

A NEW MODEL FOR REALISTIC RANDOM PERTURBATIONS OF STOCHASTIC OSCILLATORS

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ABSTRACT. Classical theories predict that solutions of differential equations will leave any neighborhood of a stable limit cycle, if white noise is added to the system. In reality, many engineering systems modeled by second order differential equations, like the van der Pol oscillator, show incredible robustness against noise perturbations, and the perturbed trajectories remain in the neighborhood of a stable limit cycle for all times of practical interest. In this paper, we propose a new model of noise to bridge this apparent discrepancy between theory and practice. Restricting to perturbations from within this new class of noise, we consider stochastic perturbations of second order differential systems that –in the unperturbed case– admit asymptotically stable limit cycles. We show that the perturbed solutions are globally bounded and remain in a tubular neighborhood of the underlying deterministic periodic orbit. We also define stochastic Poincaré map(s), and further derive partial differential equations for the transition density function.

1. INTRODUCTION

It is well understood that many engineering and physical systems, like oscillators, can be modeled by deterministic dynamical systems having stable limit cycles (periodic orbits) as attractors.

A prototypical example is the van der Pol oscillator, that is governed by the second order differential equation:

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + x = 0 . \quad (1)$$

It is well known that, for positive α , every solution of (1), except the origin, is attracted to the unique orbitally stable limit cycle, and that the strength of the damping, α , is intimately related to the rate at which trajectories approach this limit cycle (see [12]).

However, in practice, noise is inevitable, and this motivates including random perturbation effects in the differential equations models. Among the many ways in which this has been done, we will focus on the case when the randomness takes the form of a forcing term. For example, when we add random noise to (1), we will obtain the following equation:

$$\ddot{x}_\varepsilon - \alpha(1 - x_\varepsilon^2)\dot{x}_\varepsilon + x_\varepsilon + \varepsilon\xi = 0 , \quad (2)$$

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where ξ represents the random perturbation, and ε is a small (positive) value. In (2), and hereafter, x_ε will denote the “solution” when (1) has been subject to random perturbations as in (2). (For later reference, note that noise has been added to the original second order problem, prior to converting it into a first order system; see below).

A commonly used model of random perturbation ξ is white noise; i.e., $\xi = dW_t$, where W_t is the standard 1-dimensional Brownian motion. In this case, (2) becomes the classical stochastic van der Pol oscillator (weakly perturbed, for ε small). Other models of noise have been studied in [3], [4], [19], [20], and [10].

The presence of noise in a differential equation brings in several new challenges that require different approaches from those of deterministic dynamics. Of course, the key fact is that the dynamics will depend on the noise, not on the initial conditions. One of the most dramatic impacts of this fact is that (for any model of noise of which we are aware) the stable limit cycle gets destroyed. Finally, it is also worth realizing that noise causes changes in both phase and amplitude of the solutions. The impact on the phase is usually termed *phase noise*, or *time jitter* in the engineering literature [11, 17], and considerable progress has been made, both in mathematics and engineering, toward understanding phase noise. For example, it is well appreciated that phase noise can become arbitrarily large even for perturbations that remain small [6, 7, 16]. Moreover, for white noise, a fundamentally important and striking result (see [2, 9]) states that –with probability arbitrarily close to 1– trajectories asymptotically escape from any neighborhood of the deterministic limit cycle!

However, in real life, things do not appear to be nearly as bad. We give three examples to support this statement. First, consider the circuits (oscillators) commonly used in cellular phones: these have a base frequency of around 1GHz, oscillating in excess of 10^9 times per second. While being subject to unavoidable random ambient disturbances, a cell phone oscillator typically works continuously for days, even months or years, without experiencing any break down. Second, in laboratory studies¹ on a cantilevered piezoelectric energy harvester, which is a electroelastic system converting ambient vibrations generated by stochastic perturbations into electricity through the direct piezoelectric effect [8], no breakdown caused by random perturbation was actually ever observed. Finally, the reports in [5] indicate that trajectories of a weakly perturbed van der Pol oscillator remain bounded and linger near the deterministic limit cycle. In fact, the results of this cited numerical study are consistent with our own numerical simulations of equation (2), with white noise perturbations, over long times; see Figure 1. Clearly, trajectories appear to remain in a tubular neighborhood of the deterministic limit cycle, and do not become arbitrarily large.

This discrepancy between existing theoretical predictions and practical observations is likely due to two factors: (i) the asymptotic nature of the theoretical results, which typically require an extremely long time to be observable (if at all), and/or (ii) the inadequate modeling of the noise, meaning that practical random perturbations must have bounded strength (there is no noise perturbation with infinite energy), which is different from the white noise assumption commonly used in theoretical studies. [To explain the numerical results summarized in Figure 1, we note that –although we do not force any restriction on the random number generator used to

¹We thank Prof. Erturk, of the ECE department at Georgia Tech, for sharing with us the results of the experiments carried out in his laboratory

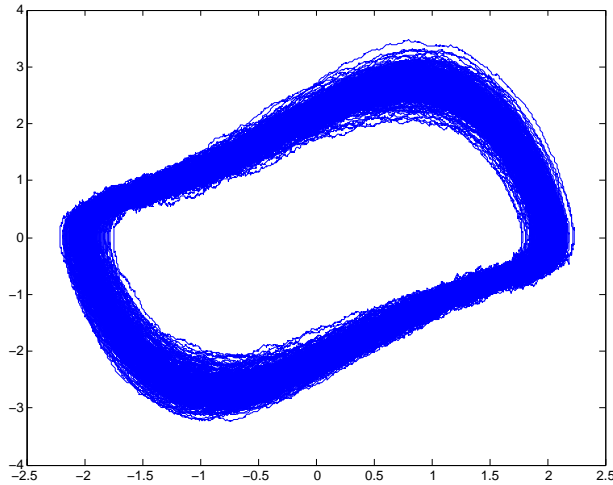


FIGURE 1. Long time behavior of (2) in numerical experiments.

mimic white noise— the pseudo random number generator used in our computation does not (and cannot) produce infinitely large perturbations.]

The above state of affairs provided us with the main motivation to carry out the present study. In particular, the above point (ii) is our key concern in this work.

We will focus on second order dissipative systems (oscillators) that possess an orbitally stable limit cycle surrounding a unique unstable equilibrium (at the origin), and we will study the impact of noise on these systems. Our main goals are: (1) to provide a new mathematical model for *realistic* random perturbations so that the trajectories of the stochastic oscillators resemble the phenomena observed in practice; and, (2) to study the behavior of solutions of these stochastic oscillators.

1.1. A new model of noise. Accounting for the possibility that standard white noise can generate infinitely large perturbations (albeit with arbitrarily small probability), while a realistic model of noise should never inject infinitely large energy into the system, here we propose a new model of noise that we believe serve as an appropriate model for random perturbations arising in practice.

Namely, we will require that the random perturbations ξ belong to the event set B defined as:

$$B = \{ \omega : \sup_{|t-s| \leq T} |W_t(\omega) - W_s(\omega)| \leq M \} . \quad (3)$$

In (3), T and M are two given positive constants, t and s are any two instants of time at most T -apart, and ω is the event of a Brownian path.

Note that B , a subset of all Brownian paths, is the collection of those Brownian motions that have bounded finite time increments. However, note that a path in B can still diverge to infinity as $t \rightarrow \infty$. Now, if one is interested in the finite time behavior of the system, then the probability of a Brownian path not in B can

be made arbitrarily small by taking M large enough, because of Hölder continuity of the Brownian motion path. However, if infinite time is considered, we observe that B has measure zero in the set of all Brownian paths defined for $t \in [0, \infty)$. Nevertheless, this does not imply there are not sufficiently many paths in B for $t \in [0, \infty)$: in fact, B contains un-countably many paths for $t \in [0, \infty)$, maintaining key characteristics of Brownian motion, including the general order of continuity of $\frac{1}{2}$.

