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THE MOMENTS SLIDING VECTOR FIELD ON THE INTERSECTION OF TWO MANIFOLDS

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Abstract. In this work, we consider a special choice of sliding vector field on the intersection of two co-dimension 1 manifolds. The proposed vector field, which belongs to the class of Filippov vector fields, will be called *moments vector field* and we will call *moments trajectory* the associated solution trajectory. Our main result is to show that the moments vector field is a well defined, and smoothly varying, Filippov sliding vector field on the intersection Σ of two discontinuity manifolds, under general attractivity conditions of Σ. We also examine the behavior of the moments trajectory at *first order exit points*, and show that it exits smoothly at these points. Numerical experiments illustrate our results and contrast the present choice with other choices of Filippov sliding vector field.

1. Introduction

Filippov convexification method, [9], is a very powerful technique to deal with piecewise smooth dynamical systems, and Filippov methodology has proven to be quite valuable in increasing our understanding of the qualitative features of piecewise smooth systems, as well as a great help in applications (e.g., see [1, 3, 18]).

However, although the case of one manifold of discontinuity separating two different vector fields is reasonably well understood, the case when the motion has to take place on the intersection Σ of two discontinuity manifolds (hence, generically, the phase space is –locally– split into four regions) still presents outstanding conceptual and practical challenges. The main difficulty is that, in general, there is no uniquely defined Filippov sliding vector field on Σ, and indeed our main goal in this work is to propose an appropriate choice of a Filippov vector field on such intersection Σ. We will do so under the assumption that Σ, in an appropriate sense, attracts nearby trajectories, and we will say that Σ *is attractive*.

In the remainder of this Introduction, we review the basic problem in which we are interested, and previous efforts, define what we mean by attractivity of Σ, introduce relevant definitions and notation, and give some preliminary results that will be useful later.

1.1. The problem and Filippov solutions.

We are interested in piecewise smooth differential systems of the following type:

$$\dot{x} = f(x)\ ,\ \ f(x) = f_i(x)\ ,\ \ x \in R_i\ ,\ i = 1, \dots, N\ ,\ \ t \in [0, T]\ . \tag{1.1}$$

Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, so that (locally) $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region R_i the function f is given by a smooth vector field f_i. Further, the regions R_i's are separated by manifolds (and their intersections) defined as 0-sets of smooth (at least $\mathcal{C}^2$) scalar valued functions h_i: $\Sigma_i := \{x \in \mathbb{R}^n : h_i(x) = 0\}$, $i = 1, \dots, p$ (and, for us, $2^p = N$).

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From (1.1), in general the vector field is not properly defined on the boundaries of the R_i's, where a classical solution ceases to exist. A successful definition of generalized solutions for problems as in (1.1) is due to Filippov, [9]. These are absolute continuous functions $x(t)$, for $t \in [0,T]$, such that $\dot{x}(t) \in \mathcal{F}(x(t))$ for almost all $t \in [0,T]$, and where $\mathcal{F}(x)$ is the convex hull of the values of $f(x)$ obtained approaching x through a region R_i. Formally:

$$\mathcal{F}(x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\mathrm{co}} \{ f(B(x,\delta)) \setminus S \} \ , \tag{1.2}$$

μ being Lebesgue measure on $\mathbb{R}^n$. Under mild conditions (boundedness and upper semicontinuity of $\mathcal{F}$), existence of Filippov solutions is guaranteed, but uniqueness is much more elusive, as it depends on the interaction of neighboring vector fields on the boundaries of the regions R_i's.

1.2. **Co-dimension 1: attractivity, existence and uniqueness.** The basic theory of Filippov (see [9]) covers fully the case of two regions separated by a manifold Σ defined as the 0-set of a smooth scalar valued function h. One has the following system:

$$\begin{gathered} \dot{x} = f_1(x) \ , \ \ x \in R_1 \ , \quad \text{and} \quad \dot{x} = f_2(x) \ , \ \ x \in R_2 \ , \\ \Sigma := \{ x \in \mathbb{R}^n \ : \ h(x) = 0 \} \ , \quad h : \ \mathbb{R}^n \to \mathbb{R} \, , \end{gathered} \tag{1.3}$$

where h is a $\mathcal{C}^k$ function, with $k \geq 2$, ∇h is bounded away from 0 for all $x \in \Sigma$, hence near Σ, and (without loss of generality) we label R_1 such that $h(x) < 0$ for $x \in R_1$, and R_2 such that $h(x) > 0$ for $x \in R_2$.

Remark 1.1. *We stress that the direction of time, the time arrow, is crucial. In this work, we will tacitly assume of proceeding forward in time. For this reason, as we clarify below, and unlike -say- the case of a boundary value problem, we believe it is important to take into account the attractivity properties of the discontinuity surface Σ, and to have these reflected into the behavior of trajectories on/near Σ.*

The interesting case is when trajectories reach Σ from R_1 (or R_2), and one has to decide what happens next. To answer this question, it is useful to look at the components of the two vector fields $f_{1,2}$ orthogonal to Σ:

$$w_1 := \nabla h(x)^\top f_1(x) \ , \ \ w_2 := \nabla h(x)^\top f_2(x) \ , \ \ x \in \Sigma \, . \tag{1.4}$$

Here, Σ is called *attractive in finite time* if for some positive constant c, we have

$$\nabla h(x)^\top f_1(x) \geq c > 0 \qquad \text{and} \qquad \nabla h(x)^\top f_2(x) \leq -c < 0 \ , \tag{1.5}$$

for $x \in \Sigma$ and in a neighborhood of Σ. In this case, trajectories starting near Σ must reach it, transversally, and remain there, giving rise to so-called *sliding motion*. A vector field associated to sliding motion is called *sliding vector field*. Filippov proposal (see (1.2)) is to take as sliding vector field on Σ a convex combination of f_1 and f_2, $f_{\mathrm{F}} := (1-\alpha) f_1 + \alpha f_2$, with α chosen so that $f_{\mathrm{F}} \in T_\Sigma$ (f_{F} is tangent to Σ at each $x \in \Sigma$):

$$\dot{x} = (1-\alpha) f_1 + \alpha f_2 \ , \quad \alpha = \frac{\nabla h(x)^\top f_1(x)}{\nabla h(x)^\top \big(f_1(x) - f_2(x) \big)} \, . \tag{1.6}$$

At the same time, Filippov theory also provides *first order exit conditions*: whenever $\alpha = 0$, respectively $\alpha = 1$, one may expect to leave Σ to enter in R_1 with vector field f_1, respectively enter R_2 with vector field f_2. [In other words, if the sliding vector field has aligned with either –but not both– f_1 or f_2, then generically (for smooth f_1, f_2) we expect to leave Σ as above].

