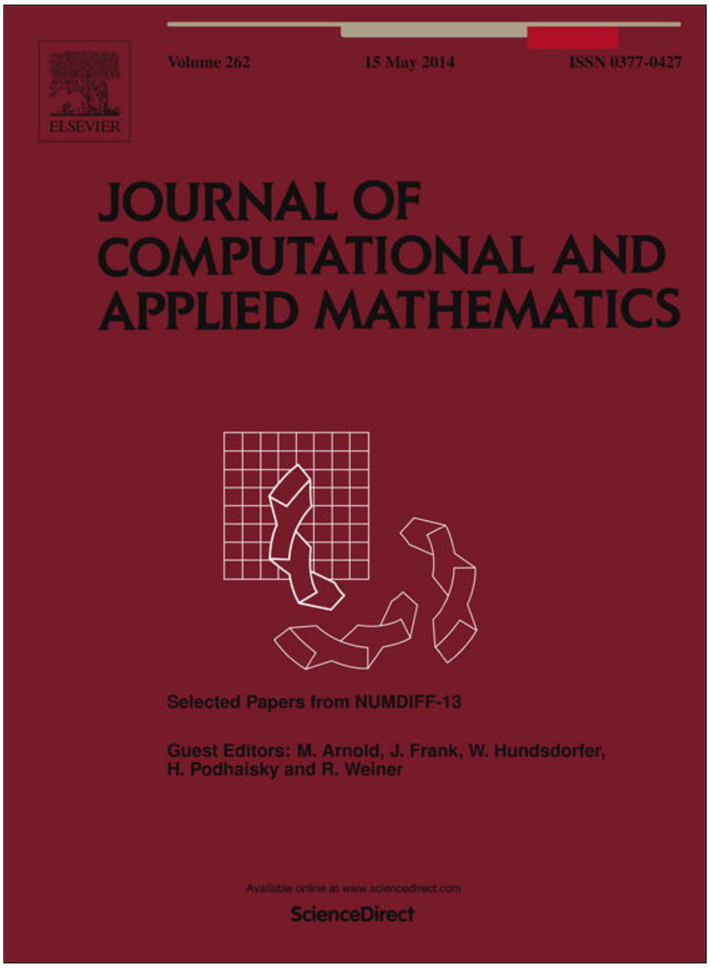
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# A comparison of Filippov sliding vector fields in codimension 2

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ABSTRACT

We consider several possibilities on how to select a Filippov sliding vector field on a codimension 2 singularity surface $\Sigma$, intersection of two codimension 1 surfaces. We discuss and compare several, old and new, approaches, under the assumption that $\Sigma$ is nodally attractive. Of specific interest is the selection of a smoothly varying Filippov sliding vector field. As a result of our analysis and experiments, the best candidates of the many possibilities explored are those based on the so-called barycentric coordinates. In the present context, one of these possibilities appear to be new.



## 1. Introduction

Our purpose in this paper is to discuss, and compare, several possibilities on how to select a Filippov sliding vector field on a codimension 2 singularity surface $\Sigma$, which is itself the intersection of two codimension 1 singularity surfaces. We give a unifying framework within which to compare the various possibilities considered, and we will highlight and clarify important connections to methods that have proven useful in computer graphics and finite elements techniques.

In this section, we review the basic problem and set up notation. Then, in Sections 2 and 3 we look at different possibilities for Filippov sliding vector fields. For convenience, we classify different choices as being either *analytic–algebraic methods* or *geometric methods*; the distinction is only for convenience of introducing the methods, but the geometric methods we consider can in fact be interpreted as special choices of analytic methods. Finally, in Section 4 we see how one may generally reformulate the problem with respect to *sub-sliding* vector fields. In Section 5 we give our conclusions.

### 1.1. The problem and Filippov solutions

We are interested in piecewise smooth differential systems of the following type:

$$\dot{x} = f(x), \quad f(x) = f_i(x),\ x \in R_i,\ i = 1, \dots, 4, \tag{1.1}$$

with initial condition $x(0) = x_0 \in R_i$, for some $i$. Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and we may as well think that $\mathbb{R}^n = \overline{\bigcup_i R_i}$. Each $f_i$ is smooth on $R_i$, $i = 1, \dots, 4$, and we will assume that each $f_i$ is actually smooth in an open neighborhood of the closure of each $R_i$, $i = 1, \dots, 4$. (Strictly speaking, this last assumption may actually be not needed, but it simplifies some of the later exposition.)

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**Table 1**
Nodal attractivity.

| Component | $i=1$ | $i=2$ | $i=3$ | $i=4$ |
|---|---|---|---|---|
| $w_i^1$ | $>0$ | $>0$ | $<0$ | $<0$ |
| $w_i^2$ | $>0$ | $<0$ | $>0$ | $<0$ |

Clearly, from (1.1), the vector field is not properly defined on the boundaries of the $R_i$'s. We are particularly interested in analyzing what happens in this case, under the scenario that solution trajectories are attracted towards these boundaries.

### 1.2. Codimension 1: attractivity, existence and uniqueness

The classical Filippov theory (see [1]) is concerned with the case of two regions separated by a surface $\Sigma$ defined as the 0-set of a smooth scalar valued function $h$:

$$\begin{aligned} &\dot{x} = f_1(x), \quad x \in R_1, \quad \text{and} \quad \dot{x} = f_2(x), \quad x \in R_2, \\ &\Sigma := \{x \in \mathbb{R}^n : h(x) = 0\}, \quad h : \mathbb{R}^n \to \mathbb{R}, \end{aligned} \tag{1.2}$$

where $\nabla h$ is bounded away from 0 for all $x \in \Sigma$, hence near $\Sigma$. As in [1], we assume that $h$ is a $\mathcal{C}^k$ function, with $k \geq 2$. Finally, without loss of generality, we label $R_1$ such that $h(x) < 0$ for $x \in R_1$, and $R_2$ such that $h(x) > 0$ for $x \in R_2$.

The interesting case is when trajectories reach $\Sigma$ from $R_1$ (or $R_2$), and one has to decide what happens next. To answer this question, it is useful to look at the components of the two vector fields $f_{1,2}$ orthogonal to $\Sigma$:

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \nabla h(x)^\top f_1(x) \\ \nabla h(x)^\top f_2(x) \end{bmatrix}, \quad x \in \Sigma. \tag{1.3}$$

Filippov theory is a first order theory (that is, based on nonvanishing $w_i$, $i = 1, 2$) providing an answer to this situation. We call $\Sigma$ *attractive in finite time* if for some positive constant $c$, we have

$$\nabla h(x)^\top f_1(x) \geq c > 0 \quad \text{and} \quad \nabla h(x)^\top f_2(x) \leq -c < 0,$$

for $x \in \Sigma$. In this case, trajectories starting near $\Sigma$ must reach it and remain there: *sliding motion*. Filippov proposal is to take as sliding vector field on $\Sigma$ a convex combination of $f_1$ and $f_2$, $f_F := (1-\alpha)f_1 + \alpha f_2$, with $\alpha$ chosen so that $f_F \in T_\Sigma$ ($f_F$ is tangent to $\Sigma$ at each $x \in \Sigma$):

$$x' = (1-\alpha)f_1 + \alpha f_2, \quad \alpha = \frac{\nabla h(x)^\top f_1(x)}{\nabla h(x)^\top f_1(x) - \nabla h(x)^\top f_2(x)}. \tag{1.4}$$

### 1.3. Codimension 2: nodal attractivity

As we said, we are concerned with (1.1), where now the $R_i$'s are (locally) separated by two intersecting smooth surfaces of co-dimension 1, $\Sigma_1 = \{x : h_1(x) = 0\}$ and $\Sigma_2 = \{x : h_2(x) = 0\}$, and we have $\Sigma = \Sigma_1 \cap \Sigma_2$. As before, we will assume that $\nabla h_1(x) \neq 0$, $x \in \Sigma_1$, $\nabla h_2(x) \neq 0$, $x \in \Sigma_2$, that $h_{1,2}$ are $\mathcal{C}^k$ functions, with $k \geq 2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for $x$ on (and in a neighborhood of) $\Sigma$.

We have four different regions $R_1, R_2, R_3$ and $R_4$ with the four different vector fields $f_i$, $i = 1, \ldots, 4$, in these regions:

$$\dot{x} = f_i(x), \quad x \in R_i, \; i = 1, \ldots, 4. \tag{1.5}$$

Without loss of generality, we can label the regions as follows:

$$\begin{aligned} &R_1 : f_1 \quad \text{when } h_1 < 0, \; h_2 < 0, \qquad R_2 : f_2 \quad \text{when } h_1 < 0, \; h_2 > 0, \\ &R_3 : f_3 \quad \text{when } h_1 > 0, \; h_2 < 0, \qquad R_4 : f_4 \quad \text{when } h_1 > 0, \; h_2 > 0. \end{aligned} \tag{1.6}$$

We further let (cf. with (1.3))

$$\begin{aligned} &w_1^1 = \nabla h_1^\top f_1, \qquad w_2^1 = \nabla h_1^\top f_2, \qquad w_3^1 = \nabla h_1^\top f_3, \qquad w_4^1 = \nabla h_1^\top f_4, \\ &w_1^2 = \nabla h_2^\top f_1, \qquad w_2^2 = \nabla h_2^\top f_2, \qquad w_3^2 = \nabla h_2^\top f_3, \qquad w_4^2 = \nabla h_2^\top f_4, \end{aligned} \tag{1.7}$$

and restrict to the case of $\Sigma$ being *nodally attractive*, a condition characterized by the constraints on the sign of $w^1$ and $w^2$ expressed in Table 1, which are assumed to be valid on $\Sigma$ and near it (uniformly away from 0).

According to the present setup, when $x$ is near $\Sigma$, a trajectory through $x$ will be attracted to $\Sigma$, and – upon reaching it – will be forced to remain on it (*sliding motion*).

**Remark 1.1.** Nodal attractivity of $\Sigma$ is just one of several different sufficient conditions under which $\Sigma$ will attract nearby trajectories. Arguably, nodal attractivity is the simplest of all these sufficient conditions and it serves as a fundamental benchmark to evaluate different means for obtaining a sliding vector field on $\Sigma$. A more comprehensive classification of attractivity conditions for $\Sigma$ is in [2], and we are currently investigating the behavior of some of the methods examined herein under these more exhaustive attractivity conditions. For completeness, we also note that the case in which $\Sigma$ does not attract nearby trajectories is of more limited interest.

Even under nodal attractivity, it is to be expected that a trajectory will typically first reach one of $\Sigma_1$ or $\Sigma_2$, and then slide on it towards $\Sigma$. (Of course, a trajectory may hit $\Sigma$ directly, without first reaching one of $\Sigma_{1,2}$ and sliding on it towards $\Sigma$, but this is a measure 0 set of initial conditions.) For completeness, and later use, below we define these *sub-sliding* vector fields.

