad \Phi_{22}(t)v_2 \to 0,$$

and clearly the second requirement is verified for any v_2. For v_1, we need to take it so that

$$\Phi_{11}(t)v_1 + \Phi_{12}(t)v_2 = e^{t\arctan(t)} \left(v_1 + \epsilon \int_0^t \left(e^{-\frac{3}{2}s\arctan(s)} f(s)\right) ds\, v_2\right) \to 0. \tag{49}$$

So, we must choose (except for normalization of v_2):

$$v_1 = -\lim_{t\to+\infty} \epsilon \int_0^t \left(e^{-\frac{3}{2}s\arctan(s)} f(s)\right) ds, \qquad v_2 = 1.$$

With this choice of v_1, the expression in (49) goes to zero and $\mathcal{S}^+ = \mathrm{Span}(v)$.

If we repeat the same reasoning for the second limit in (48), we obtain

$$w_1 = -\lim_{t\to-\infty} \epsilon \int_0^t \left(e^{-\frac{3}{2}s\arctan(s)} f(s)\right) ds, \qquad w_2 = 1.$$

Observe that $e^{-\frac{3}{2}s\arctan(s)}$ is an even function. We distinguish two cases.

(a) If f is odd, $w_1 = v_1$ and $\mathcal{S}^+ = \mathcal{S}^-$ so that (47) does not admit ED on $\mathbb{R}$.
(b) If f is even, $w_1 = -v_1$, $\mathcal{S}^+ \cap \mathcal{S}^- = \{0\}$ and (47) admits ED on $\mathbb{R}$.

Remark 27. Example (47) can be easily extended to the case of $B_{11} = -B_{22}$, and B_{11} such that

$$\int_s^t B_{11}(\tau)\,d\tau \geqslant a(t-s), \quad t \geqslant s \geqslant 0,$$

$$\int_s^t B_{11}(\tau)\,d\tau \geqslant -a(t-s), \quad t \leqslant s \leqslant 0,$$

where $a > 0$. For example, this can be verified for a function B_{11} which is odd, like the coefficient B_{11} in (47). In this class of problems, B_{11} is always leading to the unstable Lyapunov exponent (positive in forward time, negative in backward time), so that the principal matrix solution is a suitable initial condition both in forward and backward time (see Remark 2). Again, if f is even and $\epsilon \neq 0$, there is ED on $\mathbb{R}$, while if f is odd there is no ED for any ϵ. Notice that (36)–(37) do not hold; in fact, for this problem having or not ED depends in a subtle way on the term B_{12}.

2. This is generalization of the example in Remark 2 and should be compared with the case in the last remark above. Consider $B_{11}(t) = -B_{22}(t)$, and assume that there is integral separation on the whole line so that

$$\int_s^t \left(B_{11}(\tau) - B_{22}(\tau)\right) d\tau = 2\int_s^t B_{11}(\tau)\,d\tau \geqslant a(t-s) - d$$

for $a > 0$, $d \geqslant 0$, and $t \geqslant s$. Then, the scalar problems $\dot{R}_{ii} = B_{ii}(t)R_{ii}$, $i = 1, 2$, have ED on the whole line, (36) and (37) hold and the problem has ED on $\mathbb{R}$ for any ϵ and $f(t)$. If we want to compute the stable subspaces (forward and backward), then to obtain $\mathcal{S}^+$ we may use the principal matrix solution. To obtain $\mathcal{S}^-$, we cannot use the principal matrix solution and a different fundamental matrix must be employed, though in this example we actually have $\mathcal{S}^- = \mathrm{Span}(e_1)$, see Remark 26.

6. Conclusion

In this paper we have shown how techniques based on the SVD or QR decomposition of a fundamental matrix solution can be used to determine whether or not a system has exponential dichotomy on the real line. An important consequence of our results is that for a large class of problems, namely those with integral separation on both lines or more generally with stable Lyapunov exponents on both half-lines, the existence of exponential dichotomy may be determined from information on finite intervals up to exponentially small perturbations. This enlarges the class of problems for which finite interval computations are sufficient to determine exponential dichotomy from those that are asymptotically constant, asymptotically periodic, or almost periodic [27]. Another important benefit of our

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theoretical results is that they lend themselves to practical algorithmic procedures to verify whether or not a system has exponential dichotomy, as it will be reported elsewhere.

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