1.2. Our results. We shall show that selecting random perturbations ξ from B for perturbing a deterministic oscillator with attracting limit cycle, and whose right-hand-side satisfies a local Lipschitz condition², will give well defined solution trajectories that remain bounded for all times. With reference to (1) and (2), it is worth emphasizing that this does not mean that, for all t , $(x_\varepsilon(t), y_\varepsilon(t))^T$ will stay close to its deterministic counterpart $(x(t), y(t))^T$. In fact, because of the phase differences, $\sup_t \|(x_\varepsilon(t) - x(t), y_\varepsilon(t) - y(t))^T\|$ will become arbitrarily large for $t \in [0, \infty)$. On the other hand, we will show that $(x_\varepsilon(t), y_\varepsilon(t))^T$ remains close to the deterministic limit cycle for all t , and we will further show several desirable properties of the stochastic trajectories relative to our new model of noise.

To witness, if we take a short segment transversal to the limit cycle (a “section”), we will show that the stochastic trajectories will return to this section, under appropriate conditions. As a consequence, we will set forth a proposal for defining the Poincaré return map relative to the stochastic oscillators. This is very different from the scenario obtained when one uses standard white noise, in which case there is no guarantee that a trajectory will return to a given section (see [14]).

In comparison to the Poincaré map for deterministic systems, our proposal of Poincaré map for the stochastic systems has some new features that have not been studied before. Namely, unlike the deterministic case, there is no longer just a first return point for a trajectory “going around the origin once.” In fact, a solution path can (and does) intersect the given section repeatedly, and it could do so infinitely many times, while the trajectory goes around the origin just one time. As a consequence of this observation, our proposal will be to relate to each given section a *return interval* and an associated *distribution for the return points*; both return interval and distribution will depend on the section. An important outcome of the above proposal is that we will have at least three different Poincaré maps: (i) that associated to the first return points distribution, (ii) that associated to the average of the return points distribution, and (iii) that associated to the last return points distribution.

Finally, we will also investigate the evolution of the probability density function of the stochastic oscillator with noise in B . In the present case, the processes are no longer Markovian, because the random perturbations depend on their past in an interval of length T , and not only on their current values. This inhibits the possibility to write a standard Fokker-Planck equation (see below). What we shall show is that, under appropriate conditions, the probability density function can be given by rational functions depending on solutions of a pair of diffusive partial differential equations (PDEs) with vanishing boundary conditions on finite intervals.

The paper is arranged as follows. In section 2, we consider a dissipative oscillator subject to random perturbations from B , and we show local (in time) boundedness

²The local Lipschitz condition becomes a global Lipschitz condition if the solutions remain bounded.

of trajectories. In section 3, we introduce our proposal of stochastic Poincaré map, and show the main result of this paper, the global boundedness of solutions. Lastly, in section 4, we study the evolution of the probability density function in terms of the solutions of some associated PDEs.

Notation: Throughout this work, the vector norm is always the 2-norm, which will be indicated simply as $\|\cdot\|$.

1.3. Relation to previous results. A lot of effort has been devoted to study the changes that solutions undergo under the effect of white noise perturbations. But, unfortunately, the existing results, require modeling assumptions which make them inapplicable to our problem. We justify this claim below.

For a planar system of differential equations, the basic model considered is the stochastic differential equations (SDE)

$$dX = g(X)dt + f(X)dW_t, \quad (4)$$

where $X(t) = (x(t), y(t))^T$, the term W_t comprises two independent 1-dimensional Brownian motions, and the diffusion coefficient f is such that the matrix ff^T is full rank. The latter property is often referred to as “uniform ellipticity.” We refer to the excellent expositions in [1, 2, 9, 15, 25], for details and further references. But, it is worth pointing out that the system(s) of interest to us, such as (2), do not fit into the model (4). This can be readily seen if we convert the second order equation (2) into a first order system, say

$$\begin{cases} dx_\varepsilon = y_\varepsilon dt, \\ dy_\varepsilon = [\alpha(1 - x_\varepsilon^2)y_\varepsilon - x_\varepsilon]dt + \varepsilon dW_t. \end{cases} \quad (5)$$

It is very important to observe that random noise is only added to the second equation in (5). Mathematically, this is easily explained as having added the perturbation to the original second order equation (1), prior to converting it into first order system. But, a more intrinsic and deep reason is that x and y are related to the current and voltage, respectively, which have a fixed relationship for a given circuit. So, it is not physically justified to add independent noise to both equations in (5).

2. LOCAL BOUNDEDNESS OF SOLUTIONS

In this section, we introduce our model of stochastically perturbed system with noise set B , and show local (in time) boundedness of trajectories.

We consider a second order system

$$\ddot{x} = f(x, \dot{x}), \quad (6)$$

where f is a smooth function of its arguments. We rewrite (6) as the first order system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f(x, y) \end{pmatrix}, \quad \text{or simply as } \dot{X} = b(X), \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (7)$$

and we will always work under the following **assumptions** on (7):

- (i) the origin is the only equilibrium of (7), $b(0) = 0$, and it is unstable;
- (ii) the system possesses a globally orbitally stable limit cycle Γ , corresponding to a periodic solution of period $0 < T_0$;

- (iii) for X : $\|X\| \leq C$, where C is any (arbitrary, but finite) positive constant, the function b is smooth and locally Lipschitz, with Lipschitz constant L (usually, L will depend on C).

Finally, the solution of (7) with initial condition $X(0) = u$, will be written as $\phi^t(u)$.

Of course, as a consequence of the above assumptions, all orbits of (7) (except the origin) will approach Γ . In Section 3, we will quantify better the rate of approach to Γ . Finally, note that, in general, the function f may depend on a parameter α , as in (1), or even on several parameters, and it must be tacitly understood that the above assumptions must hold for all allowed values of the parameter(s); in particular, the period T_0 (which usually will depend on the problem's parameters) must remain finite as the parameter(s) varies (vary).

Our specific interest in this work is in the following perturbation of (7):

$$\begin{cases} dx_\varepsilon = y_\varepsilon dt \\ dy_\varepsilon = f(x_\varepsilon, y_\varepsilon)dt + \varepsilon dW_t \end{cases}, \quad (8)$$

where ε is a (small) positive parameter, and W_t is a 1-dimensional Wiener process so that $\xi = dW_t$ is in the event set B given in (3). An initial condition of (8) is written as $X_\varepsilon(0)$, and its solution as $X_\varepsilon(t, \omega)$, or simply $X_\varepsilon(t)$, if no confusion can arise.

As already remarked, there are good modeling reasons for considering the noise model given by the set B . The key reason, for us, has been to adopt a model of noise more in tune with what is typically observed in practice, whereby realistic ambient noise is bounded within a finite time interval (unlike, say, white noise). Indeed, on intervals of length T , noise realizations from B are locally bounded, which is meaningful since, in real world scenarios, energy is always bounded, and no perturbation can become unbounded in finite time. (Still, note that noise realizations from the set B can still become eventually unbounded, since the total increment is not constrained to remain bounded.)

As added benefit, restricting to the event set B , we will be able to show important mathematical properties of the model (8). Most notably, we will be able to propose a definition of Poincaré map, see Section 3. But, first, below we show that stochastic trajectories remain bounded in a finite time interval.

Theorem 1. *Let B be the set defined in (3), and let ω be any event from B . Then, for $\varepsilon > 0$ sufficiently small, solutions of (8) are locally bounded:*

$$\sup_{0 \leq t \leq T} \|X_\varepsilon(t, \omega)\| < \infty,$$

where T is the interval width appearing in (3).

Proof. We will argue by contradiction. So, suppose that $\sup_{0 \leq t \leq T} \|X_\varepsilon(t, \omega)\| = \infty$.