Let $\Sigma_1^{\pm} = \{x : h_1(x) = 0,\ h_2(x) \gtrless 0\}$, and similarly $\Sigma_2^{\pm}$. So, when $x$ is on $\Sigma_{1,2}$ (but not on the intersection $\Sigma$), we have the following four sub-sliding vector fields, defined as in Section 1.2:

$$\begin{aligned} &f_{\Sigma_1^+}(x) := (1-\alpha^+)f_2 + \alpha^+ f_4, \qquad f_{\Sigma_1^-}(x) := (1-\alpha^-)f_1 + \alpha^- f_3,\\ &f_{\Sigma_2^+}(x) := (1-\beta^+)f_3 + \beta^+ f_4, \qquad f_{\Sigma_2^-}(x) := (1-\beta^-)f_1 + \beta^- f_2. \end{aligned} \tag{1.8}$$

Note that – under the assumptions of nodal attractivity of Table 1 – these vector fields are well defined, with

$$\begin{aligned} &\alpha^+ := \frac{w_2^1}{w_2^1 - w_4^1}, \qquad \alpha^- := \frac{w_1^1}{w_1^1 - w_3^1},\\ &\beta^+ := \frac{w_3^2}{w_3^2 - w_4^2}, \qquad \beta^- := \frac{w_1^2}{w_1^2 - w_2^2}. \end{aligned} \tag{1.9}$$

The difficulty is how to properly define sliding motion on $\Sigma = \Sigma_1 \cap \Sigma_2$.

We will still consider the Filippov convexification method to define the vector field on $\Sigma$, whereby considering a vector field on $\Sigma$ defined as a convex combination of the vector fields $f_1, \ldots, f_4$, and such that it lies on the tangent plane to $\Sigma$, $T_\Sigma$, for any $x \in \Sigma$. That is, we will restrict to vector fields of the form

$$\begin{aligned} &f_F = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4, \qquad \lambda_i \geq 0,\ i = 1, \ldots, 4, \qquad \sum_{i=1}^{4} \lambda_1 = 1,\\ &\nabla h_1^\top f_F = \nabla h_2^\top f_F = 0. \end{aligned} \tag{1.10}$$

Adopting our previous notation, we thus have to solve the problem (for $x \in \Sigma$):

$$W\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}, \text{ and } W = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tag{1.11}$$

and we will call a solution $\lambda$ of (1.11) *admissible* if $\lambda \geq 0$.

Clearly, (1.11) is an underdetermined linear system, reflecting the fact that the mere requirement of $f_F$ being on $T_\Sigma$ is not sufficient to uniquely characterize a convex combination of the four vector fields $f_1, \ldots, f_4$. This is precisely our concern in this paper: how can one properly define $f_F$, under the conditions expressed by Table 1. Our specific interest will be to select a Filippov sliding vector field that varies smoothly with respect to $x \in \Sigma$.

### 1.4. Framework

To begin with, we have the following result on the matrix $W$. Note that $W$ depends smoothly on $x \in \Sigma$, say it is a $\mathcal{C}^k$ function of $x$, $k \geq 1$, since each of the $f_j$'s ($j = 1, \ldots, 4$) and $h_i$'s ($i = 1, 2$) are.

**Lemma 1.2.** *For $x \in \Sigma$, consider the matrix valued function $W$ of (1.11), under the sign constraints of Table 1. Then, $W$ has constant rank equal to 3, for any $x \in \Sigma$. Moreover, $\ker(W)$ is spanned by a unit vector of class $\mathcal{C}^k$.*

**Proof.** Suppose $W$ does not have rank 3. Then, there are two columns of $W$ that can be written as a linear combination of the other two. But recall that $W$ has the form $\begin{bmatrix} + & + & - & - \\ + & - & + & - \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Therefore, trivially in no case two columns of $W$ can be given as a linear combination of any of the other two columns. In particular, it follows that the matrix $W$ has a one-dimensional kernel.

As a consequence, the function $W^\top W$ – which takes values in $\mathbb{R}^{4\times 4}$ – depends smoothly on $x$, and has one eigenvalue equal to 0 and three eigenvalues not 0. But, since this is a Hermitian function of constant rank 3, then (e.g., see [3]) there exists a smoothly varying orthogonal function $U$: $U^T(W^T W)U = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, where the eigenvalues of the $(3 \times 3)$ function $M$ are not 0. In particular, calling $v$ the last column of $U$, $\ker(W) = \text{span}(v)$, as claimed. □

**Remark 1.3.** In practice, we will have that along a smooth trajectory on $\Sigma$ during sliding motion, $W$ will effectively be a smooth function of one real parameter (time).

Based upon Lemma 1.2, and under the assumptions therein, we will thus have that all solutions of (1.11) can be once and for all written as

$$\lambda = \mu + cv, \tag{1.12}$$

where $v$ (which we may, but do not have to, also take of unit length) smoothly spans $\ker(W)$, $\mu$ is any particular solution of (1.11), not necessarily admissible, and $c \in \mathbb{R}$ will need to be chosen so that $\lambda$ in (1.12) is admissible (i.e., each of its component be nonnegative). Observe that, because of Table 1, $v$ must have some positive and some negative components, and thus the admissibility interval for $c$ in (1.12) is $c \in [a, b]$, where

$$a := \max\left\{-\frac{\mu_i}{v_i} : v_i > 0\right\}, \qquad b := \min\left\{-\frac{\mu_i}{v_i} : v_i < 0\right\}, \tag{1.13}$$

where we remark that the values of $a$, $b$, depend both on $\mu$ and $v$, and of course in general on the point $x \in \Sigma$.

In the next sections, we will focus on different ways to choose $\lambda$ in (1.12), and we will further relate the various choices to each other. As already remarked, our emphasis will be to have methods which produce a smoothly varying solution vector $\lambda$.

**Remark 1.4.** In (1.12), $\mu$ can be chosen in any way we want, regardless of providing an admissible (or smooth) solution of (1.11); for example, we can select $\mu$ to be the solution of (1.11) of minimal 2-norm (see Section 2.2.1). In general, also $v$ can be any vector spanning $\ker(W)$, though here below we will always assume that it be smooth in $x \in \Sigma$. Therefore, to obtain a smoothly varying $\lambda$ from (1.12), it will be crucial to appropriately select $c$ there.

**Example 1.5** (*A Model Example*)**.** To compare the various techniques we review/introduce, we will use the following example, which is sufficiently simple to allow hand calculations, yet rich enough to illustrate all desired features.

We take the following vector fields $f_i$, $i = 1, 2, 3, 4$, taking values in $\mathbb{R}^3$:

$$f_1(x) := \begin{bmatrix} 2x_1+1 \\ -x_1+x_2x_3+1 \\ x_1+x_2+1 \end{bmatrix}, \quad x \in R_1, \qquad f_2(x) := \begin{bmatrix} 2x_1-1 \\ -x_1+x_3-1 \\ x_1+x_2x_3+2 \end{bmatrix}, \quad x \in R_2,$$

$$f_3(x) := \begin{bmatrix} 2x_1-3 \\ -x_1+x_2+2 \\ x_1+x_2x_3-1 \end{bmatrix}, \quad x \in R_3, \qquad f_4(x) := \begin{bmatrix} 2x_1+2 \\ -x_1+x_3-2 \\ x_1+x_3-2 \end{bmatrix}, \quad x \in R_4,$$

where the regions $R_i$'s are as in (1.6) and

$$h_1(x) := x_3, \qquad h_2(x) := x_2.$$

Therefore, $\Sigma = \{x \in \mathbb{R}^3 : x_2 = x_3 = 0\}$, we have the two unit normals $n_1(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $x \in \Sigma_1$, $n_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $x \in \Sigma_2$, and we can write the matrix $W$ for $x \in \Sigma$ as:

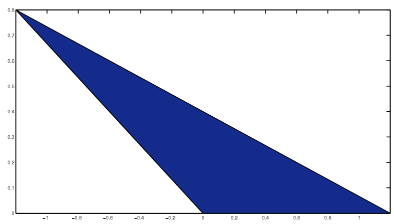
$$W(x) = \begin{bmatrix} x_1+1 & x_1+2 & x_1-1 & x_1-2 \\ -x_1+1 & -x_1-1 & -x_1+2 & -x_1-2 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{1.14}$$

Observe that the sign pattern of Table 1 for nodal attractivity holds for $x_1 \in (-1, 1)$. At the same time, we also note that the more comprehensive attractivity conditions of [2] hold also outside of this interval, namely for $|x_1| \leq 1.2$, and that when $x_1 = \pm 1.2$ the exit conditions of [2] hold, $\Sigma$ is no longer attractive, and one should exit $\Sigma$ by sliding on $\Sigma_1$, respectively $\Sigma_2$. On account of this, we would surely value any technique able to provide smoothly varying solutions $\lambda$ for all $|x_1| \leq 1.2$, relatively to the present example, and further one which when $x_1 = \pm 1.2$ renders two coefficients in $\lambda$ equal to 0. As we will see below, there are not many such choices.

Finally, one can easily obtain the general form of the admissible solutions (1.12), for example written as

$$\lambda = \begin{bmatrix} \frac{2}{3} - \frac{5}{9}x_1 \\ 0 \\ \frac{2}{3}x_1 \\ \frac{1}{3} - \frac{1}{9}x_1 \end{bmatrix} + c \begin{bmatrix} -\frac{5}{3} \\ 1 \\ 1 \\ -\frac{1}{3} \end{bmatrix}, \tag{1.15}$$

which is admissible for $(x_1, c)$ in the shaded region shown on the right.



Admissible region $(x_1, c)$ in (1.15).

## 2. Analytic–algebraic methods

Here we introduce some techniques to select $\lambda$ in (1.12). As far as we know, the construction behind the method(s) of Section 2.1 is new. The idea of Section 2.2.1 is patterned on general minimum variation principles, and the second method in that section is already in [4]. Finally, the techniques examined in Section 2.3 are patterned after a successful interpretation of the Filippov sliding vector field in co-dimension 1.

### 2.1. Mean field methods

Given the form of (1.12), and the restriction on $c$ given by (1.13), we define a *uniform* mean field method by selecting $c$ to be the midpoint of $[a, b]$ (recall that $a$ and $b$ depend on $\mu$, $v$, and $x \in \Sigma$):

$$\lambda_{MF} := \mu + \frac{a+b}{2} v. \tag{2.1}$$

Note that, in (2.1), we are taking the expected value of the random variable $\Xi$ according to the uniform distribution over $[a, b]$. This suggests a useful generalization, based on the following definition.

**Definition 2.1** (*Mean Field Methods*)**.** Let $\mu$ be a particular solution of (1.11), and $v$ be also given. Assume that the random variable $\Xi$ obeys a probability distribution over $[a, b]$, with pdf (probability density function) $g(\xi)$. Then, we define the family of mean field methods according to

$$c := \int_a^b \xi g(\xi)\, d\xi \quad \text{and} \quad \lambda_g := \mu + \left(\int_a^b \xi g(\xi)\, d\xi\right) v. \tag{2.2}$$

We have the following result, telling us that the (pointwise) value of $\lambda_g$ is independent of $\mu$.

**Lemma 2.2.** *For given $v$, the value of $\lambda_g$ in (2.2) is independent of the particular solution $\mu$. Moreover, choosing $c$ and $\lambda_g$ as in (2.2) always gives an admissible solution.*

**Proof.** Suppose that we have chosen $c$ as in (2.2) for a given $\mu$, and let $\widetilde{\mu}$ be another solution of (1.11), giving admissibility interval $\widetilde{c} \in [\widetilde{a}, \widetilde{b}]$.