We note that, during sliding motion, the right-hand side of (1.6) is a smooth vector field. This allows to study the dynamics during sliding motion using classical tools from the theory of dynamical systems with smooth vector fields; in particular, stability and bifurcation studies for equilibria on Σ, and for periodic orbits that may lie at least partly on Σ, have been extensively studied (e.g., see [3]). □

1.3. **Co-dimension 2: general attractivity by subsliding.** Our specific interest in this work is the case of (1.1) with $N = 4$. Now we will assume that the R_i's are (locally) separated by two intersecting smooth manifolds of co-dimension 1. That is, we have

$$\Sigma_1 = \{x:\ h_1(x) = 0\}\ ,\ \Sigma_2 = \{x:\ h_2(x) = 0\}\ ,\ h_i:\ \mathbb{R}^n \to \mathbb{R}\ ,\ i = 1, 2\ ,\ \Sigma = \Sigma_1 \cap \Sigma_2\ , \tag{1.7}$$

and we will also use the following notation

$$\Sigma_1^{\pm} = \{x:\ h_1(x) = 0\ ,\ h_2(x) \gtrless 0\}\ ,\ \Sigma_2^{\pm} = \{x:\ h_2(x) = 0\ ,\ h_1(x) \gtrless 0\}\ . \tag{1.8}$$

We will always assume that $h_{1,2}$ are $\mathcal{C}^k$ functions, with $k \geq 2$, that $\nabla h_1(x) \neq 0, x \in \Sigma_1$, $\nabla h_2(x) \neq 0, x \in \Sigma_2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for x on (and in a neighborhood of) Σ.

So, we have four different regions R_1, R_2, R_3 and R_4 with the four different smooth vector fields f_i, $i = 1, \dots, 4$, in these regions:

$$\dot{x} = f_i(x)\ ,\ x \in R_i\ ,\ i = 1, \dots, 4\ . \tag{1.9}$$

Without loss of generality, we will label these regions as follows:

$$\begin{aligned} &R_1:\ f_1 \quad \text{when} \quad h_1 < 0\ , h_2 < 0\ , \qquad R_2:\ f_2 \quad \text{when} \quad h_1 < 0\ , h_2 > 0\ ,\\ &R_3:\ f_3 \quad \text{when} \quad h_1 > 0\ , h_2 < 0\ , \qquad R_4:\ f_4 \quad \text{when} \quad h_1 > 0\ , h_2 > 0\ . \end{aligned} \tag{1.10}$$

We are specifically interested in the case when trajectories starting near Σ will reach it, transversally (and in finite time), a case referred to as having Σ attractive for nearby dynamics. To characterize this situation, it is again convenient to consider the components of the vector fields orthogonal to Σ. That is, we let (cfr. with (1.4))

$$\begin{aligned} &w_1^1 = \nabla h_1^\top f_1\ ,\ w_2^1 = \nabla h_1^\top f_2\ ,\ w_3^1 = \nabla h_1^\top f_3\ ,\ w_4^1 = \nabla h_1^\top f_4\ ,\\ &w_1^2 = \nabla h_2^\top f_1\ ,\ w_2^2 = \nabla h_2^\top f_2\ ,\ w_3^2 = \nabla h_2^\top f_3\ ,\ w_4^2 = \nabla h_2^\top f_4\ , \end{aligned} \tag{1.11}$$

and we will use the notation $w_i = (w_i^1, w_i^2) \in \mathbb{R}^2$, $i = 1, 2, 3, 4$, for those four points in $\mathbb{R}^2$.

Example 1.2. *The simplest case of attractive Σ is when it is* nodally attractive*. This means that on each of $\Sigma_{1,2}^{\pm}$ there is sliding motion toward the intersection Σ. These sliding motions on $\Sigma_{1,2}^{\pm}$ occur with Filippov sliding vector fields given as in* (1.6)*, henceforth labeled $f_{F_{1,2}^{\pm}}$. Namely,*

$$\begin{aligned} &f_{F_1^+} = (1-\alpha^+)f_2 + \alpha^+ f_4\ ,\ \alpha^+ = \left[\frac{\nabla h_1^\top f_2}{\nabla h_1^\top (f_2 - f_4)}\right]_{x\in\Sigma_1^+} = \frac{w_2^1}{w_2^1 - w_4^1}\ ,\\ &f_{F_1^-} = (1-\alpha^-)f_1 + \alpha^- f_3\ ,\ \alpha^- = \left[\frac{\nabla h_1^\top f_1}{\nabla h_1^\top (f_1 - f_3)}\right]_{x\in\Sigma_1^-} = \frac{w_1^1}{w_1^1 - w_3^1}\ ,\\ &f_{F_2^+} = (1-\beta^+)f_3 + \beta^+ f_4\ ,\ \beta^+ = \left[\frac{\nabla h_2^\top f_3}{\nabla h_2^\top (f_3 - f_4)}\right]_{x\in\Sigma_2^+} = \frac{w_3^2}{w_3^2 - w_4^2}\ ,\\ &f_{F_2^-} = (1-\beta^-)f_1 + \beta^- f_2\ ,\ \beta^- = \left[\frac{\nabla h_2^\top f_1}{\nabla h_2^\top (f_1 - f_2)}\right]_{x\in\Sigma_2^-} = \frac{w_1^2}{w_1^2 - w_2^2}\ . \end{aligned} \tag{1.12}$$

Finally, at first order, we note that nodal attractivity is guaranteed by the signs of Table 1 for the entries of w_i^j, $i = 1, \dots, 4$, $j = 1, 2$. □

The next characterization of attractivity for Σ was called *attractivity through sliding* in [7].

Definition 1.3 (Partial Nodal Attractivity; [7])**.** *We say that Σ is* partially nodally attractive*, or* attractive through sliding*, if the following conditions hold:*

Table 1. Nodal Attractivity.