Let C_1 be the maximum of $\|\phi^t(u)\|$, for $0 \leq t \leq T$, along the deterministic limit cycle:

$$C_1 = \max_{0 \leq t \leq T} \|\phi^t(u)\|, \quad u \in \Gamma.$$

Then, there must exist a constant $C > 2C_1$, for which the stopping time

$$\tau(\omega) = \inf\{t : \|X_\varepsilon(t, \omega)\| = C\},$$

must satisfy

$$\tau(\omega_0) \leq T,$$

for some $\omega_0 \in B$. In other words, for such ω_0 , we have

$$\sup_{0 \leq s \leq \tau(\omega_0)} \|X_\varepsilon(s, \omega_0)\| = C. \quad (9)$$

Consider this event ω_0 . From [18], we know that there exists a strong solution $X_\varepsilon(t, \omega_0)$ up to time $\tau(\omega_0)$ (see [22] for the definition of strong solution). Let L be the local Lipschitz constant of b when $\|X\| \leq C$. Hence, for $t \leq \tau(\omega_0)$, we have

$$\begin{aligned} \|X_\varepsilon(t, \omega_0) - X(t)\| &= \left\| \int_0^t b(X_\varepsilon(s, \omega_0)) - b(X(s)) ds + \varepsilon \begin{pmatrix} 0 \\ W_t(\omega_0) \end{pmatrix} \right\|, \\ &\leq L \int_0^t \|X_\varepsilon(s, \omega_0) - X(s)\| ds + \varepsilon |W_t(\omega_0)|. \end{aligned}$$

From Gronwall's Lemma, and since $\omega_0 \in B$, we obtain

$$\begin{aligned} \sup_{0 \leq s \leq \tau(\omega_0)} \|X_\varepsilon(s) - X(s)\| &\leq \varepsilon e^{L\tau(\omega_0)} \sup_{0 \leq s \leq \tau(\omega_0)} |W_s(\omega_0)| \\ &\leq \varepsilon e^{L\tau(\omega_0)} M. \end{aligned} \quad (10)$$

Also, since $\tau(\omega_0) \leq T$, using the triangular inequality in (10) we get

$$\begin{aligned} \sup_{0 \leq s \leq \tau(\omega_0)} \|X_\varepsilon(s)\| &\leq \sup_{0 \leq s \leq \tau(\omega_0)} \|X(s)\| + \varepsilon e^{\tau(\omega_0)L} M \\ &\leq C_1 + \varepsilon e^{LT} M. \end{aligned}$$

Therefore, if $\varepsilon < \frac{C_1}{2e^{LT}M}$, then

$$\sup_{0 \leq s \leq \tau(\omega_0)} \|X_\varepsilon(s)\| \leq \frac{3}{2} C_1 < C,$$

contradicting equation (9). \square

Theorem 1 establishes closeness, for short time, between stochastic and deterministic solutions (see (10)), and it gives a lead on how to define the return map (Poincaré map) in stochastic systems. We do this next.

3. STOCHASTIC POINCARÉ MAP AND GLOBAL BOUNDEDNESS

In this section, we introduce our proposal of stochastic Poincaré map, for (8). Then, using the notation resulting from the definition of Poincaré map, we will show our main theorem: global boundedness of solutions of (8).

3.1. Poincaré map. First, recall the definition of Poincaré map in the deterministic setting. We do this in the plane, since we are interested in the model (7), though of course the definition can be easily given in \mathbf{R}^d , $d > 2$.

Consider the general differential equation for $X \in \mathbf{R}^2$

$$\frac{dX}{dt} = b(X), \quad X(0) = u, \quad (11)$$

where b is a locally Lipschitz smooth vector field. Let $\phi^t(u)$ be the flow associated to (11) and suppose that (11) has a periodic solution of period T_0 , and let Γ be the orbit corresponding to this periodic solution. Therefore, $\phi^{t+T_0}(p) = \phi^t(p)$, $t \in \mathbf{R}$, $p \in \Gamma$.

The Poincaré map provides a useful tool for studying periodic orbits, whereby a periodic orbit becomes a fixed point of the Poincaré map.

Definition 2. Let p be a point on Γ and S be a local cross section at p : a smooth 1-dimensional arc, intersecting Γ (only) at p , transversally. Given an open and connected neighborhood U of p , $U \subset S$, for every point $u \in U$, define the first return time $\tau(u)$ as

$$\tau(u) = \inf\{t > 0 : \phi^t(u) \in S\} .$$

Then, the Poincaré map $P : U \rightarrow S$ is defined by

$$P(u) = \phi^{\tau(u)}(u) , \quad u \in U .$$

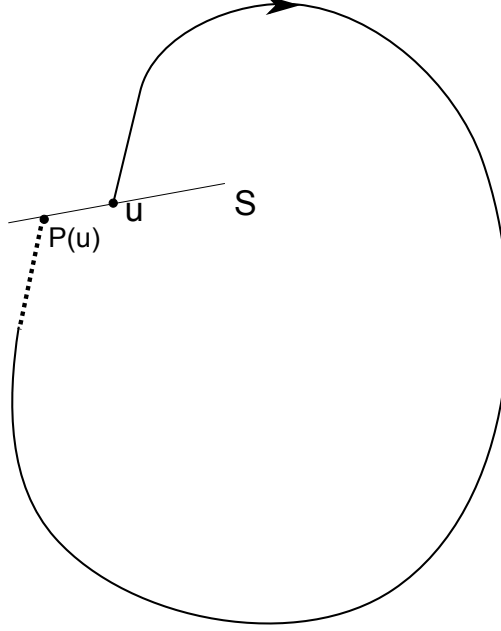


FIGURE 2. Poincaré map

Clearly $P(p) = p$ and $\tau(p) = T_0$.

With the help of the above definition, we can finally clarify assumption (ii), that we made in Section 2 relative to system (7). We say that Γ is attractive, with *rate* α_0 , $0 < \alpha_0 < 1$, if:

$$\|P(u) - p\| \leq \alpha_0 \|u - p\| , \quad \text{for all } u \in U . \quad (12)$$

As before, in case (7) depends on parameters, it has to be understood that the inequality $\alpha_0 < 1$ must hold uniformly with respect to parameters variation.

3.2. Stochastic Poincaré map. Consider now the general SDE associated to (11):

$$dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \varepsilon \begin{pmatrix} 0 \\ W_t(\omega_0) \end{pmatrix} , \quad X_\varepsilon(0) = u, \quad (13)$$

where $X_\varepsilon(\cdot), u \in \mathbf{R}^2$ and W_t is a standard Wiener process in R^1 . In this case, it is known (see [22]) that a unique strong solution exists locally, that is for times before the stochastic solution blows up to infinity.

If we try to define a (stochastic) Poincaré map for (13), we face some intrinsic challenges.

- (1) With nonzero probability, the stochastic trajectory will not return “after one loop” to a given section, even though the unperturbed trajectory is periodic.
- (2) Even if the stochastic trajectory returns to a given section, the first return point doesn’t represent all return points to that section. In fact, the trajectory can intersect a given section several times, even infinitely many times. [There is no monotonicity of motion with respect to a given section].

Selecting random perturbations from B in (3), and relative to the model (8), allows us to solve the above difficulties. In fact, the following two facts hold as a consequence of Theorem 1 and of (the proof of) Theorem 4 below.

- (1) First, for all events in B , the stochastic trajectory of (8) will return to a given section.
- (2) Second, although the stochastic trajectory may repeatedly enter and exit (or even stay for a while in) a certain section, it will have to leave such section within a finite time. See Figure 3 for an illustration of this fact.

Note that a stochastic path can (and does) intersect a given section repeatedly, and it could do so even infinitely many times, before leaving the section.

By virtue of points (1) and (2), and this last observation above, our **proposal** is to

Associate to a given section both a return interval and a distribution for the return points; both return interval and distribution will depend on the section.

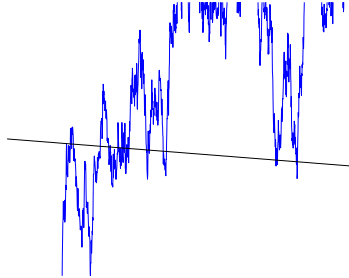


FIGURE 3. Return points of stochastic trajectory: the first return interval (solid segment)

Let us set forth our proposal more precisely.

Consider a section S and a neighborhood U of $p \in \Gamma$ as in Definition 2. For $\omega \in B$, and $u \in U$, let X_ε be the stochastic trajectory of (8), such that $X_\varepsilon(0) = u$.

To begin with, we introduce the first return time

$$\tau_\varepsilon(u, \omega) = \inf\{t : X_\varepsilon(t) \in S, t \in [\frac{1}{2}T_0, \frac{3}{2}T_0]\} , \quad (14)$$

and the last return time

$$\sigma_\varepsilon(u, \omega) = \sup\{t : X_\varepsilon(t) \in S, t \in [\frac{1}{2}T_0, \frac{3}{2}T_0]\} . \quad (15)$$

(Both of these values are well defined, for sufficiently small ε , because of Theorems 1 and 4.)

Then, we define the return “interval” $E(u, \omega)$ or simply E :

$$E = \{X_\varepsilon(t) : X_\varepsilon(t) \in S, t \in [\tau_\varepsilon, \sigma_\varepsilon]\} . \quad (16)$$

(Again, Theorems 1 and 4 ensure that E is well defined.)