Then, there exists a value $\tau \in [a, b]$ such that $\widetilde{\mu} = \mu + \tau v$. But

$$\widetilde{\mu} + \widetilde{c}v \geq 0 \Leftrightarrow \mu + (\widetilde{c} + \tau)v \geq 0 \Leftrightarrow \widetilde{c} + \tau \in [a, b] \Leftrightarrow \widetilde{c} \in [a - \tau, b - \tau].$$

In particular, $[\widetilde{a}, \widetilde{b}]$ and $[a, b]$ have the same length. From this, it follows that if $\xi$ has pdf $g(\xi)$ over $[a, b]$, then $\widetilde{\xi}$ will have pdf $\widetilde{g}(\widetilde{\xi}) := g(\widetilde{\xi} + \tau)$, $\widetilde{\xi} \in [\widetilde{a}, \widetilde{b}] = [a - \tau, b - \tau]$. Therefore, by (2.2),

$$\begin{aligned}
\widetilde{\lambda}_g &= \widetilde{\mu} + \left(\int_{a-\tau}^{b-\tau} \widetilde{\xi}\widetilde{g}(\widetilde{\xi})\, d\widetilde{\xi}\right) v = \widetilde{\mu} + \left(\int_{a-\tau}^{b-\tau} \widetilde{\xi} g(\widetilde{\xi} + \tau)\, d\widetilde{\xi}\right) v \\
&= \widetilde{\mu} + \left(\int_a^b (\xi - \tau) g(\xi)\, d\xi\right) v = \widetilde{\mu} + \left(\int_a^b \xi g(\xi)\, d\xi - \tau \int_a^b g(\xi)\, d\xi\right) v \\
&= \widetilde{\mu} - \tau v + \left(\int_a^b \xi g(\xi)\, d\xi\right) v = \mu + \left(\int_a^b \xi g(\xi)\, d\xi\right) v = \lambda_g.
\end{aligned}$$

Finally, that choosing $c$ and $\lambda_g$ as in (2.2) produces an admissible solution is clear. □

The following example shows that, in general, $\lambda_{MF}$ (i.e., where the probability distribution function is the uniform distribution), although obviously admissible, and trivially continuous in case $\mu$ is, is not as smooth as $W$.

**Example 2.3.** Let us refer to Example 1.5. By the configuration of this problem, it is easy to obtain

$$\frac{a(x_1) + b(x_1)}{2} = \begin{cases} \frac{1}{6}x_1 - \frac{1}{15}, & \text{if } x_1 \in \left[-\frac{6}{5}, 0\right], \\ \frac{1}{18}x_1 - \frac{1}{15}, & \text{if } x_1 \in \left[0, \frac{6}{5}\right], \end{cases}$$

which gives $\lambda_{MF}$ not differentiable at $x_1 = 0$, whereas $W$ is analytic in $x_1$. □

So, it is natural to ask: "How can we choose a distribution function $g$ in order to make $\lambda_g$ in (2.2) as smooth as $W$?"
We propose to consider the following family of distribution functions:

$$g_\alpha(\xi) := \frac{\alpha(\xi - a)^{\alpha-1}}{(b-a)^\alpha}, \quad \xi \in [a, b], \ \alpha \in (0, +\infty). \tag{2.3}$$

This family of pdf's belongs to the Beta distribution family with parameters $(\alpha, 1)$, and we restrict to this family of pdf's because of their natural formulation on compact intervals.

For (2.3), we have

$$g_\alpha \geq 0, \quad \int_a^b g_\alpha(\xi)d\xi = 1, \qquad \int_a^b \xi g_\alpha(\xi)d\xi = \frac{1}{\alpha+1}a + \frac{\alpha}{1+\alpha}b,$$

from which $c$ in (2.2) is given by

$$c = (1-\gamma)a + \gamma b, \quad \gamma = \frac{\alpha}{\alpha+1}, \tag{2.4}$$

that is, for every $\alpha \in (0, +\infty)$, the expectation of the random variable $\xi$ with measure $g_\alpha(\xi)$ is the convex combination of $a, b$ with weights $\frac{1}{\alpha}, \frac{\alpha}{1+\alpha}$.

Although not necessarily any choice of $\alpha$ in (2.3) gives an admissible solution as smooth as $W$ (e.g., taking $\alpha = 1$ gives $\lambda_{\mathrm{MF}}$), we will see in Section 3 that in fact it is possible to choose $\alpha$ to obtain a smoothly varying, admissible, $\lambda_g$.

### 2.2. *Minimum norm*

Here we look at two very natural approaches: to choose the Filippov sliding vector field $f_F$ in such a way to minimize $\|\lambda\|$, or to minimize $\|f_F\|$ directly. Below, the norm is the 2-norm.

#### 2.2.1. *Minimizing* $\lambda$

Here we seek the minimum norm solution of (1.11).

Without directly imposing the positivity constraints, it is simple to obtain the minimum 2-norm solution; e.g., by using the SVD (singular value decomposition) of $W$: $W = USV^\top$, where $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{4\times 4}$ are orthogonal and $S = [\Sigma,\ 0]$ with $\Sigma = \mathrm{diag}(\sigma_i, i = 1, 2, 3)$ (note, $\sigma_i \neq 0$):

$$\lambda_{\min} = Vy, \quad y = \begin{bmatrix} d_1/\sigma_1 \\ d_2/\sigma_2 \\ d_3/\sigma_3 \\ 0 \end{bmatrix}, \ d = U^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which can also be rewritten from the form (1.12) as

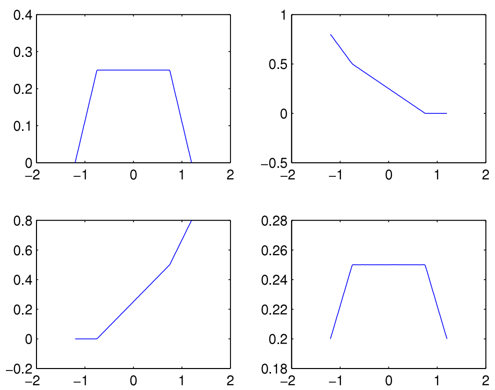
$$\lambda_{\min} := (I - vv^\top)\mu. \tag{2.5}$$

It is easy to realize that $\lambda_{\min}$ is as smooth as $W$.[1] However, it is equally simple to realize that in general this solution may not be admissible (i.e., it is not generally true that $\lambda_{\min} \geq 0$).

Using again the structure (1.12), the min 2-norm admissible solution, $\widehat{\lambda}_{\min}$, is simply given by $\lambda_{\min}$ above if $\lambda_{\min}$ is admissible, and by whichever of $\mu + av$ or $\mu + bv$ gives minimum 2-norm otherwise. Unfortunately, now $\widehat{\lambda}_{\min}$ may fail to vary smoothly.

**Example 2.4.** Take Example 1.5, at $x_1 = -0.9$.

Then, $\lambda_{\min} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{20} \\ -\frac{1}{20} \\ \frac{1}{4} \end{bmatrix}$, which is clearly not admissible. In this case, the admissible solution of minimum 2-norm is $\widehat{\lambda}_{\min} = \begin{bmatrix} \frac{1}{6} \\ \frac{3}{5} \\ 0 \\ \frac{7}{30} \end{bmatrix}$, with $f_{\min} = \begin{bmatrix} -1.7667 \\ 0 \\ 0 \end{bmatrix}$. (Coincidentally, these correspond to $\lambda_{\mathrm{ave}}$ and $f_{\mathrm{ave}}$ as in Example 2.11.) However, as can be seen in the figure on the right, $\widehat{\lambda}_{\min}$ is not as smooth as $W$.




$\widehat{\lambda}_{\min}$ components for Example 1.5.

#### 2.2.2. *Minimizing* $f$ *[Minimum Variation]*

This approach was already suggested in [4]. The goal is to find $f$ as in (1.10) of minimal norm. That is, one solves

$$\min \|f\|^2, \ \text{subject to } W\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

[1] Use the argument in the proof of Lemma 1.2.

Writing $\lambda = \mu + cv$ as in (1.12), we have to determine the minimum of

$$\|F_\mu\|^2 + 2cF_\mu^\top F_v + c^2\|F_v\|^2, \quad \text{where } F_\mu := \sum_{i=1}^{4} \mu_i f_i,\ F_v := \sum_{i=1}^{4} v_i f_i.$$

The minimum is attained for $c = -\frac{F_\mu^\top F_v}{\|F_v\|^2}$, and so the vector field afforded by this approach is

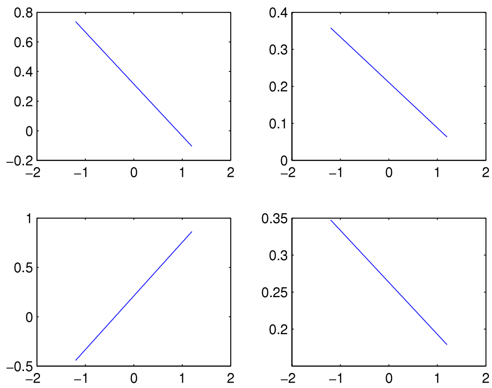
$$f_{\mathrm{MV}} := F_\mu^\top \left( I - \frac{F_v}{\|F_v\|} \frac{F_v^\top}{\|F_v\|} \right) F_\mu, \tag{2.6}$$

which can be fit into the class of vector fields (1.10) by taking

$$\lambda_{\mathrm{MV}} = \mu - \frac{F_\mu^T F_v}{\|F_v\|^2} v. \tag{2.7}$$

Unfortunately, this approach is also affected by similar limitations as those encountered for $\lambda_{\min}$. To be precise, now it may happen that $f_{\mathrm{MV}}$ is not a Filippov vector field (in the sense that $\lambda_{\mathrm{MV}}$ in (2.7) is not admissible), and by restricting the minimization search so that $\lambda_{\mathrm{MV}}$ is admissible may render a non-smooth $f_{\mathrm{MV}}$.

**Example 2.5.** Consider again Example 1.5. Here, the resulting $f_{\mathrm{MV}} = 0$.

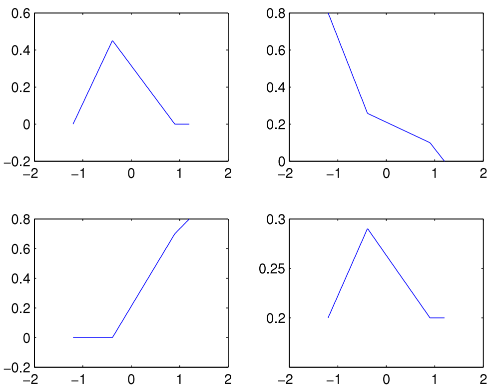


Looking at the $\lambda_{\mathrm{MV}}$ components on the right, we notice that they are smooth, but not always positive for $x_1 \in (-1, 1)$.