Component	$i=1$	$i=2$	$i=3$	$i=4$
$w_i^1,\, i=1:4$	>0	>0	<0	<0
$w_i^2,\, i=1:4$	>0	<0	>0	<0

(a): $\begin{bmatrix} w_j^1(x) \\ w_j^2(x) \end{bmatrix}$ *does not have the same sign of* $\begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}$ *for* $x \in R_j$, $j = 1,2,3,4$ *;*

(b): *at least one of the following conditions is satisfied on* Σ*, and in a neighborhood of* Σ*:*

(1^+)**:** $\det \begin{bmatrix} w_2^1 & w_4^1 \\ 1 & 1 \end{bmatrix} > 0$ *together with* (1_a^+)*:* $(1-\alpha^+)w_2^2 + \alpha^+ w_4^2 < 0$*;*

(1^-)**:** $\det \begin{bmatrix} w_3^1 & w_1^1 \\ 1 & 1 \end{bmatrix} < 0$ *together with* (1_a^-)*:* $(1-\alpha^-)w_1^2 + \alpha^- w_3^2 > 0$*;*

(2^+)**:** $\det \begin{bmatrix} w_4^2 & w_3^2 \\ 1 & 1 \end{bmatrix} < 0$ *together with* (2_a^+)*:* $(1-\beta^+)w_3^1 + \beta^+ w_4^1 < 0$*;*

(2^-)**:** $\det \begin{bmatrix} w_1^2 & w_2^2 \\ 1 & 1 \end{bmatrix} > 0$ *together with* (2_a^-)*:* $(1-\beta^-)w_1^1 + \beta^- w_2^1 > 0$*;*

(c): *if any of* $(1^\pm)$ *or* $(2^\pm)$ *is satisfied, then* $(1_a^\pm)$ *or* $(2_a^\pm)$ *must be satisfied as well.*

Above, we note that the quantities $\alpha^\pm$*,* $\beta^\pm$ *(as given in* (1.12)*), are well defined whenever the relevant conditions* $(1^\pm)$*,* $(2^\pm)$ *hold.* □

The next result gives a handy rewriting of $(1_a^\pm)$, $(2_a^\pm)$ in Definition 1.3.

Lemma 1.4. *Let any of* $(1^\pm)$ *and/or* $(2^\pm)$ *in Definition 1.3 hold. Then, the corresponding conditions* $(1_a^\pm)$*,* $(2_a^\pm)$ *are equivalent, respectively, to the following:*

$$\widetilde{(1_a^+)} : \det \begin{bmatrix} w_2 & w_4 \end{bmatrix} < 0; \qquad \widetilde{(1_a^-)} : \det \begin{bmatrix} w_3 & w_1 \end{bmatrix} < 0;$$
$$\widetilde{(2_a^+)} : \det \begin{bmatrix} w_4 & w_3 \end{bmatrix} < 0; \qquad \widetilde{(2_a^-)} : \det \begin{bmatrix} w_1 & w_2 \end{bmatrix} < 0.$$

Proof. Let us prove equivalence between (1_a^+) and $\widetilde{(1_a^+)}$. The others are analogous.

Since (1^+), (1_a^+), (1.12), hold, we get that

$$\frac{-w_4^1 w_2^2 + w_2^1 w_4^2}{\det \begin{bmatrix} w_2^1 & w_4^1 \\ 1 & 1 \end{bmatrix}} < 0 \ , \quad \text{from which} \det \begin{bmatrix} w_2 & w_4 \end{bmatrix} < 0 \ .$$

Conversely, if $\det \begin{bmatrix} w_2 & w_4 \end{bmatrix} < 0$, since (1^+) holds, we get (1_a^+) at once. □

Remark 1.5. *Partial nodal attractivity (which of course includes nodal attractivity as a special case) implies that one has sliding motion on (at least) one of* $\Sigma_{1,2}^\pm$*, directed towards* Σ*, and no sliding motion on any of* $\Sigma_{1,2}^\pm$*, away from* Σ*. A typical solution trajectory starting near* Σ *will approach (in finite time) the intersection* Σ*, by first sliding on one of* Σ_1 *or* Σ_2*, directed towards* Σ *(of course, a trajectory may also reach* Σ *directly from within one of the regions* R_i*'s, but this is a less likely event).*

Remark 1.6. *We also note that partial nodal attractivity is not an exclusive characterization of attractivity of* Σ*. Namely,* Σ *may also be* spirally attractive*. In this case, there is no attractivity toward* Σ *through sliding on any of* $\Sigma_1^\pm$*,* $\Sigma_2^\pm$*, and trajectories reach* Σ *by spiraling around it. See* [4] *for the characterization of spirally attractive* Σ*.*

1.4. **Co-dimension 2: general ambiguity.** At this point, we may envision having the following scenario for a solution trajectory of a system (1.9), with attractive $\Sigma = \Sigma_1 \cap \Sigma_2$.

- It starts in a region R_i for some $i = 1, 2, 3, 4$, until
- it reaches transversally one of $\Sigma^{\pm}_{1,2}$;
- then, it begins sliding on $\Sigma^{\pm}_{1,2}$ toward Σ, until
- it reaches transversally the intersection Σ. What happens then?

Now, when Σ is attractive, a trajectory starting on Σ cannot leave Σ. But, how should a solution trajectory evolve on Σ? In the class of Filippov solutions, we will need to have that $\dot{x} \in \mathcal{F}(x)$ as in (1.2), and further that $\dot{x}$ lies on the tangent plane to Σ, for any $x \in \Sigma$. That is, Filippov solutions will be such that

$$\begin{aligned} &\dot{x} \;\in \left\{ \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \;,\;\; \lambda_i \geq 0 \;,\; i = 1, \dots, 4 \;,\;\; \sum_{i=1}^{4} \lambda_1 = 1 \right\} \;, \\ &\nabla h_1^\top \dot{x} \;=\; \nabla h_2^\top \dot{x} \;=\; 0 \,. \end{aligned} \tag{1.13}$$

But, from (1.13), it is plainly apparent that there is no uniqueness of a sliding vector field on Σ, so that sliding motion on Σ is not uniquely defined.

In this work, we propose a way to select a smooth sliding vector field on Σ, from the class of Filippov convex combinations (1.13), whenever Σ is attractive through sliding. In other words, we will select a *smooth Filippov sliding vector field* f_{F}: for $x \in \Sigma$, this is of the form

$$\begin{aligned} &f_{\mathrm{F}} \;=\; \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \;,\;\; \lambda_i \geq 0 \;,\; i = 1, \dots, 4 \;,\;\; \sum_{i=1}^{4} \lambda_1 = 1 \;, \\ &\nabla h_1^\top f_{\mathrm{F}} \;=\; \nabla h_2^\top f_{\mathrm{F}} \;=\; 0 \,, \end{aligned} \tag{1.14}$$

where the coefficients λ_i's depend smoothly on $x \in \Sigma$. Therefore, with previous notation, we will have to solve the problem (for $x \in \Sigma$):

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \qquad \text{where} \qquad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \;,\; \text{and } W = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix} \;,\;\; \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \;. \tag{1.15}$$

Obviously, (1.15) is an underdetermined linear system, reflecting the fact that the mere requirement of f_{F} being on T_Σ is not generally sufficient to uniquely[1] characterize a convex combination of the four vector fields $f_1, \dots, f_4$. We propose the following definition of *admissible* solution of (1.15).