Our proposal is to associate to E in (16) a “return distribution” function $P_{u, \omega}$, by which we can solve for the sample average of the set E . Since the distribution conveys all information on the return points, the Poincaré map (call it P_ε) should be constructed as a **point-to-distribution** map, for each stochastic path:

$$P_\varepsilon : u \in U \rightarrow P_{u, \omega} .$$

At the same time, from the foregoing, it is natural to define three different point-to-point maps, all of which can be computed in a numerical simulation: first return map, last return map and average return map.

Definition 3. *Let the cross section S , neighborhood U , and Poincaré map P , be defined as in Definition 2, for the unperturbed system (7).*

Let $\omega \in B$, and X_ε be the solution of (8). Then, we define stochastic Poincaré maps $P_\varepsilon, P_\varepsilon : U \rightarrow S$, as follows.

- For $\varepsilon = 0$, $P_0 = P$.
- For $\varepsilon \neq 0$, and any $u \in U$, then we define:
 - the first return map

$$P_\varepsilon(u, \omega) = X_\varepsilon(\tau_\varepsilon(u, \omega)) , \quad (17)$$

where τ_ε is defined in (14);

- the last return map

$$P_\varepsilon(u, \omega) = X_\varepsilon(\sigma_\varepsilon(u, \omega)) , \quad (18)$$

where σ_ε is defined in (15);

- and the average return map

$$P_\varepsilon(u, \omega) = \int_E y dP_{u, \omega}(y) , \quad (19)$$

where E is defined in (16) and $P_{u, \omega}$ is the return distribution function associated to E .

Remark 1. In the recent work [14], the authors proposed a Poincaré map definition for the van der Pol oscillator, subject to standard white noise perturbation on εdW_t . Assuming that trajectories return to a given section (although there is a positive probability that they will not return), the authors further looked at the first return map and for sufficient small ε , argued that this map can be viewed as a Gaussian perturbation of the deterministic map. By comparison, restricting to noise from within B , we actually proved that trajectories always return (for ε sufficiently small) to a given section. Furthermore, our proposal of Poincaré map takes into account all return points, and it gives a more detailed description relative to a given section, description which is not available by a simple Gaussian process.

3.3. Global boundedness. By exploiting the Poincaré map, we will show our main theorem, which we state next.

Theorem 4. *Consider the system (8), where ω is in B defined by (3). Assume that (12) holds, and that the assumptions of Lemmata 5 and 6 below hold (in particular,*

(24)). Then, for $\varepsilon > 0$ sufficiently small, the stochastic trajectories of (8) are globally bounded:

$$\sup_{t \geq 0} \|X_\varepsilon(t, \omega)\| < \infty .$$

To prove this result, we will proceed according to the following steps.

- (1) We define the Poincaré section S as a line section, and show closeness between unperturbed and stochastic solutions during the first return time; see Lemma 5.
- (2) We construct the first return map P_ε (see Definition 3), and sharpen the result of Lemma 5 about closeness of the stochastic trajectory and the unperturbed Poincaré map for the first return to the given section; see Lemma 6.
- (3) Combining the above closeness results, and asymptotic stability of the deterministic limit cycle, we show that there exists a neighborhood (an interval) U_ε of $p \in \Gamma$, $U_\varepsilon \subset S$, invariant under the stochastic Poincaré map: $P_\varepsilon(U_\varepsilon) \subset U_\varepsilon$. This will complete the proof.

First of all, we define the Poincaré section. Through the polar representation of points in the plane, we take the section to be a line segment placed at a given angular value θ_0 :

$$S = \{(x, y) \in \mathbf{R}^2 : x = r(\theta_0) \cos(\theta_0), y = r(\theta_0) \sin(\theta_0)\} , \quad (20)$$

where $a(\theta_0) \leq r(\theta_0) \leq b(\theta_0)$, and a and b are chosen sufficiently small so that the line segment S intersects Γ transversally at just one point. With this, we can identify points of the stochastic trajectory that returns to this section:

$$\theta_\varepsilon(t) = \theta_\varepsilon(0) = \theta_0 .$$

Naturally, the neighborhood $U \subset S$ as in Definition 3, now becomes an open subinterval of S containing the intersection with Γ . This way of proceeding will be validated in Lemma 5.

To illustrate, in Figure 4, we show a typical stochastic trajectory of (2) starting from the section S , and traveling around the origin once before returning to S .

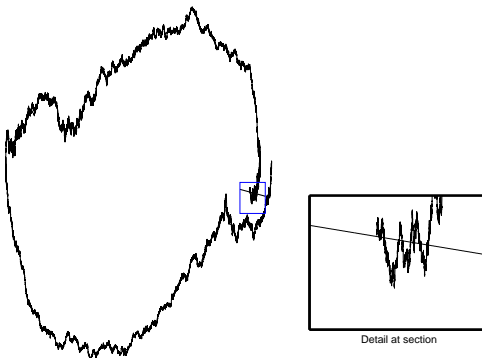


FIGURE 4. One realization for the stochastic van der Pol oscillator: on the section are all return points

With this, we can be more specific about the meaning of first return time with respect to the section (cfr. with (17)):

$$\tau_\varepsilon^1 = \inf\{t : \theta_\varepsilon(t) = \theta_\varepsilon(0) - 2\pi\} . \quad (21)$$

Without loss of generality, here below we will also make a simplification in the definition of B :

“we require $T = 2T_0$ in (3), where $T_0 > 0$ is the period of the periodic solution of (7)”.

This choice is legitimate, since we can always modify M to ensure that Theorem 1 holds for $T = 2T_0$. So, to reiterate, the event set B is henceforth given by

$$B = \{ \omega : \sup_{|t-s| \leq T} |W_t(\omega) - W_s(\omega)| \leq M, \text{ and } T = 2T_0 \} . \quad (22)$$

Finally, let L be the local Lipschitz constant of $b(X)$ in (7) for $\|X\| \leq \sup_{0 \leq t \leq T} \|X_\varepsilon(t, \omega)\|$.

We further consider the annular neighborhood of radius $\varepsilon e^{LT} M$ around the periodic trajectory Γ ; see Figure 5. Finally, we let times t_1 and t_2 be defined as follows (again, see Figure 5):

$$\begin{aligned} t_1 &= \inf\{t > \frac{1}{2}T_0 : \|X(t) - X(T_0)\| = \varepsilon e^{LT} M\} , \\ t_2 &= \sup\{t < \frac{3}{2}T_0 : \|X(t) - X(T_0)\| = \varepsilon e^{LT} M\} . \end{aligned} \quad (23)$$

Note that t_1, t_2 depend on ε , and that by continuity of the strong solution $X_\varepsilon(t)$, $\tau_\varepsilon^1 \in (t_1, t_2)$. Also, note that for $\varepsilon > 0$ sufficiently small, with t_1, t_2 as in (23), then for $i = 1, 2$, we have:

$$\int_0^1 b(X(t_i + s(T_0 - t_i))) ds \neq 0 . \quad (24)$$

We are now ready to show closeness after the first return time.

Lemma 5. *Let B be as in (22), $\omega \in B$, and let L, T_0 , and τ_ε^1 be as above. Then, for $\varepsilon > 0$ sufficiently small, we have:*

$$\tau_\varepsilon^1 < 2T_0 ,$$

and

$$\sup_{0 \leq t \leq \tau_\varepsilon^1} \|X_\varepsilon(t, \omega) - X(t)\| < C_0 \varepsilon ,$$

where $C_0 = e^{LT} M$.

Proof. Proceeding as in the derivation of (10), we have

$$\sup_{0 \leq s \leq T} \|X_\varepsilon(s) - X(s)\| \leq \varepsilon e^{LT} M . \quad (25)$$

Consider the deterministic system (7). For either $i = 1$ or $i = 2$, we have

$$X(T_0) - X(t_i) = \int_{t_i}^{T_0} b(X(s)) ds = \int_0^1 b(X(t_i + s(T_0 - t_i))) ds (T_0 - t_i) .$$

Since (24) holds, we can solve for $|T_0 - t_i|$ from this last equation:

$$|T_0 - t_i| = \frac{\|X(T_0) - X(t_i)\|}{\left\| \int_0^1 b(X(t_i + s(T_0 - t_i))) ds \right\|} . \quad (26)$$

Now, recall that $\tau_\varepsilon^1 \in (t_1, t_2)$. Thus, by choosing ε :

$$\varepsilon < \min_{i=1,2} \left[\frac{T_0 \left\| \int_0^1 b(X(t_i + s(T_0 - t_i))) ds \right\|}{4e^{LT} M} \right],$$

and using this bound in (26), we get $\tau_\varepsilon^1 < T = 2T_0$.