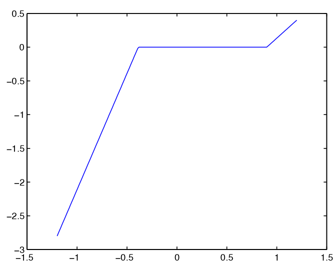
By imposing positivity constraints, that is solving

$$\min \|f\|^2, \text{ subject to } W\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda \geq 0, \tag{2.8}$$

we highlight in the figures below how this generally produces a lack of smoothness in $\lambda$ and a resulting lack of smoothness in $f$.



Components of $\lambda$ for (2.8).



First component of $f$ relatively to (2.8).

For completeness, we remark that – in general – it is not true that $f_{\mathrm{MV}} = 0$ even without imposing the admissibility constraints.

**Remark 2.6.** Whereas it is surely possible to select a different norm, rather than the 2-norm we have chosen, it is not clear to us how "a-priori" one may choose a norm so to obtain a smoothly varying admissible solution through the above minimization processes.

### 2.3. Averaging

Here we attempt to indirectly define a Filippov sliding vector field by averaging the dynamics near $\Sigma$ in a similar way to what has proven to be successful in the case of sliding motion on a co-dimension 1 surface.

We recall that when $\Sigma$ has co-dimension 1, a simple averaging process of the Euler discretization method converges to the Filippov sliding vector field in (1.4). In that case, the idea seems to have been originally introduced by Utkin in [5] (see also [6,7] for added generality). The idea is simple, but we need to re-interpret it appropriately in order to appreciate how we may extend it.

Let $x_0 \in \Sigma$, let $n(x_0)$ be the (unit) normal to $\Sigma$ at $x_0$ and represent points in a $\delta$-neighborhood of $x_0$, of base point $x_0$ (i.e., whose orthogonal projection is $x_0$), as $\{x \in \mathbb{R}^n : x = x_0 + n(x_0)c(x)\}$, where the scalar valued function $c(x)$ represent the distance along the normal direction, hence $c(x) = h(x)$. This way we can define a strip $\mathcal{C}$ of width $2\delta$ around $\Sigma$.

Now, suppose we have fields $f_1$ and $f_2$, defined on and around $\Sigma$. Take a point $x^{(0)} \in R_1$, of base point $x_0 \in \Sigma$, such that $h(x^{(0)}) = -\delta$, and consider the value given by a Euler step, $x^{(1)} = x^{(0)} + \tau_0 f_1(x^{(0)})$, with $\tau_0$ chosen so that $x^{(1)}$ is in $R_2$ and $h(x^{(1)}) = \delta$ (this is always possible, given that $h_x^T f_1 > 0$). From $x^{(1)}$, we take another Euler step, $x^{(2)} = x^{(1)} + \tau_1 f_2(x^{(1)})$, with $\tau_1$ so that $x^{(2)} \in R_1$ and $h(x^{(2)}) = -\delta$. Now consider $(x^{(2)} - x^{(0)})/(\tau_0 + \tau_1) = \frac{\tau_0}{\tau_0+\tau_1} f_1(x^{(0)}) + \frac{\tau_1}{\tau_0+\tau_1} f_2(x^{(1)})$. A standard calculation (e.g., see [6]) gives that

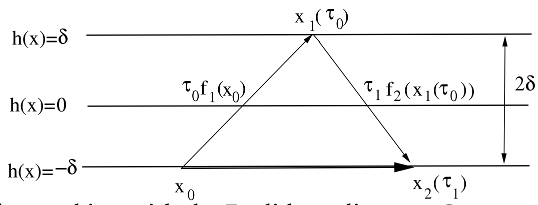
$$\lim_{\delta\to 0}(x^{(2)} - x^{(0)})/(\tau_0 + \tau_1) = \alpha f_1(x_0) + (1-\alpha) f_2(x_0),$$
$$\alpha = h^T(x_0)f_1(x_0)/(h^T(x_0))(f_1(x_0) - f_2(x_0)),$$

that is (1.4).

**Remarks 2.7.**

(i) We note that this averaging process is logically one-dimensional, since the iterates are effectively controlled by the scalar values $h(x)$, rather than just by $x$.
(ii) We also note that the limiting value is the same for any point at distance $\delta$ from $\Sigma$, relatively to the same base point $x_0 \in \Sigma$. In other words, we could have started just as well from the point $x_0 + n(x_0)\delta$.
(iii) Finally, we stress that the process is (and must be) stopped after two Euler steps.

We can visualize this process as if it is taking place on an interval of length $2\delta$ for the $h$-axis around the origin ($h = 0$), and we bounce from one end of the interval to the other. See the figure on the right.




In co-dimension 2, we attempt to generalize the above approach by working with the Euclidean distance. So, we consider a "cylinder-like" region $\mathcal{C}$ surrounding $\Sigma$ (which serves as the "axis" of the cylinder) and "radius" $\delta$, as defined by the requirement that

$$x \in \mathcal{C} \iff \|h(x)\|^2 = (h_1(x))^2 + (h_2(x))^2 = \delta^2.$$

It will be useful to better explain the structure of $\mathcal{C}$ by considering points within distance $\delta$ from a base point $x_0 \in \Sigma$. In other words, if $N(x_0) = [n_1, n_2]_{x_0}$ represent the matrix of the unit normals at $x_0 \in \Sigma$, we will have $x = x_0 + N(x_0)c(x)$, and $\|x - x_0\|^2 \le \delta^2$. Hence, all points in $\mathcal{C}$ (hence, at distance $\delta$ from $\Sigma$), of same base point (orthogonal projection) $x_0 \in \Sigma$, will belong to a section $R_\delta(x_0)$ of $\mathcal{C}$, for which we will have

$$c = \delta\big(N^\top(x_0)N(x_0)\big)^{-1}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \quad \theta \in S^1. \tag{2.9}$$

Through (2.9), we can thus bijectively map all points in $\mathcal{C}$ of same base point $x_0$ to points on the unit circle, i.e., to angles $\theta$. (Note that, in general, the neighborhood is ellipsoidal.)

**Example 2.8.** Consider Example 1.5. Here, $\Sigma$ is a plane, and the two normals are $n_1 = e_3$ and $n_2 = e_2$. From (2.9) we get $c = \delta\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$, that is a circular neighborhood. All points in $\mathcal{C}$ are distinguished by the value of the first component $x_1$, and by the angle $\theta$, and the vector fields, evaluated on $\mathcal{C}$, assume the form

$$f_1(x) = \begin{bmatrix} 2x_1 + 1 \\ -x_1 + 1 + \delta^2\cos\theta\sin\theta \\ x_1 + 1 + \delta\cos\theta \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} 2x_1 - 1 \\ -x_1 - 1 + \delta\sin\theta \\ x_1 + 2 + \delta^2\cos\theta\sin\theta \end{bmatrix},$$
$$f_3(x) = \begin{bmatrix} 2x_1 - 3 \\ -x_1 + 2 + \delta\cos\theta \\ x_1 - 1 + \delta^2\cos\theta\sin\theta \end{bmatrix}, \qquad f_4(x) = \begin{bmatrix} 2x_1 + 2 \\ -x_1 - 2 + \delta\sin\theta \\ x_1 - 2 + \delta\sin\theta \end{bmatrix}.$$

With the above in mind, we will now distinguish between two different averaging processes: (i) averaging the dynamics induced by the original vector fields $f_{1,2,3,4}$, or (ii) averaging the dynamics induced by the sub-sliding vector fields of (1.8), $f^{\pm}_{\Sigma_{1,2}}$.

*2.3.1. Averaging original dynamics*

Here we look at the dynamics of the Euler map under the original vector fields, by requiring successive iterates to remain in $\mathcal{C}$.

We generate points on $\mathcal{C}$ by the following iterative process.

**Algorithm 1.**

(i) Given a point $x^{(0)} \in \mathcal{C}$, let $x^{(0)} \in R_{i_0}$ (one of the regions $R_1, R_2, R_3, R_4$) and let $f_{i_0}$ be the corresponding vector field. Then, take a Euler step with stepsize $\tau_0$ so that the value

$$x^{(1)} = x^{(0)} + \tau_0 f_{i_0}(x^{(0)}) \tag{2.10}$$

is also in $\mathcal{C}$ (see Lemma 2.9). (In the (measure 0) eventuality that $x^{(0)}$ or one of the iterates below is on $\Sigma_1$ or $\Sigma_2$, we modify this construction by taking the Filippov sliding vector field $f^{\pm}_{\Sigma_{1,2}}$ on these co-dimension 1 surfaces.)

(ii) Repeat this process. That is, for $k = 0, 1, 2, \ldots$, let

$$x^{(k+1)} = x^{(k)} + \tau_k f_{i_k}(x^{(k)}), \quad \tau_k : \|h(x^{(k+1)})\|^2 = \delta. \tag{2.11}$$

**Lemma 2.9.** *Let the assumptions on $\Sigma$ of Table 1 hold. Then, for given $\delta > 0$, the above iteration (2.11) is well defined. That is, there exists a unique $\tau_k > 0$ in (2.11).*

**Proof.** We consider the first step, assuming that $x^{(0)}$ is not on either of $\Sigma_1$, $\Sigma_2$. The other steps, as well as the case of $x^{(0)} \in \Sigma_{1,2}$, are handled similarly. We have $\|h(x^{(0)})\|^2 = \delta^2$, and seek $\tau_0$ such that $\|h(x^{(1)})\|^2 = \delta^2$. From Taylor expansion with remainder in the Lagrange form, we have

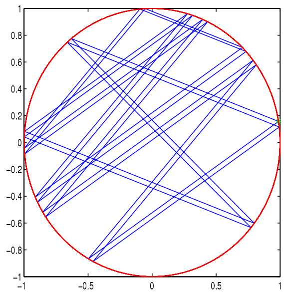
$$h(x^{(1)}) = h(x^{(0)}) + \tau_0 \nabla h^{\top}(\eta_0) f_{i_0}(\eta_0), \quad (\eta_0)_j \in \left[(x^{(0)})_j, (x^{(1)})_j\right],\ j = 1, \ldots, n.$$

Now, requiring $h(x^{(1)})^T h(x^{(1)}) = \delta^2$, gives $\tau_0 = 0$, which is unacceptable, or

$$\tau_0 = -2\delta \frac{h^{\top}(x^{(0)})\left[\nabla h^{\top}(\eta_0) f_{i_0}(\eta_0)\right]}{\|\nabla h^{\top}(\eta_0) f_{i_0}(\eta_0)\|^2},$$

which is strictly positive on account of Table 1 and of the labeling of the regions $R_1, \ldots, R_4$. □

It is insightful to visualize this iterative process as if we bounce from point to point on a circle of radius $\delta$ around the origin by taking Euler steps of appropriate stepsizes; see the figure on the right.




In order to obtain an average vector field from the above iteration, we now collect together in four different groups all stepsizes generated in (2.11) above, according to which one is the vector field for which they are being Euler steps. That is, from (2.11) we will call $\tau_k = \tau_k^{(1)}$, if $f_{i_k} = f_1$, and similarly for $\tau_k^{(2)}, \tau_k^{(3)}, \tau_k^{(4)}$, with the obvious modification required if we are using one of the $f^{\pm}_{\Sigma_{1,2}}$. It must be appreciated that the values of the $\tau_k$'s depend on $\delta$.