Definition 1.7. *Under the conditions of partial nodal attractivity of Definition 1.3, we say that a solution λ of* (1.15) *is* admissible, *if $\lambda \geq 0$ and λ depends smoothly on $x \in \Sigma$.* □

Remark 1.8. *The problem of understanding sliding motion on Σ has been of considerable interest in the last 15 years. To date, the choice that has received most attention is one based on* bilinear interpolation. *This consists in selecting the Filippov vector field below:*

$$\begin{aligned} &(a) \qquad f_B := (1-\alpha)\left((1-\beta) f_1 + \beta f_2\right) + \alpha\left((1-\beta) f_3 + \beta f_4\right) \;, \\ &(b) \qquad (\alpha, \beta) \in (0,1)^2 \;:\; W\lambda_B = 0 \qquad \text{with } \lambda_B := \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix} \;. \end{aligned} \tag{1.16}$$

[1] There are special cases when the aforementioned ambiguity is not present, as when two of the original vector fields are identical (e.g., see [17]), but in general we must expect to have an underdetermined system.

This bilinear interpolation method was originally introduced in [2] *for nodally attractive* Σ*, it was further mentioned in* [3]*, it was later studied in* [8, 7]*, and it is effectively the sliding technique underpinning the singular perturbation approach of* [15] *and of* [14]*. As proven in* [7]*, when the conditions of Definition 1.3 hold, this bilinear method gives an admissible solution* λ_B *and a smoothly varying Filippov vector field on* Σ*. To be precise, and for later reference, we note that one needs to solve the nonlinear system* (1.16)-(b)*, that is* $W\lambda_B = 0$*, for* (α, β)*. In general, this system may have more than one admissible solution; the quoted result in* [7] *guarantees that there is only one admissible solution (i.e., values of* α *and* β *in* $[0, 1]$*), whenever* Σ *is attractive as in Definition 1.3.*

But, unfortunately, there are potential difficulties caused by the choice (1.16) of vector field. These become apparent when Σ loses attractivity at generic *first order exit points* (see below), where one of the sub-sliding vector fields (on Σ_1 or Σ_2) has itself become tangent to Σ. As we will see in Lemma 1.10, at generic exit points Σ ceases to be attractive, and one might expect a trajectory to exit Σ on the lower co-dimension manifold. However, as proven in [7], at generic exit points there could be two solutions of (1.16)-(b), giving distinct (α, β) in $[0, 1]^2$, and different vector fields. Again referring to [7], one such solution always necessarily gives the sliding vector field on the lower co-dimension manifold, but the other solution corresponds to the sliding vector field that the trajectory was obeying. As a consequence, even assuming that one is able to obtain all roots of (1.16)-(b) rather than just following one by continuation, in general there is a catch: either one discontinuously changes the value of (α, β) in order to exit from Σ (and loses smoothness), or the loss of attractivity of Σ will go unnoticed to the bilinear vector field one is using (which remains well defined) and one ends up sliding on Σ, even though Σ is no longer attractive (see Section 4 for illustration of this fact). To us, this seems undesirable, since -if perturbations off Σ obey the dynamics of the original piecewise smooth system (1.9)- in general we expect that the perturbed solution trajectories will not return to Σ, when Σ is not attractive.

Definition 1.9 (First order exit points; [7])**.** *Let* $\dot{x}$ *be as in* (1.13)*, and let* $f_{\Sigma_{1,2}^{\pm}}$ *be as in* (1.12) *(whenever there is a well defined sliding motion on* $\Sigma_{1,2}^{\pm}$*). We say that* $x \in \Sigma$ *is a* generic first order exit point *if one (and just one) of the* $f_{F_{1,2}^{\pm}}$ *is itself in the class* (1.13)*, that is it is tangent to* Σ*. The corresponding* $f_{F_{1,2}^{\pm}}$ *is called an* exit vector field*.* □

As Lemma 1.10 below clarifies (see also [7]), first order exit points are points where Σ ceases to be partially nodally attractive.

Lemma 1.10. *If a point* $x_e \in \Sigma$ *is a first order exit point relative to* Σ_1^+*, then*

$$\det \begin{bmatrix} w_2 & w_4 \end{bmatrix} = 0. \tag{1.17}$$

Analogously, if the first order exit points correspond to a sliding regime on Σ_1^- *we have* $\det \begin{bmatrix} w_3 & w_1 \end{bmatrix} = 0$*, relatively to* Σ_2^+ *we have* $\det \begin{bmatrix} w_4 & w_3 \end{bmatrix} = 0$*, and relatively to* Σ_2^- *have* $\det \begin{bmatrix} w_1 & w_2 \end{bmatrix} = 0$*.*

Proof. If $x_\mathrm{e} \in \Sigma$ is a potential exit point for subsliding on Σ_1^+, then (at x_e) $f_{\mathrm{F}_1^+}$ is not just in the plane tangent to Σ_1 but also to Σ. That is, at x_e we must have

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$

which of course implies

$$\begin{bmatrix} w_2 & w_4 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{hence} \qquad \det \begin{bmatrix} w_2 & w_4 \end{bmatrix} = 0 ,$$

since $f_{\mathrm{F}}{}_1^+ = \lambda_2 f_2 + \lambda_4 f_4$, $\lambda_2 + \lambda_4 = 1$, is the Filippov sliding vector field on Σ_1^+. Similarly for the other cases. □

As a consequence of Lemma 1.10, at a generic first order exit point for one of the $\Sigma_{1,2}^{\pm}$, we would like a solution trajectory to leave Σ and to begin sliding (away from Σ) on the relevant sub-manifold $\Sigma_{1,2}^{\pm}$ with corresponding exit vector field. For this reason, we will further restrict our search for admissible λ, solutions of (1.15), in such a way that they will render the exit vector field at generic first order exit points.

Definition 1.11 (Smooth Exits)**.** *Let λ in* (1.14)-(1.15) *be admissible and such that, at a generic first order exit point, λ renders also the exit vector field*[2]*. Then, f_F will be called a* smoothly exiting *vector field.* □

1.5. **Co-dimension 2: A proposal for a smooth Filippov sliding vector field.** To reiterate, under the assumption of attractivity of Σ as in Section 1.3, we will want to select an admissible (positive and smooth) solution of (1.15) in such a way that it will further lead to a smoothly exiting vector field at generic first order exit points.

Our proposal is based on a rather general principle: *To regularize the system* (1.15) *by adding to it one extra condition, linear in λ, so to obtain an invertible system giving a solution λ enjoying the above properties*. We will adopt the choice we begun exploring in [5] for the case of nodal attractivity of Σ, a choice we had called *moments regularization*. [In [5], we also showed that, under nodal attractivity assumption, this regularization gave for λ the mean value barycentric coordinates of Floater ([10, 11]), associated to the points w_i, $i = 1, 2, 3, 4$.]