Hence, for $\omega \in B$, and for ε sufficiently small, inequality (25) holds up to time τ_ε^1 . \square

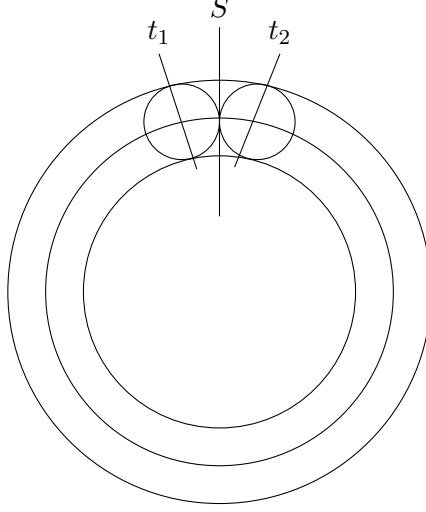


FIGURE 5. The middle circle represents Γ . The stochastic trajectories $X_\varepsilon(\omega, t)$, $\omega \in B$, are inside the annulus.

Next, let P_ε denote the first return map as in (17), corresponding to the section S defined as in (20). We show the closeness on the section S .

Lemma 6. *With same notation and assumptions as in Lemma 5, let $X_\varepsilon(\omega, t)$, be the solution of (8) for $0 \leq t \leq \tau_\varepsilon^1$, with initial value $X_\varepsilon(0)$ in U , a sufficiently small open interval of S .*

Then, for $\varepsilon > 0$ sufficiently small,

$$\|X_\varepsilon(\tau_\varepsilon^1) - P_0(X_\varepsilon(0))\| \leq 5C_0\varepsilon.$$

Proof. From Lemma 5, taking $t = \tau_\varepsilon^1$, (25) becomes

$$\|X_\varepsilon(\tau_\varepsilon^1) - \phi^{\tau_\varepsilon^1}(X_\varepsilon(0))\| \leq C_0\varepsilon.$$

Also, we have both

$$\phi^{\tau_\varepsilon^1}(X_\varepsilon(0)), P_0(X_\varepsilon(0)) \in \cup_{t \in [t_1, t_2]} \mathcal{B}(X(t), C_0\varepsilon),$$

where t_1 and t_2 are defined in (23), and $\mathcal{B}(X(t), C_0\varepsilon)$ are circles with center $X(t)$ and radius $C_0\varepsilon$. Therefore, we have

$$\|X(\tau_\varepsilon^1) - P_0(X_\varepsilon(0))\| \leq 4C_0\varepsilon,$$

and so

$$\|X_\varepsilon(\tau_\varepsilon^1) - P_0(X_\varepsilon(0))\| \leq \|X_\varepsilon(\tau_\varepsilon^1) - X(\tau_\varepsilon^1)\| + \|X(\tau_\varepsilon^1) - P_0(X_\varepsilon(0))\| \leq 5C_0\varepsilon.$$

\square

Finally, we show global boundedness and complete the proof of Theorem 4, by using the stability of the deterministic limit cycle, as expressed by (12), and local boundedness of X_ε .

In the proof below, we will need to compare the stochastic and deterministic solutions. For this reason, we will use the following notation. For all $k = 1, 2, \dots$, we write

$$\phi^{\Delta_\varepsilon^k} \left(X_\varepsilon \left(\sum_{i=1}^k \tau_\varepsilon^i \right) \right), \text{ where } \Delta_\varepsilon^k := \tau_\varepsilon^{k+1} - \tau_\varepsilon^k,$$

for the solution of the deterministic equation (7) at time τ_ε^{k+1} , which started at time τ_ε^k with initial condition $X_\varepsilon(\sum_{i=1}^k \tau_\varepsilon^i)$. Above, we have recursively defined the values τ_ε^k as the k -th “first return” times:

$$\tau_\varepsilon^k = \inf\{t > \tau_\varepsilon^{k-1} : \theta_\varepsilon(t) = \theta_\varepsilon(0) - 2\pi\}, \quad k = 1, 2, \dots, \quad \tau_\varepsilon^0 = 0. \quad (27)$$

These values τ_ε^k will be shown to be well defined in the proof below.

Proof of Theorem 4. First, let us show that

$$\sup_k \|X_\varepsilon(\sum_{i=1}^k \tau_\varepsilon^i)\| < \infty,$$

by showing that there exists an interval $U_\varepsilon \subset S$,

$$U_\varepsilon = \{X : \|X - p\| \leq R_0\}, \text{ such that } P_\varepsilon(U_\varepsilon) \subset U_\varepsilon.$$

We show this last fact by induction, in the process showing that the times τ_ε^k 's are well defined.

If $k = 1$, from Lemma 5, τ_ε^1 exists, and $\tau_\varepsilon^1 \in (t_1, t_2)$ with t_1 and t_2 given in (23), in particular $\tau_\varepsilon^1 \in (\frac{1}{2}T_0, \frac{3}{2}T_0)$. From Lemma 6, we have

$$\|X_\varepsilon(\tau_\varepsilon^1) - P_0(X_\varepsilon(0))\| \leq R_1\varepsilon,$$

where $R_1 = 5C_0$, and $C_0 = e^{LT}M$ with L the Lipschitz constant of b for X : $\|X\| \leq \sup_{0 \leq t \leq 2T_0} \|X_\varepsilon(t, \omega)\|$.

Also, since $X_\varepsilon(0) \in U_\varepsilon$, denoting with $p \in S$ the fixed point of P_0 and using (12), we have

$$\begin{aligned} \|X_\varepsilon(\tau_\varepsilon^1) - p\| &\leq \|X_\varepsilon(\tau_\varepsilon^1) - P_0(X_\varepsilon(0))\| + \|P_0(X_\varepsilon(0)) - p\| \\ &\leq R_1\varepsilon + \alpha_0 R_0. \end{aligned}$$

So, if $\varepsilon < \frac{(1-\alpha_0)R_0}{R_1}$, then $X_\varepsilon(\tau_\varepsilon^1) \in U_\varepsilon$.

By induction, suppose that for $j = 1, \dots, N$, the times τ_ε^j are well defined, that satisfy

$$\tau_{j-1} + \frac{1}{2}T_0 \leq \tau_\varepsilon^j \leq \tau_{j-1} + \frac{3}{2}T_0,$$

where we have set

$$\tau_{j-1} := \sum_{i=0}^{j-1} \tau_\varepsilon^i,$$

and that

$$\|X_\varepsilon(\tau_N) - p\| \leq \sum_{k=0}^{N-1} \alpha_0^k R_1\varepsilon + \alpha_0 R_0.$$

Note that, since $0 < \alpha_0 < 1$, then $\sum_{k=0}^{N-1} \alpha_0^k \leq \sum_{k=0}^{\infty} \alpha_0^k = \frac{1}{1-\alpha_0}$. Therefore, choosing $\varepsilon < \frac{(1-\alpha_0)^2 R_0}{R_1}$ will give $X(\tau_N) \in U_\varepsilon$.