Suppose[2] that the trajectory generated by $x^{(0)}$ is periodic in the angle $\theta$; that is, suppose that we generate iterates whose associated angles satisfy $\theta(x^{(0)}), \theta(x^{(1)}), \ldots, \theta(x^{(N_0)}) = \theta(x^{(0)})$, and note that $N_0$ itself generally may depend on $\delta$. Under this situation, it is reasonable to consider the following quantity:

$$\lambda^i_{\text{ave}}(x^{(0)}, \delta) := \frac{\sum_{k=0}^{N_0-1} \tau_k^{(i)}}{\sum_{k=0}^{N_0-1} \tau_k}, \quad i = 1, 2, 3, 4. \tag{2.12}$$

[2] We conjecture that, for fixed $\delta > 0$, and constant vector fields, this supposition is correct, but lack a complete proof of this fact; based on what follows, we lack motivation to embark in such possible proof.

Note that this would give an admissible solution. But, as we said, we need the orbits to be periodic. Moreover, we must demand that (2.12) has a limit as $\delta \to 0$, a property which is not clear at all if it is true. In fact, both periodicity and existence of the limit are quite hard to prove in general and/or to verify in a practical problem. Furthermore, as we see in Example 2.10, even if the orbit is periodic and the limit exists, in general the value of points in $\mathcal{C}$ with same projection $x_0 \in \Sigma$ differs. As a consequence, this averaging technique turns out to be unsatisfactory as a way to define a Filippov sliding vector field. We say this because an obvious requirement of this way of proceeding must be that the limiting values of $\lambda_{\text{ave}}(x^{(0)}, \delta)$ be the same for all $x^{(0)} \in R_\delta(x_0)$.

**Example 2.10.** Consider Example 1.5, with $x_1 = 0.5$ there; so, we let $x_0 = (0.5, 0, 0) \in \Sigma$. We take two different points in $R_\delta(x_0)$, namely (see Example 2.8) corresponding to: (a) $\theta = \texttt{eps}$, and (b) $\theta = 0.7815$ (here, eps is the machine precision, and $\texttt{eps} \approx 2.2204e-016$). In these cases, the generated orbits are periodic and for $\lambda_{\text{ave}}$ given in (2.12) the limiting values as $\delta \to 0$ exist and give:

$$\text{(a)} \begin{bmatrix} 0 \\ 0.2333 \\ 0.5667 \\ 0.2000 \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} 0.3889 \\ 0 \\ 0.3333 \\ 0.2778 \end{bmatrix},$$

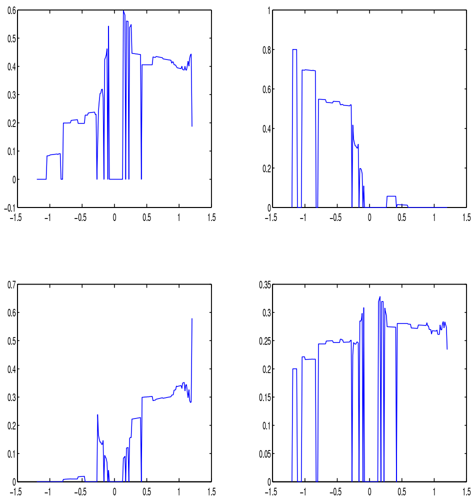
with average periods of 95.2704 and 96.2323 respectively.

To move out of the impasse above, we also considered a second averaging process, over the angle $\theta$, for all points with same base point on $\Sigma$. That is, calling $x(\theta)$ the points in $\mathcal{C}$ with same base point $x_0$, and subject to the same limitations previously mentioned on the proper definition of $\lambda_{\text{ave}}(x(\theta))$, we considered the following quantity,

$$\lambda_{\text{ave}}(x_0) := \frac{1}{2\pi}\int_0^{2\pi} \lambda_{\text{ave}}(x(\theta))\, d\theta, \tag{2.13}$$

which – as long as it is well defined – is surely giving an admissible solution, identical for all points in $\mathcal{C}$ with same base point $x_0$. Alas, even when well defined, the above turns out to be unsatisfactory.

**Example 2.11.** Let us refer again to Example 1.5, with $x_1 = -0.9$.



In this case we obtain $f_{\text{ave}} = \begin{bmatrix} -1.7667 \\ 0 \\ 0 \end{bmatrix}$, and $\lambda_{\text{ave}} = \begin{bmatrix} 0.1667 \\ 0.6000 \\ 0 \\ 0.2333 \end{bmatrix}$, which is surely admissible. But, as the figure on the right exemplifies, this $\lambda_{\text{ave}}$ solution is clearly not differentiable in $x_1$, despite $W$ being analytic in it. As a consequence, this possible way to interpret how to select a Filippov sliding vector field does not appear to be a viable choice.

*2.3.2. Averaging sub-sliding dynamics*

In the nodally attractive case considered in this work, we can take also an alternative point of view in order to build an average sliding vector field. As before, we consider the 2-norm to define the cylinder $\mathcal{C}$ around $\Sigma$, of radius $\delta$.

The point of the construction below is to realize that – because of nodal attractivity – a trajectory of the dynamical system (1.1) starting at a point in $\mathcal{C}$ will typically *hit* one of the sub-sliding surfaces $\Sigma_{1,2}^{\pm}$ before reaching $\Sigma$ itself. This allows us to effectively reduce the dimensionality of the averaging process, by looking at the points in $\mathcal{C}$ which end up first on one of $\Sigma_{1,2}^{\pm}$. At that point, the averaging process will be the same as we had in co-dimension 1.

Recalling (1.8)–(1.9), we will look for a sliding vector field on $\Sigma$ of the following form

$$f := c_1^+ f_{\Sigma_1^+} + c_1^- f_{\Sigma_1^-} + c_2^+ f_{\Sigma_2^+} + c_2^- f_{\Sigma_2^-}. \tag{2.14}$$

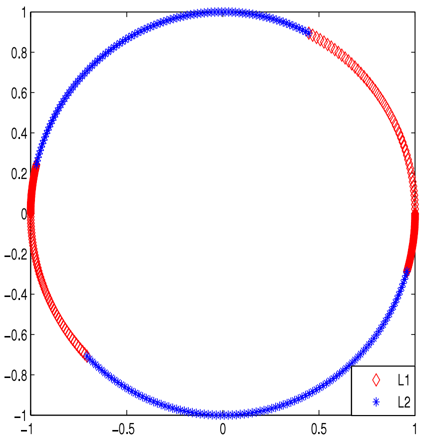
To understand how to select the coefficients $c_{1,2}^{\pm}$, we reason as follows.

Let $x_0 \in \Sigma$ be given, and consider the $\delta$-section $R_\delta(x_0)$ in $\mathcal{C}$, defined as before; see (2.9). For fixed value of $\delta$, consider the Euler segments starting at a point $x^{(0)} \in R_\delta(x_0)$, defined so to remain in $\mathcal{C}$, but monitoring the first time that any such segment crosses one of the $\Sigma_{1,2}^{\pm}$. In other words, we define (see (2.10)) $x^{(1)}(\tau) = x^{(0)} + \tau f_{i_0}(x^{(0)})$, $\tau \le \tau_0$; if this segment reaches $\mathcal{C}$ without first having crossed one of the $\Sigma^{\pm}_{1,2}$, then we take $\tau = \tau_0$ as in (2.10), and continue by taking Euler segments (see (2.11)) to generate $x^{(k+1)}(\tau) = x^{(k)} + \tau f_{i_k}(x^{(k)})$, $\tau \leq \tau_k$, until the first time one of these segments crosses one of the $\Sigma^{\pm}_{1,2}$. (The probability 0 eventuality that one of these segments first reaches $\Sigma$ directly is presently ignored, and see Remarks 2.12-(i).) It is quite easy to see that, because of nodal attractivity, for any starting point in $R_\delta(x_0)$ there is a first Euler segment crossing one of $\Sigma^{\pm}_{1,2}$. We stress that this process generally depends on $\delta$.

By doing what described above, and recalling the form of $R_\delta(x_0)$, we effectively obtain a partition of $S^1$, that is of $[0, 2\pi]$, into arcs: an angle from each of these arcs is associated to whichever sub-surface $\Sigma^{\pm}_{1,2}$ is reached first by the Euler segments starting from that angle in $R_\delta(x_0)$. So, for given $\delta$, we will have four arc-lengths, which we call $\theta^{\pm}_{1,2}$; e.g., $\theta^{+}_1$ is the length of the arc of $S^1$ whose associated points have a Euler segment first reaching $\Sigma^{+}_1$, etc. Again, let us stress that these $\theta^{\pm}_{1,2}$ generally depend on $\delta$.

Now, as soon as one of the sub-surfaces $\Sigma^{\pm}_{1,2}$ is reached by a Euler segment, we reduce the dimensionality of the process and go back to the case of co-dimension 1. For example, suppose that for a certain angle $\theta$, the Euler iterates starting with $x^{(0)} \in R_\delta(x_0)$ reach $\Sigma^{+}_1$ first; then, we restrict consideration to the co-dimension 1 surface $\Sigma_1$, with Filippov vector fields given by $f^{+}_{\Sigma_1}$ and $f^{-}_{\Sigma_1}$ in (1.8)–(1.9); but, in co-dimension 1 the averaging process is well understood, and in this case it will give a Filippov sliding vector field at $x_0 \in \Sigma$. With this, we will now have (all quantities below generally depend on $\delta$)

$$f_{\Sigma_1} := (1 - a_1)f_{\Sigma_1^+} + a_1 f_{\Sigma_1^-}, \qquad f_{\Sigma_2} := (1 - a_2)f_{\Sigma_2^+} + a_2 f_{\Sigma_2^-},$$
$$a_1 := \frac{n_2^\top f_{\Sigma_1^+}}{n_2^\top (f_{\Sigma_1^+} - f_{\Sigma_1^-})}, \qquad a_2 := \frac{n_1^\top f_{\Sigma_2^+}}{n_1^\top (f_{\Sigma_2^+} - f_{\Sigma_2^-})}. \tag{2.15}$$




Next, we compute the following ratios, defining the percentage of points in $R_\delta(x_0)$ contributing to $f_{\Sigma_1}$, respectively to $f_{\Sigma_2}$, see the figure on the right. We make the dependence on $\delta$ explicit:

$$L_1(\delta) := \frac{\theta_1^+(\delta) + \theta_1^-(\delta)}{2\pi}, \qquad L_2(\delta) := \frac{\theta_2^+(\delta) + \theta_2^-(\delta)}{2\pi}. \tag{2.16}$$

Finally, we let $\delta \to 0$, and propose taking

$$L_1 = \lim_{\delta \to 0} L_1(\delta), \qquad L_2 = \lim_{\delta \to 0} L_2(\delta), \tag{2.17}$$

and from this the overall sliding vector field at $x_0 \in \Sigma$ as

$$f_{\text{mean}} = L_1 f_{\Sigma_1} + L_2 f_{\Sigma_2}.$$

With this rewriting, the coefficients $c^{\pm}_{1,2}$ in (2.14) are:

$$c_1^+ := L_1(1 - a_1), \qquad c_1^- := L_1 a_1, \qquad c_2^+ := L_2(1 - a_2), \qquad c_2^- := L_2 a_2. \tag{2.18}$$

Therefore, by making definition (2.14) explicit in terms of the $f_i$'s, this "average" solution of (1.11) is

$$\lambda_{\text{mean}} := \begin{bmatrix} (1-\alpha^-)c_1^- + (1-\beta^-)c_2^- \\ (1-\alpha^+)c_1^+ + \beta^- c_2^- \\ \alpha^- c_1^- + (1-\beta^+)c_2^+ \\ \alpha^+ c_1^+ + \beta^+ c_2^+ \end{bmatrix}. \tag{2.19}$$

**Remarks 2.12.**

(i) The case in which a Euler segment crosses $\Sigma$ directly, ahead of crossing either (but not both) $\Sigma_1$ or $\Sigma_2$, is not a concern in defining the values in (2.16), and then (2.17), because, for each given $\delta$, there are just four angles giving this eventuality. Hence, they do not contribute to the arc lengths we used.