1.5.1. *Moments regularization.* We consider the following system (cfr. with (1.15)) to be satisfied for $x \in \Sigma$:

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \qquad \text{where} \qquad M := \begin{bmatrix} W \\ \mathbb{1}^\top \\ d^\top \end{bmatrix} , \tag{1.18}$$

with W defined in (1.15) and

$$d := \begin{bmatrix} d_1 \\ -d_2 \\ -d_3 \\ d_4 \end{bmatrix} , \qquad \text{where} \qquad d_i := \|w_i\|_2 \ , \ \ i = 1, \dots, 4 \ . \tag{1.19}$$

Definition 1.12 (Moments method)**.** *We call* moments method *the method resulting from solving* (1.18) *for λ, and using this in the selection of sliding vector field in* (1.14)*. We call* moments solution *the solution λ of* (1.18)*, call* moments vector field *the resulting vector field* (1.14)*, and call* moments trajectory *the solution of the differential equation on Σ obtained when using the moments vector field.* □

Below, we validate the moments method, by showing that, for $x \in \Sigma$ and Σ attractive as in Definition 1.3, the matrix M in (1.18) is non-singular, that the unique solution of (1.18) is admissible, and that the resulting smoothly varying Filippov sliding vector field f_{F} is further smoothly exiting at generic first order exit points. Let us emphasize that our construction will give **a** Filippov solution (1.14) of the general piecewise smooth system (1.9). Let us also emphasize that the overall solution trajectory, in general, will only be piecewise smooth: our concern is that it be smooth on the intersection Σ, but of course –in general– it will be only continuous at entry points in a sliding region.

[2] this means that two of the four entries of λ are 0

Remark 1.13. *Of course, the formulation of the moments method we validate in this paper is valid precisely for the case of Σ of co-dimension 2 examined herein. The extension of the moments method to the case of Σ of co-dimension 3 (intersection of three co-dimension 1 surfaces) requires an appropriately modified formulation; details are in* [6].

A plan of the paper is as follows. In Section 2, we associate a quadrilateral to the attractivity configuration of Σ. In Section 3, this geometrical configuration is exploited to prove invertibility of the matrix M in (1.18), and admissibility of the unique solution λ. In Section 4, we show on two examples how the moments method compares to the bilinear interpolant technique of (1.16), and in particular we highlight the different behaviors at first order exit points. In Section 5, we rigorously prove that the moments vector field is smoothly exiting at generic first order exit points, and we briefly discuss other possibilities enjoying this property. Finally, in Section 6, we give some conclusions.

2. Geometrical pattern for the dynamical problem

In this section, we give a useful geometrical reinterpretation of the algebraic problem (1.15), when Σ is attractive. Later, this configuration will be exploited to establish solvability of the system (1.18).

We begin by observing that the general Filippov convexification construction based on (1.14)-(1.15) is effectively saying that the origin must be in the convex hull of the four points w_i, $i = 1, \dots, 4$. However, the convex hull of the four points w_i's is a very large set, and may fail to give a good geometrical correspondence with the dynamics of the problem.

Example 2.1. *Consider the following model problem of the type* (1.9)*:*

$$\dot{x} = f_i(x)\ ,\quad i = 1, 2, 3, 4, \qquad \text{where}$$

$$f_1 = \begin{bmatrix} x_3 - 1 \\ x_3 \\ x_1 - 1 \end{bmatrix}\ ,\ f_2 = \begin{bmatrix} 2 \\ -1 \\ x_2 - 1 \end{bmatrix}\ ,\ f_3 = \begin{bmatrix} -1 \\ 2 \\ x_1 x_2 - 1 \end{bmatrix}\ ,\ f_4 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\ ,$$

where (see (1.7)*)* $\Sigma_1 = \{x_1 = 0\}$*,* $\Sigma_2 = \{x_2 = 0\}$*, and so* $\Sigma = \{x_1 = x_2 = 0\}$*, and therefore (see* (1.11)*)*

$$w_1 = \begin{bmatrix} x_3 - 1 \\ x_3 \end{bmatrix}\ ,\ w_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\ ,\ w_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\ ,\ w_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}\ .$$

In this case, on Σ, there is a unique Filippov sliding vector field: $\dot{x}_3 = -1$.

Consider the initial condition $(0, 0, 2)$ and the time interval $0 \leq t \leq 2$. In Figure 1, we show the four snapshots of the vertices w_i's, at times $t = 0$, $t = 1$, $t = 5/3$, and $t = 9/5$. For $t = 0$, we are in a configuration of nodal attractivity, which persists for as long as $t < 1$. However, as soon as $t \geq 1$, the vertex w_1 plays no role in the convex hull of the four points (dotted segment). Also, observe that as soon as $t > 5/3$, Σ is no longer attracting nearby trajectories (hence, a perturbation off Σ will move away from Σ), though the convex hull has not changed. □

Motivated by the above, our goal is to consider a geometric configuration that better reflects the dynamics of the problem (and attractivity of Σ). To this end, we propose to consider the quadrilateral Q, determined by w_1, w_2, w_4, w_3, in this order. Accordingly, we are proposing to reinterpret an admissible Filippov solution as one that obtains weights λ to be put on the vertices of Q in such a way that the origin be the *barycenter* of Q relative to λ [3]. For later reference, we summarize our proposal of quadrilateral Q. Later in this work, Q will always refer to this quadrilateral.

[3] In this context, we can reinterpret (1.18) as a physical equilibrium requirement about the moments provided by the weights λ with respect to origin, hence the proposed name of *moments method* we adopted for our technique.