Now, when $k = N + 1$, consider equation (7) with initial condition $X_\varepsilon(\tau_N)$. By Gronwall inequality, we have

$$\sup_{\tau_N \leq s \leq \tau_N + T} \|X_\varepsilon(s) - \phi^{\Delta_\varepsilon^N} \left(X_\varepsilon(\tau_N) \right)\| \leq C_0 \varepsilon,$$

hence $\tau_N + \frac{1}{2}T_0 < \tau_\varepsilon^{N+1} < \tau_N + \frac{3}{2}T_0$. Also, since

$$\|X_\varepsilon(\tau_{N+1}) - \phi^{\Delta_\varepsilon^N} \left(X_\varepsilon(\tau_N) \right)\| \leq C_0 \varepsilon,$$

similarly to the proof of Lemma 6, then

$$\|X_\varepsilon(\tau_\varepsilon^{N+1}) - P_0(X_\varepsilon(\tau_\varepsilon^N))\| \leq R_1 \varepsilon. \quad (28)$$

Using contractility of the Poincaré map as expressed by (12), and $X_\varepsilon(\tau_\varepsilon^N) \in U_\varepsilon$, we have

$$\|P_0(X_\varepsilon(\tau_\varepsilon^N)) - p\| \leq \alpha_0 \|X_\varepsilon(\tau_\varepsilon^N) - p\|. \quad (29)$$

Combining inequalities (28) and (29), we obtain

$$\begin{aligned} \|X_\varepsilon(\tau_\varepsilon^{N+1}) - p\| &\leq \|X_\varepsilon(\tau_\varepsilon^{N+1}) - P_0(X_\varepsilon(\tau_\varepsilon^N))\| + \|P_0(X_\varepsilon(\tau_\varepsilon^N)) - p\| \\ &\leq R_1 \varepsilon + \alpha_0 \|X_\varepsilon(\tau_\varepsilon^N) - p\| \leq \sum_{k=0}^N \alpha_0^k R_1 \varepsilon + \alpha_0 R_0. \end{aligned}$$

In particular, if $\varepsilon < \frac{(1-\alpha_0)^2 R_0}{R_1}$, then $X(\tau_{N+1}) \in U_\varepsilon$, and this completes the induction process.

Finally, since for all k , $\tau_{k-1} + \frac{1}{2}T_0 < \tau_\varepsilon^k < \tau_{k-1} + \frac{3}{2}T_0$, then τ_ε^k must be in between two consecutive multiples of the period T_0 . As a consequence of this, for any time t we can write $t = \tau_\varepsilon^k + s$, for some k , and with $0 \leq s \leq T_0$. Using again Theorem 1, we then obtain

$$\sup_{t \geq 0} \|X_\varepsilon(t)\| < \infty,$$

which completes the proof. \square

Remark 2. The main implication of Theorem 4 is that, although random perturbation in B will not be bounded for all times, the stochastic trajectories will remain within a tubular neighborhood of the deterministic limit cycle.

Remark 3. To illustrate the situation, consider a system (7) which is unambiguously representable in polar coordinates (for example, the van der Pol oscillator), use polar coordinates (ρ, θ) for the deterministic problem, and $(\rho_\varepsilon(t), \theta_\varepsilon(t))$ to model amplitude and phase in the stochastically perturbed version. Theorem 4 implies that -as long as the perturbation is selected from within the set B - the amplitude ρ_ε remains bounded:

$$\sup_{t \geq 0, \omega \in B} |\rho_\varepsilon(t) - \rho(t)| < \infty.$$

In turns, this helps explaining why we observe no catastrophic breakdown in cell-phone service, in agreement with practical experience.

At the same time, we must emphasize that the phase perturbation does become unbounded:

$$\sup_{t \geq 0, \omega \in B} |\theta_\varepsilon(t) - \theta(t)| = \infty.$$

In turns, this helps explaining why we may (and do) lose cell phone connection during lengthy conversations; see details in [6].

To sum up, although perturbations occur in both amplitude and phase, there is a clear distinction among the two: in particular, the strong stability property of the deterministic limit cycle ensures that the amplitude perturbations remain bounded.

4. CONNECTION WITH PARTIAL DIFFERENTIAL EQUATIONS

In this section, we attempt deriving PDEs for the transition density function associated to the trajectories of (8), with $\omega \in B$. First, we review some known results and give needed notations.

4.1. Diffusion process and partial differential equations. For completeness, here we review the standard derivation of PDEs for diffusion processes; for details, see [22].

Consider a d -dimensional Markov family $\{X_t, \mathcal{F}_t\}$, which is a diffusion process with drift vector $b = (b_1, \dots, b_d)$ and diffusion matrix $a = \{a_{ik}\}_{1 \leq i, k \leq d}$.

This means that for any $f \in C^2$, one has

$$\lim_{t \rightarrow 0} \frac{1}{t} [\mathbb{E} f(X_t | X_0 = x) - f(x)] = (Lf)(x), \quad \forall x \in \mathbf{R}^d,$$

where the infinitesimal operator L is given by

$$(Lf)(x) = \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d a_{ik}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i}.$$

Suppose that the Markov family of X_t has a transition density function

$$P(X_t \in dy | X_0 = x) = p(t, x, y) dy; \quad \forall x \in \mathbf{R}^d, \quad t > 0.$$

Then, $p(t, x, y)$ satisfies the forward Kolmogorov (Fokker-Planck) equation, for fixed $x \in \mathbf{R}^d$:

$$p_t(t, x, y) = L^* p(t, x, y); \quad (t, y) \in (0, \infty) \times \mathbf{R}^d,$$

and the backward Kolmogorov equation, for fixed $y \in \mathbf{R}^d$:

$$p_t(t, x, y) = L p(t, x, y); \quad (t, x) \in (0, \infty) \times \mathbf{R}^d,$$

where the adjoint operator L^* is given by

$$(L^* f)(x) = \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d \frac{\partial^2 (a_{ik}(x) f(x))}{\partial x_i \partial x_k} - \sum_{i=1}^d \frac{\partial (b_i(x) f(x))}{\partial x_i}, \quad \forall x \in \mathbf{R}^d.$$

4.2. Killed diffusions. Let us also introduce the killed diffusion PDE, by considering the one dimensional diffusion process

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \quad (30)$$

where b, σ are Lipschitz functions, and W_t is a standard Wiener process. Consider events set C :

$$C = \{\omega : \sup_{0 \leq s \leq t} |X_s| \leq M_0\}.$$

Define the first exit time $\tau_C = \inf\{t : |X_t| = M_0\}$. The killed diffusion is defined as

$$X_t^C = \begin{cases} X_t, & \text{if } t < \tau_C; \\ X_{\tau_C}, & \text{if } t \geq \tau_C. \end{cases} \quad (31)$$

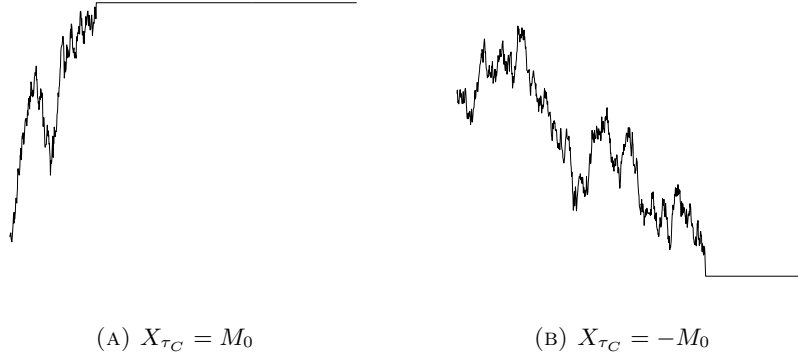


FIGURE 6. Killed diffusion

Consider the transition density function $p(t, x, y)$ of X_t^C :

$$p(t, x, y)dy = P(X_t^C \in dy | X_0 = x) .$$

In Lemma 7, we give the Fokker-Planck equation for the killed diffusion X_t^C , which is a PDE with vanishing boundary conditions on a finite interval. For the proof of Lemma 7, we refer to [13] and [24].

Lemma 7. *For fixed x , $p(t, x, y)$ solves*

$$\begin{cases} p_t = -(bp)_y + \frac{1}{2}(\sigma^2 p)_{yy} , & |y| < M_0 , \\ p(t, x, y) = 0, & |y| = M_0 , \\ p(0, x, y) = \delta_0(x - y) . \end{cases}$$

□

4.3. Derivation of PDE. However, there are difficulties in following the above standard steps to derive the evolution conditioned on B , because:

- $X_\varepsilon(t)$ is not a Markov process, since it depends both on values in the past and in the future; in fact, $X_\varepsilon(t)$ depends on the full set of values in the time interval $(t - T, t + T)$.