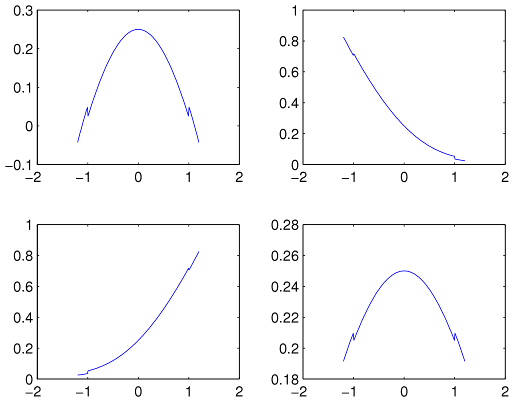
(ii) The limit in (2.17) as $\delta \to 0$ exists as a consequence of the fact that (for any $i = 1, 2, 3, 4$) $\|f_i(x) - f_i(x_0)\|$ is arbitrarily small for $x \in R_\delta(x_0)$.

(iii) In principle, it is possible to attempt averaging for neighborhoods of $\Sigma$ defined by norms other than the 2-norm we used. We made some (limited) experiments also with the $\infty$-norm and the 1-norm, and our results were qualitatively similar to those we reported for the 2-norm.

**Example 2.13.** Let us consider again Example 1.5, with $x_1 = -0.9$.

In this case we obtain $\begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} 0.1992 \\ 0.0383 \\ 0.2179 \\ 0.5446 \end{bmatrix}$, and $\lambda_{\text{mean}} = \begin{bmatrix} 0.0636 \\ 0.6618 \\ 0.0618 \\ 0.2127 \end{bmatrix}$

and $f_{\text{mean}} = \begin{bmatrix} -2.1582 \\ 0 \\ 0 \end{bmatrix}$, whereas a plot of all components of $\lambda_{\text{mean}}$ in function of $x_1$ is given on the right.



As the above figure makes clear, the components vary smoothly as long as the nodal attractivity assumptions hold; i.e., $x_1 \in (-1, 1)$. But, they do not extend nicely outside of this interval, a fact which appears to limit this averaging process and the construction of $\lambda_{\text{mean}}$ to purely nodally attractive configurations.

## 3. Geometric methods

Here we look at techniques which can be naturally framed within the context of rebuilding polygons in the plane, and finding a representation (i.e., coordinates) for points internal to the polygon in terms of convex combination of the vertices. As it turns out, these are the most interesting techniques.

So, the idea is to think of the values $w_j^i$, $i = 1, 2, j = 1, 2, 3, 4$, in (1.7) as giving the four different points $w_j = (w_j^1, w_j^2)$, $j = 1, 2, 3, 4$, then consider the polygon made up by joining the vertices in the following order

$$\Pi := w_1 w_2 w_4 w_3.$$

Given our assumptions on the $w_j^i$'s, it is easy to realize that the origin is inside the polygon. Thus, our task is to find the *coordinates* of the origin with respect to the given vertices.

Although not derived from this interpretation, the technique in [8,2] belongs to this class of methods. The appropriate framework within which to interpret these techniques, and to derive another very promising one, turns out to be that of *barycentric coordinates*, widely used in computer graphics.

**Definition 3.1** (*Barycentric Coordinates*)**.** Let $\Omega$ be a closed convex polygon in the plane, with vertices $w_1, \ldots, w_n$, $n \geq 3$, and let $z \in \Omega$. The functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, are called *barycentric coordinates* for $z$, if they satisfy the three properties of positivity, convexity, and interpolation:

$$\text{(a) } \lambda_i(z) \geq 0, \quad i = 1, \ldots, n, \qquad \text{(b) } \sum_{i=1}^{n} \lambda_i(z) = 1, \qquad \text{(c) } \sum_{i=1}^{n} \lambda_i(z) w_i = z. \tag{3.1}$$

In the special case of $n = 3$, barycentric coordinates are unique and are called triangular coordinates. For $n \geq 4$, there is no unique choice of barycentric coordinates. In the context of interest to us, we have $n = 4$, $z = 0$, and we seek $\lambda_i(0)$ to be smoothly varying functions of the vertices $w_1, \ldots, w_4$.

Even though barycentric coordinates are not unique for $n \geq 4$, they share some general properties that follow from the three defining axioms (3.1). In particular, they satisfy the Lagrange property $\lambda_i(w_j) = \delta_{ij}$, and they are linear along each edge of $\Omega$. To see this, observe that the axioms (3.1) imply linear precision, i.e. for any linear function $f$ one has $\sum_{i=1}^{n} \lambda_i(z) f(w_i) = f(z)$.

Below, we will look at three instances of quadrilateral barycentric coordinates of the origin relatively to the polygon of vertices $w_1, w_2, w_4, w_3$ (in this order). Note that, under nodal attractivity assumption, the origin is inside the polygon.

### *3.1. Bilinear interpolation*

An important choice of barycentric coordinates is based upon bilinear interpolation. In this case, one seeks $\lambda$ in (3.1) of the form:

$$\lambda = \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix}, \quad \alpha, \beta \in [0, 1]. \tag{3.2}$$

We will call $\lambda_B$ the choice above. In our context, this choice was first proposed in [9], and then thoroughly investigated and justified in [2], where it was proven to give a smoothly varying solution $\lambda$ so that the Filippov sliding vector field in (1.10) is well defined. (The results in [2] validate this choice under more general attractivity assumptions than just nodal attractivity.)

Quite clearly, the structure (3.2) derives from the convexity requirement on the solution components,

$$(\lambda_1+\lambda_2)+(\lambda_3+\lambda_4) = (1-\alpha)+\alpha = (1-\alpha)(1-\beta)+(1-\alpha)\beta+\alpha(1-\gamma)+\alpha\gamma,$$

where $\alpha, \beta, \gamma \in [0, 1]$, and then $\lambda_B$ is obtained by selecting $\gamma = \beta$. This choice can be understood as a (nonlinear) regularization of the system (1.11), as below.

**Definition 3.2.** A vector $\lambda \in \mathbb{R}^4$ is said to satisfy the B-condition if $\lambda_1\lambda_4 = \lambda_2\lambda_3$. Equivalently, letting $R := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, one has $\lambda^\top R\lambda = 0$.

**Lemma 3.3.** *A solution $\lambda$ of (1.11) is $\lambda_B$ if and only if it satisfies the B condition.*

**Proof.** It is straightforward from the construction that $\lambda_B$ satisfies the B condition. Now, suppose $\lambda$ verifies the B condition. Then, let us define

$$\alpha := \frac{\lambda_3}{\lambda_1+\lambda_3}, \qquad \beta := \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

A trivial computation gives

$$(1-\alpha)(1-\beta) = \frac{\lambda_1}{\lambda_1+\lambda_3}\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{(\lambda_1)^2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)} = \lambda_1,$$

and similarly for the other components. □

This $\lambda_B$ can be also obtained by appropriate choices of $c$ in (1.12), and as a mean field solution associated to a special value of $\alpha$ in the pdf (2.3).

**Theorem 3.4.** *Consider the form (1.12), $\lambda = \mu + cv$, where $\mu$ is any particular solution of (1.11), $v$ spans* ker $W$, *and $c \in [a, b]$ (admissibility interval). Then, the bilinear interpolant solution $\lambda_B$ is obtained with $c = -\frac{R^\top \mu R}{R^\top v R}$, and it is the mean-field solution associated to the pdf (2.3) with $\alpha = \gamma/(1-\gamma)$, $\gamma := -\frac{1}{b-a}\left(a + \frac{R^\top \mu R}{R^\top v R}\right)$.*

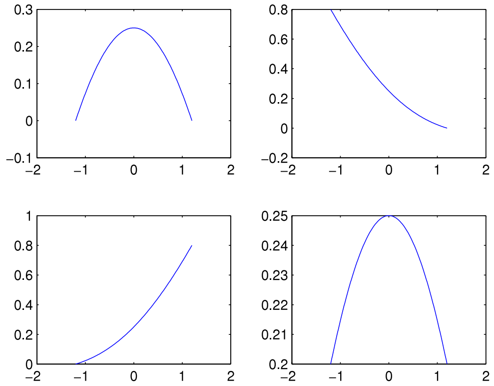
**Proof.** Since $\lambda_B$ solves (1.11), then $R^\top \lambda_B R = 0$. Therefore, the value of $c$ in (1.12) is $c = -\frac{R^\top \mu R}{R^\top v R}$.

From (2.4) and the above, we must then have $(1-\gamma)a + \gamma b = -\frac{R^\top \mu R}{R^\top v R}$. □

**Example 3.5.** In Example 1.5 with $x_1 = -0.9$ we have

$$\lambda_B = \begin{bmatrix} 0.1056 \\ 0.6367 \\ 0.0367 \\ 0.2211 \end{bmatrix} \quad \text{and} \quad f_B = \begin{bmatrix} -1.9989 \\ 0 \\ 0 \end{bmatrix},$$

whereas a plot of $\lambda_B$ as a function of $x_1$ is shown on the right. Note that two of the components of $\lambda_B$ vanish at $\pm 1.2$ (see Example 1.5).




*3.2. Moments solution: mean value coordinates*

Another, less transparent, instance of barycentric coordinates is obtained upon selecting the $\lambda_i$'s in such a way that the total moment of $w_1, w_3$ equals the total moment of $w_2, w_4$, all taken with respect to the origin. More precisely, we regularize (1.11) by adding to it the following condition:

$$d_1\lambda_1 - d_2\lambda_2 - d_3\lambda_3 + d_4\lambda_4 = 0, \quad \text{where } d_i := \sqrt{(w_i^1)^2 + (w_i^2)^2},\ i = 1, \dots, 4.$$

So, we are looking for a solution of the system

$$\begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ d_1 & -d_2 & -d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \widehat{W} \\ d^\top \\ e^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.3}$$

Below, we will show that there is always a unique solution of (3.3), as smooth as $\widehat{W}$. We will call this solution the *moments solution* and label it as $\lambda_m$. As far as we know, this choice of selecting $\lambda$ in (1.10) is new.

First, we have the following lemma.