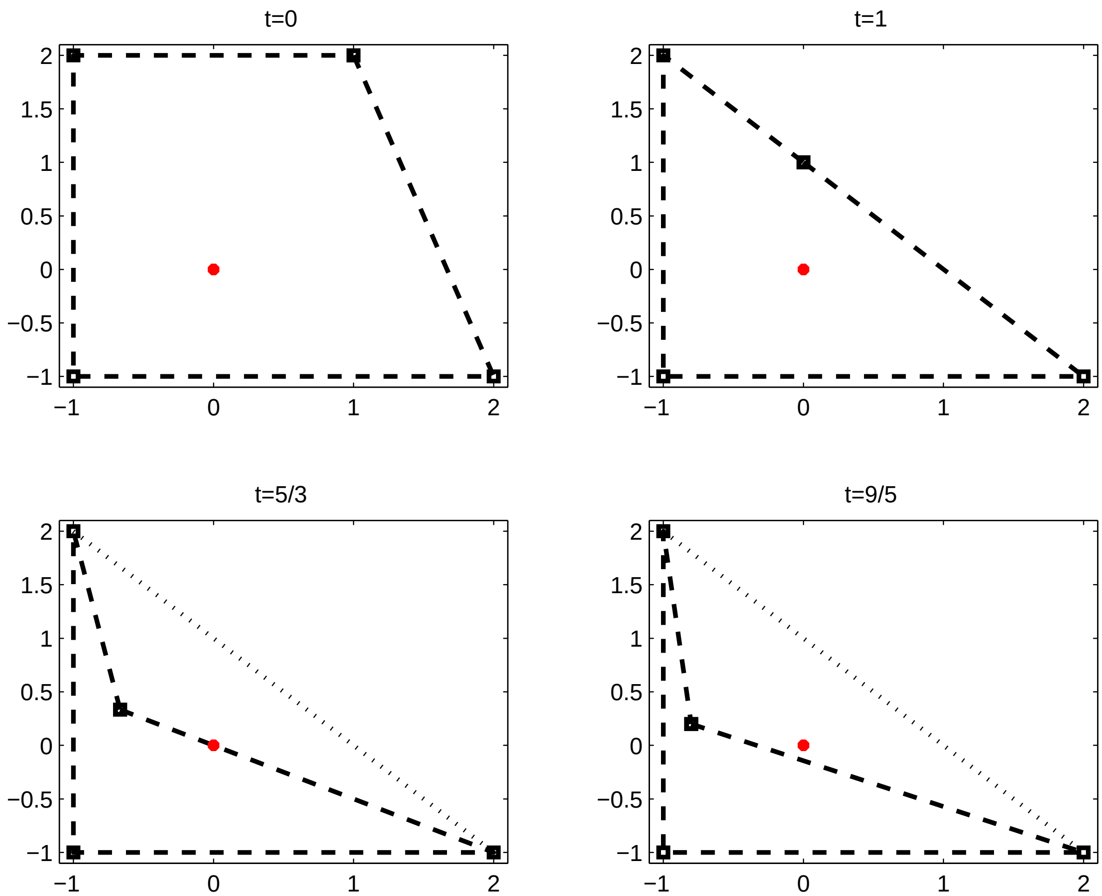

FIGURE 1. Dynamics of Example 2.1: Convex hull versus quadrilateral Q.

Definition 2.2. *Given the four points w_1, w_2, w_3, w_4, as in* (1.11)*, we define the quadrilateral Q associated to W to be the quadrilateral obtained by joining the four points in the order w_1 to w_2, to w_4, to w_3, and back to w_1.* □

The following result is a simple consequence of the characterizations of attractivity of Σ and the definition of quadrilateral Q. [For part (i), in the case of Σ attractive through sliding, the result follows at once from Definition 1.3. In the case of spiral attractivity, it follows immediately from [4, Table 3 or 4]). For part (ii), see Lemma 1.4.] Also, note that, in case (i), sliding motion on Σ should be taking place.

Lemma 2.3. *Let W and Q be defined as usual, for $x \in \Sigma$.*

(i) *If Σ is attractive (through sliding, or by spiraling), then the origin is in the interior of Q. In particular, if the origin is external to Q, then Σ cannot be attractive.*
(ii) *If x is a generic first order exit point, then the origin belongs to one side (and one only) of Q.* □

We emphasize that that the quadrilateral Q tells us that "if $0 \notin \bar{Q}$ then Σ is not attractive, and a trajectory with initial conditions off Σ will not be attracted to Σ": this is our key reason to consider Q.

Below, we give some results on the interplay between the quadrilateral Q and the algebraic problem (1.18). These results will be used in Section 3 to establish solvability of (1.18).

Definition 2.4. *The quadrilateral Q is called* non-degenerate*, if and only if these two conditions hold:*

(a) *the vertices are not all aligned (equivalently, at most three vertices are aligned), and*
(b) *if one vertex of Q is at the origin, then there cannot be two other vertices aligned with it; in particular, no two vertices can be at the origin.* □

Remark 2.5. *In agreement with Lemma 2.3, it is an important observation that, in each of the sliding configurations allowed by Definition 1.3,*[4] *the points w_i, $i = 1, \dots,$, will always give that Q is non-degenerate. In fact, the origin is always in the interior of Q. Furthermore, at generic first order exit points, the origin is along one edge (and one only) of Q, and in particular the origin cannot be a vertex of Q.*

Next, we give a key algebraic result that will be used in Section 3.

Lemma 2.6. *If Q is non-degenerate, then the matrix $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ in (1.15) has full rank 3. Furthermore, there is a nontrivial vector v, as smooth as W, spanning $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$.*

Proof. Since we are assuming the quadrilateral relative to W to be non-degenerate, then there exist three vectors in $\{w_i : i = 1, 2, 3, 4\}$ such that the corresponding triangle has nonzero area: this implies that the columns corresponding to those three vectors in $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ are linearly independent. The statement about the span of $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ is because the symmetric function $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \begin{bmatrix} W^\top & \mathbb{1} \end{bmatrix}$ has exactly one zero eigenvalue which is simple (of algebraic multiplicity 1). Therefore, the eigenvector associated to this 0 eigenvalue can be chosen smooth (e.g., see [13]), and it provides a basis for $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. □

We next give a more precise algebraic characterization of the vector $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ relatively to non-degenerate quadrilaterals. This result will be used in Section 5.

Notation 2.7. *We will write $\mathcal{A}_{ijk}$ for the signed area of the triangle of vertices w_i, w_j, w_k, in this order, $i, j, k = 1, 2, 3, 4$, and where the indices are distinct. For example, $\mathcal{A}_{123} = \frac{1}{2} \det \begin{bmatrix} w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{bmatrix}$, and the sign of the determinant indicates whether the triangle is traced clockwise or counterclockwise.* □

Lemma 2.8. *Let Q be non-degenerate, and let W be the usual matrix: $W = [w_1\ w_2\ w_3\ w_4]$. Then, if $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, v can have at most one zero component.*

Proof. By Lemma 2.6, there is at least one triangle determined by vertices of Q with nonzero area: without loss of generality, we assume it to be $\mathcal{A}_{123}$. Therefore, by Cramer's rule (and

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[4] there are 13, not equivalent ones, [7]

elementary rules of the determinant), we can write this vector v as

$$v = \begin{bmatrix} \mathcal{A}_{243} \\ \mathcal{A}_{134} \\ \mathcal{A}_{142} \\ \mathcal{A}_{123} \end{bmatrix} . \tag{2.1}$$

If, by contradiction, more than one of these components were zero, then the four vertices would be aligned: but this contradicts that Q be non-degenerate. □

Additionally, (2.1) also shows smoothness of v, because the (signed) area of a triangle is a smooth function of the triangle vertices (that is, the determinant is a smooth function of the matrix entries).