Because of the above difficulty, we restrict to a subset of B which allows us to restart the process at certain times, and which is more amenable to analysis. As in (22), take $T = 2T_0$, where T_0 is the period of the deterministic limit cycle. Then, we consider the events set

$$B_0 = \{\omega : \sup_{0 \leq t \leq T_0} |W_{t+T_0} - W_{tT_0}| \leq \frac{1}{2}M\} . \quad (32)$$

Clearly, B_0 in (32) is a subset of B in (22). With respect to B , B_0 has the advantage that, on each time interval of width T_0 , the Wiener process increments are independent of that previous time interval of width T_0 . Moreover, for first time interval, by introducing the absolute running maximum

$$M_t = \sup_{0 \leq s \leq t} |W_s| , \quad t \leq T_0 ,$$

$(X_\varepsilon(t), W_t, M_t)$ forms a Markov process, since condition B_0 is nothing but the restriction to those events for which M_t is bounded.

Motivated by the above, we will restrict to B_0 . Then, on the first time interval, $(X_\varepsilon(t), W_t, M_t)$ will be analyzed on separated subintervals $(0, t)$ and (t, T_0) , where the first subinterval can be analyzed as a killed diffusion process and the second one can be analyzed by a standard PDE approach.

To be more precise, we will solve for the transition density function conditioned on events B_0 :

$$p(t, X, X_\varepsilon(0) \mid B_0) dX = P(X_\varepsilon(t) \in dX \mid B_0, X_\varepsilon(0)) , \quad (33)$$

where P represents probability function.

We divide our approach in three steps.

- (i) From 0 to $t \leq T_0$, we introduce the new process $z_t = W_t$, and solve for the density function of $(X_\varepsilon(t), z_t)$ at (X, z) :

$$P(X_\varepsilon(t) \in dX, z_t \in dz, M_t \leq \frac{1}{2}M \mid X_\varepsilon(0)) . \quad (34)$$

As in killed diffusions, the corresponding equation is a PDE with vanishing boundary conditions.

- (ii) For the remaining time from t to T_0 , we will solve for the probability function

$$P(\sup_{t \leq s \leq T_0} |W_s| \leq \frac{1}{2}M \mid W_t = z) .$$

By the Markov property of $(X_\varepsilon(t), W_t, M_t)$, we will then form the transition density function on $(0, T_0)$.

- (iii) Finally, for any time t , by the Markov property, we will derive the transition density function by connecting to the value obtained at the right-end point of the previous time interval.

We are now ready to give details of our approach. For our basic model (8), with dW_t from B_0 in (32), introduce the new process $z_t = z_0 + W_t$, so that equation (8) becomes

$$\begin{cases} dx_\varepsilon = y_\varepsilon dt , \\ dy_\varepsilon = f(x_\varepsilon, y_\varepsilon) dt + \varepsilon dW_t , \\ dz_t = dW_t . \end{cases} \quad (35)$$

For a test function $g(x, y, z) \in C^2(R^3)$, the infinitesimal operator corresponding to the process $(X_\varepsilon(t), z_t)$ is

$$(Lg)(x, y, z) = yg_x + f(x, y)g_y + \frac{1}{2}(\varepsilon^2 g_{yy} + 2\varepsilon g_{yz} + g_{zz}) . \quad (36)$$

Now we begin the derivation on each time interval. Let

$$\tau_z = \inf\{t : |z_t| = \frac{1}{2}M\} , \quad M_t = \sup_{0 < s < t} |z_s| ,$$

and consider events up to N time intervals

$$B_0^N = \{\omega : \sup_{0 \leq t \leq T_0} |W_{t+kT_0} - W_{kT_0}| \leq \frac{1}{2}M , \quad k \leq N\} .$$

We first derive transition density for $X_\varepsilon(t)$ conditioned on B_0^N :

$$u_N(t, X, X_\varepsilon(0)) dX = P(X_\varepsilon(t) \in dX \mid X_\varepsilon(0), B_0^N) , \quad \text{for } N = 1, 2, \dots \quad (37)$$

As discussed above, this derivation goes through three steps.

Step one. We begin with transition density function on $(0, t)$, which plays a core role in this derivation. Since $(X_\varepsilon(t), z_t)$ is also a diffusion process, condition

transition density for $(X_\varepsilon(t), z_t)$ with event $\{M_t \leq \frac{1}{2}M\}$ is the same as a killed diffusion process, where we only cut off on the z_t part. To be more precisely, we define following killed diffusion by

$$(X_\varepsilon^M(t), z_t^M) = \begin{cases} (X_\varepsilon(t), z_t), & \text{if } t < \tau_z ; \\ (X_\varepsilon(\tau_z), z_{\tau_z}), & \text{if } t \geq \tau_z . \end{cases} \quad (38)$$

The transition density of $(X_\varepsilon^M(t), z_t^M)$ is the same as (34).

In details, consider B_0^1 . For $\tau_z > t$ and fixed $X_\varepsilon(0)$, we derive the transition density function u of process $(X_\varepsilon(t), z_t)$ with events $\{M_t \leq \frac{1}{2}M\}$:

$$\begin{aligned} u(t, X, z, X_\varepsilon(0))dXdz &= P(X_\varepsilon(t) \in dX, z_t \in dz, M_t \leq \frac{1}{2}M \mid X_\varepsilon(0), z_0 = 0) \\ &= P(X_\varepsilon^M(t) \in dX, z_t^M \in dz \mid X_\varepsilon(0), z_0 = 0) . \end{aligned}$$

For fixed $(X_\varepsilon(0), z_0)$, we also denote $u(t, X, z, X_\varepsilon) = u(t, X, z)$. The corresponding Fokker-Planck equation becomes

$$\begin{cases} \frac{\partial u}{\partial t} = L^*u, & (X, z) \in D , \\ u(t, X, z) = 0, & |z| = \frac{1}{2}M , \\ u(0, X, z) = \delta_{(0,0,0)}(X - (x_\varepsilon(0), y_\varepsilon(0)), z - z_0) , \end{cases} \quad (39)$$

where

$$D = \mathbf{R} \times \mathbf{R} \times (-\frac{1}{2}M, \frac{1}{2}M) .$$

Remark 4. This degenerate equation is similar to that in [21].

We delay justification of equation (39) until the end.

Step two. Here we discuss the event on (t, T_0) . At time t , restricting to the events in B_0^1 is equivalent to requiring that the process z_t remains bounded up to time T_0 .

Consider the probability of z_t remaining bounded until time t while starting at point z :

$$v(t, z) = P(\tau_z > t \mid z_0 = z) .$$

Here v represents probability of the events $\{\sup_{0 \leq s \leq t} |W_s + z| \leq \frac{1}{2}M\}$. Then (see [22]), $v(t, z)$ satisfies the following PDE:

$$\begin{cases} v_t = \frac{1}{2}v_{zz}, & -\frac{1}{2}M < z < \frac{1}{2}M, t > 0 , \\ v(t, \frac{1}{2}M) = v(t, -\frac{1}{2}M) = 0, & t > 0 , \\ v(0, z) = 1, & -\frac{1}{2}M < z < \frac{1}{2}M . \end{cases} \quad (40)$$

Here, (40) can be solved. The remaining probability becomes

$$v(T_0 - t, z) = P(\sup_{t \leq s \leq T_0} |W_s| \leq \frac{1}{2}M \mid W_t = z) ,$$

and probability of B_0^1 is

$$v(T_0, 0) = P(B_0^1) .$$

Combining step one and two: Since $(X_\varepsilon(t), z_t, M_t)$ forms a Markov process, we obtain the joint transition density function for $(X_\varepsilon(t), z_t)$ with B_0^1 :

$$p(t, X, z, X_\varepsilon(0), B_0^1)dXdz = P(X_\varepsilon(t) \in dX, z_t \in dz, B_0^1 \mid X_\varepsilon(0), z_0 = 0) ,$$

which satisfies

$$p(t, X, z, X_\varepsilon(0), B_0^1) = u(t, X, z) P\left(\sup_{t \leq s \leq T_0} |W_s| \leq \frac{1}{2}M \mid W_t = z\right).$$

And the marginal density becomes

$$p(t, X, X_\varepsilon(0), B_0^1) = \int_{-\frac{1}{2}M}^{\frac{1}{2}M} p(t, X, z, X_\varepsilon(0), B_0^1) dz.$$

The derivation becomes as following: for the first time interval $(0, T_0)$, recall u_1 defined in (37) represents transition density function for $X_\varepsilon(t)$ conditioned on B_0^1 , which statisfies

$$\begin{aligned} u_1(t, X, X_\varepsilon(0)) &= \frac{p(t, X, X_\varepsilon(0), B_0^1)}{P(B_0^1)} \\ &= \frac{\int_{-\frac{1}{2}M}^{\frac{1}{2}M} u(t, X, z) v(T_0 - t, z) dz}{v(T_0, 0)}, \end{aligned}$$

where u satisfies equation (39) with $X_\varepsilon(0)$, $z_0 = 0$ and v is the solution of equation (40).