**Lemma 3.6.** *Under the nodal attractivity assumptions of Table 1, the matrix* $\begin{bmatrix} \widehat{W} \\ d^\top \end{bmatrix}$ *has full rank* 3*, and thus its kernel is* 1*-dimensional.*

**Proof.** The sign pattern of the above matrix is

$$\begin{bmatrix} + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}.$$

Then, we claim that any linear combination with coefficients $a_1, a_2$ of the first and second rows cannot match the third row. Obviously, the claim is correct if either of $a_1$ or $a_2$ is 0. Now, if $a_1, a_2 > 0$, then $d_4$ cannot be obtained; if $a_1 > 0, a_2 < 0$, then $d_2$ cannot be obtained; if $a_1 < 0, a_2 > 0$, then it is $d_3$ that cannot be obtained, and if $a_1, a_2 < 0$, then $d_1$ cannot be obtained. □

To prove that (3.3) gives an admissible solution, it is very convenient to establish the equivalence of (3.3) to the so-called mean value coordinates introduced by Floater; see [10].

**Definition 3.7** (*Mean Value Coordinates*)**.** Let $\Omega$ be a planar polygon of vertices $w_1, \ldots, w_n$. For $x \in \Omega$, let

$$\lambda_i(x) := \frac{\nu_i(x)}{\sum_{j=1}^{n} \nu_j(x)}, \qquad \nu_i(x) := \frac{\tan\left(\frac{\alpha_{i-1}(x)}{2}\right) + \tan\left(\frac{\alpha_i(x)}{2}\right)}{\|w_i - x\|}, \tag{3.4}$$

and $\alpha_i(x)$ be the angle at $x$ in the triangle $[x, w_i, w_{i+1}]$. Then, the $\lambda_i(x)$ are called mean value coordinates of $x$.

We refer to the cited work of Floater, [10], for a proof that mean value coordinates are well defined for points inside our polygon. Here, we show that they are equivalent to the moments solution in our context, where we have the polygon of vertices $w_1, w_2, w_4$ and $w_3$, and seek mean value coordinates of the origin.

**Lemma 3.8.** *The mean value coordinates satisfy* (3.3)*.*

**Proof.** We already know that the mean value coordinates verify (1.11), so we are left to prove that they fulfill the third equation of (3.3). But this follows immediately from (3.4), since

$$\begin{aligned} d_1\lambda_1 - d_2\lambda_2 - d_3\lambda_3 + d_4\lambda_4 = {} & \tan\left(\frac{\alpha_2}{2}\right) + \tan\left(\frac{\alpha_1}{2}\right) - \left(\tan\left(\frac{\alpha_4}{2}\right) + \tan\left(\frac{\alpha_2}{2}\right)\right) - \left(\tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{\alpha_3}{2}\right)\right) \\ & + \left(\tan\left(\frac{\alpha_3}{2}\right) + \tan\left(\frac{\alpha_4}{2}\right)\right) = 0. \quad \square \end{aligned}$$

Finally, we have the following.

**Theorem 3.9.** *The mean value coordinates* (3.4) *are the unique solutions of* (3.3)*. In particular,* (3.3) *is a nonsingular system.*

**Proof.** From Lemma 3.8, we know that the mean value coordinates vector $\lambda_m$ is a solution of (3.3), with positive components, and – in particular – it is a nontrivial solution of $\begin{bmatrix} \widehat{W} \\ d^\top \end{bmatrix} \lambda = 0$. Hence, see Lemma 3.6, $\lambda_m$ spans the kernel of $\begin{bmatrix} \widehat{W} \\ d^\top \end{bmatrix}$. Since any solution $\mu$ of (3.3) must satisfy $\mu \in \ker\left(\begin{bmatrix} \widehat{W} \\ d^\top \end{bmatrix}\right)$, and $e^\top \mu = 1$, then (3.3) has the unique solution $\lambda_m$. □

**Remarks 3.10.**

(i) An important consequence of the above is that $\lambda_m$ is as smooth as $W$. In fact, $\lambda_m$ is a solution of (3.3), which – on account of Theorem 3.9 – is an invertible linear system, and so its solution is as smooth as the coefficients, that is as $W$. See also Example 3.12.

(ii) In light of the above equivalence, we favor implementing the moments method as we proposed in this work, that is solving (3.3), rather than by forming (3.4). Indeed, in the present context, solving (3.3) is much simpler.

The following result summarizes the relation between the moments solution, the general form of admissible solution in (1.12), and the mean field solution associated to a special value of $\alpha$ in the pdf (2.3).

**Theorem 3.11.** *Consider the form (1.12), $\lambda = \mu + cv$, where $\mu$ is any particular solution of (1.11), $v$ spans* ker $W$, *and $c \in [a, b]$ (admissibility interval). Then, the moments solution $\lambda_m$ is obtained with $c = -\frac{d^\top \mu}{d^\top v}$, where $d := \begin{bmatrix} d_1 \\ -d_2 \\ -d_3 \\ d_4 \end{bmatrix}$, in (1.12), and is the mean-field solution associated to the pdf (2.3) with $\alpha = \gamma/(1-\gamma)$, $\gamma := -\frac{1}{b-a}\left(a + \frac{d^\top \mu}{d^\top v}\right)$.*

**Proof.** Since $\lambda_m$ is a solution of (1.11), then $d^\top \lambda_m = 0$. Therefore, the value of $c$ in (1.12) is $-\frac{d^\top \mu}{d^\top v}$, as stated.

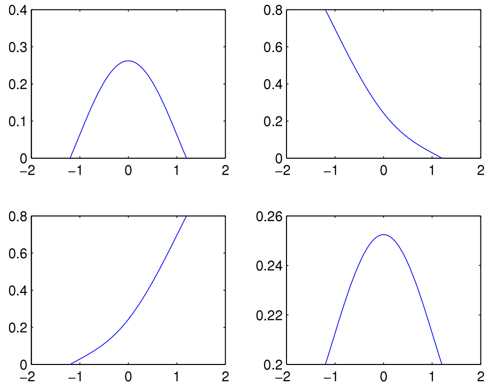
From (2.4) and the above, we must then have $(1-\gamma)a + \gamma b = -\frac{d^\top \mu}{d^\top v}$, from which the result follows. □

**Example 3.12.** Let us consider Example 1.5, with $x_1 = -0.9$.

In this case we get

$$\lambda_m = \begin{bmatrix} 0.0949 \\ 0.6431 \\ 0.0431 \\ 0.2190 \end{bmatrix}, \qquad f_m = \begin{bmatrix} -2.0395 \\ 0 \\ 0 \end{bmatrix},$$

whereas a plot of $\lambda_m$ in function of $x_1$ is shown on the right. Note that two of the components of $\lambda_m$ vanish at $\pm 1.2$ (see Example 1.5).



### 3.3. Wachspress solution

Another choice of planar barycentric coordinates is due to Wachspress (see [10,11]). Rephrased in our context, this gives an admissible value of $\lambda$ in (1.11), which we will call $\lambda_W$, defined by the requirement (see Fig. 1):

$$\lambda_i := \frac{\mu_i}{\sum_{i=1}^{4} \mu_i}, \qquad \mu_1 := \frac{\cot \gamma_3 + \cot \beta_1}{d_1^2}, \text{ etc.} \tag{3.5}$$

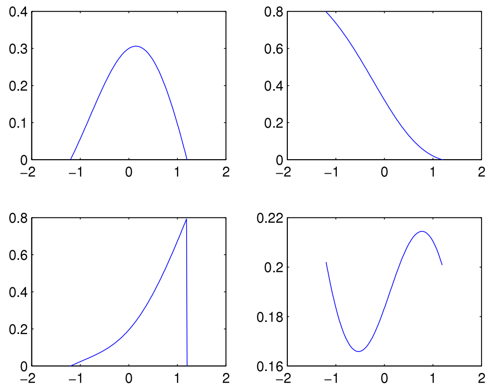
We refer to the original derivation of Wachspress, [11], for a justification of this choice.

**Example 3.13.** Let us consider Example 1.5, with $x_1 = -0.9$.

In this case we get

$$\lambda_W = \begin{bmatrix} 0.0832 \\ 0.6483 \\ 0.0506 \\ 0.2180 \end{bmatrix}, \qquad f_W = \begin{bmatrix} -2.0833 \\ 0 \\ 0 \end{bmatrix},$$

whereas a plot of $\lambda_W$ in function of $x_1$ is shown on the right.



We note that Wachspress coordinates extend smoothly beyond the nodal attractivity interval $(-1, 1)$, but the plot of the third component betrays that Wachspress coordinates are not well defined when the origin belongs to a side of the polygon, a fact already remarked by Floater in [10]. This fact makes $\lambda_W$ less appealing than $\lambda_B$ and $\lambda_m$ beyond the case of nodally attractive $\Sigma$.

### 3.4. Another geometric solution

A final choice of geometric coordinates is the one based on the construction adopted in [12]. This choice does not generally give a Filippov solution (that is, it does not select a value of $\lambda$ in (1.11)), but still selects a value of $\lambda$ giving a smoothly varying vector field on $\Sigma$. The difference with respect to the standard Filippov choice is that one first projects the vector fields onto

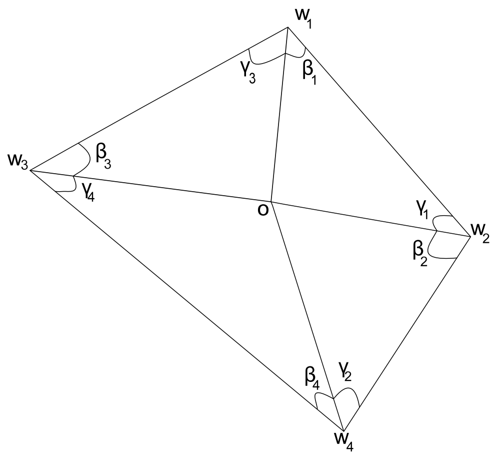



**Fig. 1.** Figure for the definition of the Wachspress solution.

the tangent plane at $x_0 \in \Sigma$, then seeks a convex combination of the same. In our notation, calling $\lambda_P$ the resulting values of these convex coefficients, one proceeds as follows.

One seeks a sliding vector field (not necessarily of Filippov type) of the form

$$f_P := \sum_{i=1}^{4} \lambda_i v_i, \quad v_i = f_i - N(N^\top N)^{-1} w_i, \; N = \begin{bmatrix} \nabla h_1 & \nabla h_2 \end{bmatrix}.$$

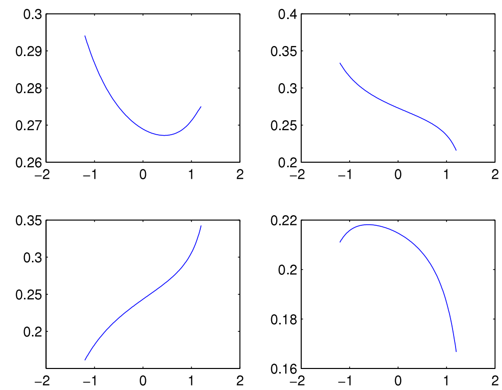
In its simplest form, in [12], selection of $\lambda$ was done as follows:

$$\lambda_i = \frac{\mu_i}{\sum_j \mu_j}, \quad \text{where } \mu_i = \frac{\prod_{j \neq i} a_j^\top w_j}{\prod_{j \neq i} a_j^\top w_j - a_i^\top w_i}, \quad i = 1, \ldots, 4$$

$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad a_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

**Example 3.14.** Let us consider again Example 1.5, with $x_1 = -0.9$.

In this case we get $\lambda_P = \begin{bmatrix} 0.2789 \\ 0.2940 \\ 0.2021 \\ 0.2250 \end{bmatrix}$, and $f_P = \begin{bmatrix} -1.9713 \\ 0 \\ 0 \end{bmatrix}$, whereas a plot of $\lambda_P$ in function of $x_1$ is shown on the right.