Remark 2.9. *In light of Lemmata 2.6 and 2.8, clearly any solution of* $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *can be written as*

$$\lambda = \lambda_p + cv \, ,$$

where λ_p *is any particular solution, and* $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, *and thus we note that* v *cannot have all components of the same sign. Therefore, in particular, if* λ_p *is admissible (hence* $\lambda_p \geq 0$*), in order for* λ *to be admissible we must have* $a \leq c \leq b$*, where* $a \leq 0$ *and* $b \geq 0$ *are defined as*

$$a := \max\left\{ -\frac{\lambda_{p,i}}{v_i} : v_i > 0 \right\} \ , \quad b := \min\left\{ -\frac{\lambda_{p,i}}{v_i} : v_i < 0 \right\} \ .$$

3. Moments Solution under general attractivity conditions

Let us now assume the quadrilateral Q is non-degenerate and the origin is internal to it or on at most one of its edges. In particular, this is the situation when Σ is attractive through sliding. Then, we will show that M in (1.18) is nonsingular, and the moments solution λ is admissible. In Section 5, we will further show that the moments vector field is smoothly exiting at generic first order exit points.

So, consider system (1.18), repeated here for convenience:

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \ , \qquad \text{where} \qquad M := \begin{bmatrix} W \\ \mathbb{1}^\top \\ d^\top \end{bmatrix} \ , \tag{3.1}$$

and recall that, see Lemma 2.6, $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ has dimension 1 and it is smoothly spanned by a vector v, which we will take as in (2.1).

The following general result will be used below.

Lemma 3.1. *Let* $A \in \mathbb{R}^{(n-1)\times n}$ *be of rank* $(n-1)$*, and let its null space be spanned by the vector* v*. Let* $d \in \mathbb{R}^n$ *be given and consider the matrix* $B = \begin{bmatrix} A \\ d^\top \end{bmatrix}$*. Then,* B *is nonsingular if and only if* $d^\top v \neq 0$*.*

Proof. Suppose B is nonsingular, and by contradiction that $d^\top v = 0$. Then obviously $Bv = 0$, hence B would be singular. If $d^\top v \neq 0$, since $\ker(A)$ is spanned just by v, then there cannot be any vector y: $By = 0$. □

Using Lemma 3.1 and Laplace expansion of the determinant with respect to the fourth row of M, from (3.1) we get (for v in (2.1)):

$$\det M = d^\top v \ . \tag{3.2}$$

Now, let M_{adj} be the adjugate[5] of M. Since $MM_{\text{adj}} = M_{\text{adj}}M = \det(M)I$, if M is invertible, to obtain the unique solution of (3.1) we must look at the third row, $M_{\text{adj}}(3,:)$, of M_{adj}.

Direct computation gives

$$M_{\text{adj}}(3,:) = \left[\det\begin{bmatrix} w_2 & w_3 & w_4 \\ -d_2 & -d_3 & d_4\end{bmatrix},\ -\det\begin{bmatrix} w_1 & w_3 & w_4 \\ d_1 & -d_3 & d_4\end{bmatrix},\ \det\begin{bmatrix} w_1 & w_2 & w_4 \\ d_1 & -d_2 & d_4\end{bmatrix},\ -\det\begin{bmatrix} w_1 & w_2 & w_3 \\ d_1 & -d_2 & -d_3\end{bmatrix}\right] \ ,$$

and further

$$\sum_{j=1}^{4} M_{\text{adj}}(3,j) = d^\top v = \det M \ ,$$

and therefore the unique solution of (3.1), if indeed it exists unique, must be given by

$$\lambda_{\text{M}} := \frac{1}{d^\top v} M_{\text{adj}}(3,:)^\top \ . \tag{3.3}$$

What we will prove below is that each entry in $M_{\text{adj}}(3,:)$ has the same sign (some entries may be 0, but not all of them can be), from which it will follow that $\det M \neq 0$, and further that the entries of λ_{M} are all nonnegative (and sum to 1), which is what we had set out to prove.

We use a geometrical technique. To begin with, assume that for all $i = 1,2,3,4$, $w_i \neq 0$, and express each w_i in polar coordinates:

$$w_i = d_i \widehat{w}_i, \quad \widehat{w}_i := \begin{bmatrix}\cos\theta_i \\ \sin\theta_i\end{bmatrix} \ , \ i = 1,2,3,4 \ . \tag{3.4}$$

Note that just as the original vertices w_i's gave us the quadrilateral Q, now we have obtained the quadrilateral $\widehat{Q}$ defined by the vertices $\widehat{w}_1, \widehat{w}_2, \widehat{w}_4, \widehat{w}_3$ (in this order) on the unit circle; in so doing, we have respected the signs of the original vertices coordinates. In particular, if Q was non-degenerate, so is the associated quadrilateral $\widehat{Q}$ on the unit circle, and if the origin was internal to Q, it is still internal to the new quadrilateral $\widehat{Q}$.

In this new notation, we have (note the changes of sign on the second equality)

$$M_{\text{adj}}(3,:)^\top = \begin{bmatrix} d_2d_3d_4 \det\begin{bmatrix}\cos\theta_2 & \cos\theta_3 & \cos\theta_4 \\ \sin\theta_2 & \sin\theta_3 & \sin\theta_4 \\ -1 & -1 & 1\end{bmatrix} \\ -d_1d_3d_4 \det\begin{bmatrix}\cos\theta_1 & \cos\theta_3 & \cos\theta_4 \\ \sin\theta_1 & \sin\theta_3 & \sin\theta_4 \\ 1 & -1 & 1\end{bmatrix} \\ d_1d_2d_4 \det\begin{bmatrix}\cos\theta_1 & \cos\theta_2 & \cos\theta_4 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_4 \\ 1 & -1 & 1\end{bmatrix} \\ -d_1d_2d_3 \det\begin{bmatrix}\cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 \\ 1 & -1 & -1\end{bmatrix}\end{bmatrix} = \begin{bmatrix} d_2d_3d_4 \det\begin{bmatrix}-\widehat{w}_2 & -\widehat{w}_3 & \widehat{w}_4 \\ 1 & 1 & 1\end{bmatrix} \\ d_1d_3d_4 \det\begin{bmatrix}\widehat{w}_1 & -\widehat{w}_3 & \widehat{w}_4 \\ 1 & 1 & 1\end{bmatrix} \\ -d_1d_2d_4 \det\begin{bmatrix}\widehat{w}_1 & -\widehat{w}_2 & \widehat{w}_4 \\ 1 & 1 & 1\end{bmatrix} \\ -d_1d_2d_3 \det\begin{bmatrix}\widehat{w}_1 & -\widehat{w}_2 & -\widehat{w}_3 \\ 1 & 1 & 1\end{bmatrix}\end{bmatrix} \ ,$$