Step three. “Refreshing”. X_t under B_0 can be seen as a refreshed process at each time kT_0 . And by applying the Markov property, we can derive general transition density function for $X_\varepsilon(t)$.

Consider the events set B_0^{N+1} . We denote w as the transition density function for $(X_\varepsilon(t), z_t)$ with events B_t :

$$w(t, X, z, X_\varepsilon(0), z_0) dX dz = P(X_\varepsilon(t) \in dX, z(t) \in dz, B_t \mid X_\varepsilon(0), z_0),$$

where

$$B_t = \{\omega : \sup_{\substack{0 \leq s < T_0 \\ s + kT_0 \leq t}} |W_{s+kT_0} - W_{kT_0}| \leq M\}.$$

Again, for fixed \bar{X}_0 , we denote $w = w(t, X, z)$, and by Chapman–Kolmogorov equation

$$w(t, X, z) = \int_{\mathbf{R}^2} \dots \int_{\mathbf{R}^2} u(t, \bar{X}, \bar{X}_N) \prod_{i=0}^{N-1} u(T_0, \bar{X}_{i+1}, \bar{X}_i) d\bar{X}_1 d\bar{X}_2 \dots d\bar{X}_N,$$

where we used the notation $\bar{X}_i = (X_i, z_i)$, and $\bar{X}_0 = (X_\varepsilon(0), 0)$. Combining events from time t to $(N+1)T_0$, we have

$$\begin{aligned} u_{N+1}(t, X, X_\varepsilon(0)) &= \frac{\int_{-\frac{1}{2}M}^{\frac{1}{2}M} w(t, X, z) v((N+1)T_0 - t, z) dz}{P(B_0^{N+1})} \\ &= \frac{\int_{-\frac{1}{2}M}^{\frac{1}{2}M} w(t, X, z) v((N+1)T_0 - t, z) dz}{v(T_0, 0)^{N+1}}. \end{aligned}$$

Arbitrary t . From the independent increments property of B_0 for each time interval, we can derive the transition density function for $X_\varepsilon(t)$ conditioned on B_0 . Indeed, for any t , there exists $N = 0, 1, \dots$, such that $t \in [NT_0, (N+1)T_0]$. Then, we simply have

$$p(t, X, X_\varepsilon(0) \mid B_0) = u_{N+1}(t, X, X_\varepsilon(0)).$$

Finally, we justify equation (39).

Proof of (39). The basic approach we use is standard; e.g., see [13] and [23].

The boundary conditions can be given as $u(0, x, y, z) = \delta_{(0,0,0)}(x - x_\varepsilon(0), y - y_\varepsilon(0), z - z(0))$ and $u(t, x, y, \pm \frac{1}{2}M) = 0$. Next, we follow the same steps used to derive the Fokker-Planck equation for the diffusion process.

To simplify notation, denote $Y = (x, y, z)$ and $Y_t = (x_\varepsilon(t), y_\varepsilon(t), z(t))$. Consider any test function $h(x, y, z) = h(Y) \in C^2$ with compact support. Then,

$$\int_D h(Y) \frac{\partial u(t, Y)}{\partial t} dY = \int_D h(Y) \lim_{\Delta t \rightarrow 0} \frac{u(t + \Delta t, Y) - u(t, Y)}{\Delta t} dY, \quad (41)$$

where $D = \mathbf{R} \times \mathbf{R} \times (-\frac{1}{2}M, \frac{1}{2}M)$. Again, consider the process (Y_t, M_t) . Since (Y_t, M_t) is a Markov process, if we denote its density function with \bar{p} , which is defined by

$$\bar{p}(t, Y, Y(0)) dY = P(Y_t \in dY, M_t \leq \frac{1}{2}M | Y(0)),$$

Therefore the Chapman–Kolmogorov equation implies

$$\begin{aligned} u(t + \Delta t, Y) &= \bar{p}(t + \Delta t, Y, Y(0)) \\ &= \int_D u(t, Z) \bar{p}(\Delta t, Y, Z) dZ. \end{aligned}$$

Above, the last equality comes from the Markov property. Hence, equation (41) becomes

$$\begin{aligned} & \int_D h(Y) \frac{\partial u(t, Y)}{\partial t} dY \\ &= \int_D h(Y) \lim_{\Delta t \rightarrow 0} \frac{\int_D u(t, Z) \bar{p}(\Delta t, Y, Z) dZ - u(t, Y)}{\Delta t} dY \\ &= \lim_{\Delta t \rightarrow 0} \frac{\int_D \int_D h(Y) u(t, Z) \bar{p}(\Delta t, Y, Z) dZ dY - \int_D h(Y) u(t, Y) dY}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\int_D \int_D h(Y) u(t, Z) \bar{p}(\Delta t, Y, Z) dY dZ - \int_D h(Z) u(t, Z) dZ}{\Delta t} \\ &= \int_D \lim_{\Delta t \rightarrow 0} \frac{\int_D h(Y) \bar{p}(\Delta t, Y, Z) dY - h(Z)}{\Delta t} u(t, Z) dZ. \end{aligned} \quad (42)$$

where the second and last equalities are justified by the dominated convergence theorem, and the third equality comes from Fubini's theorem. Since $E(\mathbf{1}_{\{\tau_z \leq \Delta t\}}) = o(\Delta t)$ and h is a bounded function, then

$$\begin{aligned} & \lim_{\Delta t \rightarrow 0} \frac{\int_D h(Y) \bar{p}(\Delta t, Y, Z) dY - h(Z)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{E h(Y_{\Delta t}) \mathbf{1}_{\{\tau_z > \Delta t\}} - h(Z)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left[\frac{E h(Y_{\Delta t}) - h(Z)}{\Delta t} - \frac{E h(Y_{\Delta t}) \mathbf{1}_{\{\tau_z \leq \Delta t\}}}{\Delta t} \right] = Lh(Z), \end{aligned}$$

where L is the infinitesimal operator defined by (36). Hence (42) becomes

$$\int_D h(Y) \frac{\partial u(t, Y)}{\partial t} dY = \int_D Lh(Z) u(t, Z) dZ.$$

Integrating by parts, using $u(t, Z) = 0$ on the boundary and letting $Z = Y$ on the right-hand-side, we then obtain

$$\int_D h(Y) \frac{\partial u(t, Y)}{\partial t} dY = \int_D h(Y) L^* u(t, Y) dY,$$

which gives the equation (39). \square

5. CONCLUSIONS

In this work, motivated by practical observations (real world phenomena, laboratory experiments, and numerical simulations) on typical engineering circuitries, we reconsidered what model of noise is appropriate for the mathematical modeling of stochastic perturbation of second order systems of differential equations that admit stable limit cycles. Whereas classical models consider stochastic DEs where perturbations come from standard Brownian motion paths, we restricted the class of allowed disturbances, to avoid pumping infinite energy into the system through the noise. In essence, our new model consists in selecting those Brownian paths that have bounded increments in finite time.

Of course, there are new challenges when one gives up familiar ground, such as white noise perturbations, and indeed we have encountered technical difficulties especially insofar as obtaining viable expression for the transition density function. However, by selecting the allowed perturbations from within our proposed event set, we were able to adopt many classical tools from dynamical systems, and show some interesting mathematical results, that further appear to be more in tune with practically observed circuitry behaviors.

Relative to our set of allowed stochastic perturbations, our main results have been the following.

- (i) We proved global boundedness of the stochastic trajectories, and we showed that they remain (for small values of the parameter ε appearing in front of the perturbation term) in the neighborhood of the deterministic limit cycle.
- (ii) We proposed, and ensured the existence, of stochastic Poincaré map(s) as a point-to-distribution map, and further introduced three point-to-point Poincaré maps: first, last, and average return maps.
- (iii) We associated the study of transition densities to a pair of PDEs.

In the future, we plan to further explore selected aspects of the present work. In particular, we plan to study (at least experimentally) the statistical properties of the return distribution associated to the Poincaré map, and to further perform numerical exploration of the approach based on the system of PDEs herein derived.

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