From the above plot, we note that these coordinates extend smoothly beyond the nodal attractivity interval $(-1, 1)$. However, note that none of the components of $\lambda_P$ is 0 at $\pm 1.2$ (see Example 1.5). So, although this choice does not generally give a Filippov sliding vector field, it may be of some (limited) interest in the nodally attractive case.

## 4. Nodal attractivity and stochastic basis

In this final section, we adopt the rewriting of a Filippov vector field in terms of the sub-sliding vector fields (cf. (2.14)). Indeed, we can rewrite $\lambda$ as:

$$\lambda = Mq, \quad \text{where } q := \begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix}, \quad \text{and} \quad M := \begin{bmatrix} 0 & 1-\alpha^- & 0 & 1-\beta^- \\ 1-\alpha^+ & 0 & 0 & \beta^- \\ 0 & \alpha^- & 1-\beta^+ & 0 \\ \alpha^+ & 0 & \beta^+ & 0 \end{bmatrix}.$$

Observe that $M$ is column stochastic, hence we may call any $\lambda$ derived from this form a *stochastic subsliding solution*.

This implies that we can obtain a solution of (1.11) by solving the following problem:

$$B\begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{s.t. } \lambda = M\begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} > 0, \tag{4.1}$$

where $B := WM$. Moreover, letting for $i, j = 1, 2, i < j$, $D_{ij} := \det\begin{bmatrix} w_i & w_j \end{bmatrix}$, then $B$ can be written as

$$B := \begin{bmatrix} 0 & 0 & -b_{13} & b_{14} \\ -b_{21} & b_{22} & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

where

$$b_{13} := \frac{D_{34}}{w_3^2 - w_4^2}, \qquad b_{14} := -\frac{D_{12}}{w_1^2 - w_2^2}, \qquad b_{21} := -\frac{D_{24}}{w_2^1 - w_4^1}, \qquad b_{22} := \frac{D_{13}}{w_1^1 - w_3^1}.$$

Under the nodal attractivity assumption, Table 1 assures that these $b_{ij}$'s are positive, so that the sign pattern of $B$ results $\begin{bmatrix} 0 & 0 & - & + \\ - & + & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and obviously $\text{rank}(B) = 3$. So, from (4.1) we have

$$c_1^+ = xb_{22}, \qquad c_1^- = xb_{21}, \qquad c_2^+ = yb_{14}, \qquad c_2^- = yb_{13},$$

for some $x$ and $y$ such that

$$(b_{13} + b_{14})y + (b_{21} + b_{22})x = 1,$$

and thus, for some $\gamma \in \mathbb{R}$, $y = \frac{1-\gamma}{b_{13}+b_{14}}$, $x = \frac{\gamma}{b_{21}+b_{22}}$.

In particular, we can write every solution of (4.1) as

$$q = \begin{bmatrix} 0 \\ 0 \\ \dfrac{b_{14}}{b_{13}+b_{14}} \\ \dfrac{b_{13}}{b_{13}+b_{14}} \end{bmatrix} + \gamma \begin{bmatrix} \dfrac{b_{22}}{b_{21}+b_{22}} \\ \dfrac{b_{21}}{b_{21}+b_{22}} \\ -\dfrac{b_{14}}{b_{13}+b_{14}} \\ -\dfrac{b_{13}}{b_{13}+b_{14}} \end{bmatrix} = (1-\gamma)\begin{bmatrix} 0 \\ 0 \\ \dfrac{b_{14}}{b_{13}+b_{14}} \\ \dfrac{b_{13}}{b_{13}+b_{14}} \end{bmatrix} + \gamma \begin{bmatrix} \dfrac{b_{22}}{b_{21}+b_{22}} \\ \dfrac{b_{21}}{b_{21}+b_{22}} \\ 0 \\ 0 \end{bmatrix}. \tag{4.2}$$

Set

$$s_1 := \begin{bmatrix} 0 \\ 0 \\ \dfrac{b_{14}}{b_{13}+b_{14}} \\ \dfrac{b_{13}}{b_{13}+b_{14}} \end{bmatrix}, \qquad s_2 := \begin{bmatrix} \dfrac{b_{22}}{b_{21}+b_{22}} \\ \dfrac{b_{21}}{b_{21}+b_{22}} \\ 0 \\ 0 \end{bmatrix},$$

then (4.2) rewrites as

$$q = (1-\gamma)s_1 + \gamma s_2. \tag{4.3}$$

Now, let us determine the largest admissibility interval for $\gamma$. From (4.3), we have

$$Mq = Ms_1 + \gamma M(s_2 - s_1). \tag{4.4}$$

But, both $Ms_1$ and $Ms_2$ are admissible solutions of (1.11), and so $M(s_2 - s_1)$ belongs to $\ker W$. Therefore, we can use (1.12) with

$$\mu := Ms_1,$$
$$v := M(s_2 - s_1).$$

From this, we can find the admissibility interval for $c$: $\lambda = \mu + cv$, call it $(a_S, b_S)$, see (1.13). Hence, from (4.4) we get that $\gamma \in (a_S, b_S)$ if and only if $q$ as in (4.3) provides a strictly positive solution $Mq$ of (1.11).

**Example 4.1.** Consider again Example 1.5, with $x_1 = -0.9$. We have $(a_S, b_S) = (-0.3039\ldots, 1.1144\ldots)$ and the values of $\gamma$ giving all the solutions we have derived so far are:

$$\gamma_{\mathrm{B}} = 0.5944,$$
$$\gamma_{\mathrm{MF}} = 0.4052,$$
$$\gamma_{\min} = 1.8235,$$
$$\gamma_m = 0.5034,$$
$$\gamma_{\mathrm{mean}} = 0.2375,$$
$$\gamma_{\mathrm{W}} = 0.4052,$$
$$\gamma_{\mathrm{P}} = 1.4690.$$

Note that $\gamma_{\mathrm{mean}} = L_1$ in (2.16). Also, note that $\gamma_{\min}$ and $\gamma_{\mathrm{P}}$ produce values outside of the admissibility interval, betraying that the corresponding approaches either produce Filippov solutions which are not admissible (namely, $\lambda_{\min}$), or do not produce Filippov solutions (namely, $\lambda_{\mathrm{P}}$).

## 5. Conclusions

In this paper we considered several possibilities on how to define a Filippov sliding vector field on a co-dimension 2 singularity surface $\Sigma$, intersection on two co-dimension 1 surfaces. As underlying assumption, we considered the case of nodally attractive $\Sigma$.

We broadly classified the various possibilities into two groups: algebraic/analytic and geometric. In the first group, we considered three possible ways to define a Filippov vector field: a mean-field formulation, two approaches based on minimizing the 2-norm, and two different averaging techniques. The mean-field approaches depend on the underlying probability density function (pdf), and produce a smoothly varying vector field on $\Sigma$ for an appropriate pdf. The minimization techniques we considered, in general (even if well defined) fail to produce a smoothly varying Filippov sliding vector field. The two averaging techniques we considered behave very differently: (i) averaging the original dynamics appears to have serious difficulties of convergence and smoothness, (ii) averaging the sub-sliding vector fields, instead, delivers a well defined selection; however, this specific interpretation appears to be limited to the case of nodally attractive $\Sigma$.

The geometric approaches we considered are a generally viable mean to select a Filippov sliding vector field. In particular, the techniques which can be cast in the framework of "barycentric coordinates" methods deliver a uniquely defined and smoothly varying vector field on a nodally attractive $\Sigma$. Specifically, we reinterpreted a method based on bilinear interpolation, introduced one which we called moments method, and reviewed the Wachspress method. Finally, we also revisited a method introduced in [12]. Of all of these, the bilinear interpolant and the moments method appear to be the most appropriate choices. The bilinear interpolant method has been extensively analyzed in recent works (e.g., see [2,8]), under general (not only nodal) attractivity assumptions on $\Sigma$. The moments method, instead, appear to be new in the present context (i.e., to define a Filippov sliding vector field); we further proved that this method is equivalent to the so-called mean value coordinates with which name has been used successfully in the last 10 years in the computer graphics community (see [10,13]). From the computational point of view, the expense associated with forming the moments and bilinear solution is much the same: the bulk of it is forming the values $w_j^i$'s, which is required for both methods; then, for the moments solution, we need to solve the linear system (3.3), whereas for the bilinear solution we need to solve a quadratic equation.

In future work, we anticipate considering the extension of the moments' method to the case of generally attractive $\Sigma$ (not just nodally attractive), and we will also attempt interpreting the vector field resulting from the moments method in terms of the dynamics of a suitably regularized problem (similarly to what is done in [14] for the bilinear interpolant). Finally, we will look at the case of singularity surface of co-dimension 3, a situation where we are still not aware of any technique having been proposed that successfully delivers a uniquely defined smooth Filippov vector field, not even in the nodally attractive case. Ideas based on the mean-field and moments techniques hold promise in this context. In particular, it will be interesting to see how (and if) the classes of 3-d barycentric coordinates, mean value coordinate, and spherical coordinates, that have been studied in computer graphics during the last 5 years (see [15,16]), can be used in the case of interest to us.

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Corrigendum

# Corrigendum to "A comparison of Filippov sliding vector fields in codimension 2" [J. Comput. Appl. Math. 262 (2014) 161–179]

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*The authors regret that there was a mistake in the statement of Theorem 3.4. The correction follows.*

In the statement and proof of Theorem 3.4, the value of $c$ has to be replaced with

$$c = \frac{-\mu^\top R v \pm \sqrt{(\mu^\top R v)^2 - (\mu^\top R \mu)(v^\top R v)}}{v^\top R v}.$$

We have to take the appropriate $\pm$ sign so that $c \in [a, b]$. In particular, if we let $p(x)$ be the polynomial

$$p(x) := x^2 v^\top R v + 2x\mu^\top R v + \mu^\top R \mu,$$

the correct $\pm$ sign is the opposite sign of $p(a)$.

This correction impacts also the value of $\gamma$ in the stated Theorem, and proof, which should be simply given as $\gamma = -\frac{1}{b-a}(a - c)$.

The revised proof is as follows.

**Proof.** One needs to solve for $c$ from the relation $\lambda_B^\top R \lambda_B = 0$. This gives the quadratic equation for $c$:

$$c^2 v^\top R v + 2c\mu^\top R v + \mu^\top R \mu = 0,$$

and the appropriate root is the one identified above. □

*The authors would like to apologise for any inconvenience caused by their careless proofreading.*

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