[5] the transpose of the matrix of cofactors of M

from which (again, note the changes of sign) we get

$$M_{\rm adj}(3,:)^\top = \begin{bmatrix} -d_2d_3d_4 \det \begin{bmatrix} -\widehat{w}_2 & \widehat{w}_4 & -\widehat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_3d_4 \det \begin{bmatrix} \widehat{w}_1 & \widehat{w}_4 & -\widehat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_2d_4 \det \begin{bmatrix} \widehat{w}_1 & -\widehat{w}_2 & \widehat{w}_4 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_2d_3 \det \begin{bmatrix} \widehat{w}_1 & -\widehat{w}_2 & -\widehat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \end{bmatrix}. \tag{3.5}$$

Now, each determinant in the components of the vector in (3.5) above represents the (signed) area of one of the four triangles in which the quadrilateral on the unit circle $\widehat{w}_1\widehat{w}_2\widehat{w}_4\widehat{w}_3$ is divided by its diagonals. We want to show that they all have the same signs.

The following result from convex geometry will be helpful to us.

Proposition 3.2. [16, Theorem 4.4.1 and Exercise 4.4.1] *A non-degenerate quadrilateral Q is convex if and only if its diagonals intersect in its closure.* □

Next, we prove that, for any given quadrilateral on the unit circle, containing the origin and non-degenerate, its transformed quadrilateral obtained by reflecting one of its diagonals with respect to the origin is always convex. See Figure 2 for an illustration of this fact.

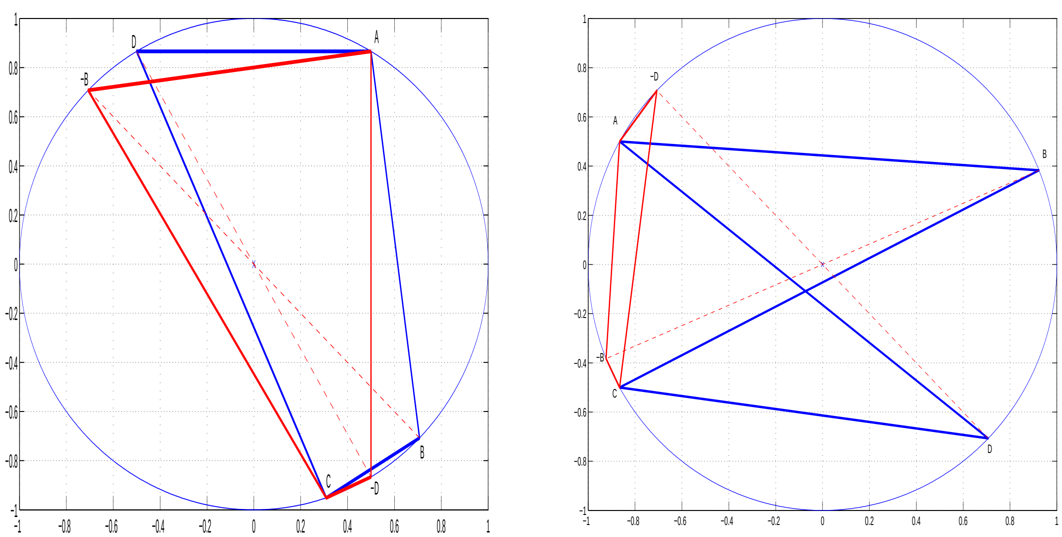

FIGURE 2. Illustration of Proposition 3.3. Transformation of the quadrilateral: left, convex case, right, nonconvex case.

Proposition 3.3. *Given a non-degenerate quadrilateral $\widehat{Q} = ABCD$ with vertices on the unit circle, and containing the origin, the transformed quadrilateral $\widetilde{Q} := A(-B)C(-D)$ is convex.*

Proof. Note that if Q reduces to a triangle, the result is trivially true. So, let us assume that all vertices of Q are distinct.

If $\widehat{Q}$ is convex, the reflected diagonal $(-B)(-D)$ still intersects the other diagonal AC in the closure of $\widetilde{Q}$.
If $\widehat{Q}$ is not convex, then it is necessarily self-intersecting (on the unit circle we can connect four points in two different ways only: to create a convex quadrilateral following any clockwise direction, or a self-intersecting one). Up to relabeling, we can assume that the origin is inside the triangle ABD. Call $\widetilde{B} := -B$, $\widetilde{D} := -D$, and consider the quadrilateral of vertices $A, \widetilde{D}, C, \widetilde{B}$, in this order.

Now, since the two angles $A\widehat{B}C$ and $A\widehat{\widetilde{D}}C$ subtend the same arc AC, being the origin inside the triangle ABD, then

$$A\widehat{B}C = A\widehat{\widetilde{D}}C = \alpha + \beta \ ,$$

where α is the angle at B in the right triangle $AB\widetilde{B}$, and β is the angle in B in the right triangle $CB\widetilde{B}$. Therefore:

$$A\widehat{\widetilde{B}}B = \frac{\pi}{2} - \alpha \ , \quad C\widehat{\widetilde{B}}B = \frac{\pi}{2} - \beta \ ,$$

and so

$$A\widehat{\widetilde{B}}C = A\widehat{\widetilde{B}}B + B\widehat{\widetilde{B}}C = \pi - (\alpha + \beta) \ ,$$

whereas $A\widehat{\widetilde{D}}C = \alpha + \beta$. Therefore $\widetilde{B}$ and $\widetilde{D}$ are on opposite sides with respect to AC because, otherwise, it would be $A\widehat{\widetilde{B}}C = A\widehat{\widetilde{D}}C$: so AC intersects $\widetilde{B}\widetilde{D}$ in the closure of $\widetilde{Q}$. By Proposition 3.2, $\widetilde{Q}$ is convex. ☐

With the help of Proposition 3.3, we can now give our main result.

Theorem 3.4. *Let Σ be defined in* (1.7), *w_i, $i = 1, \dots, 4$, be given in* (1.11), *and let Q be the quadrilateral of Definition 2.2. Assume that Q is non-degenerate, that $w_i \neq 0$, for all $i = 1, \dots, 4$, and that $0 \in \overline{Q}$, as $x \in \Sigma$. Then, the matrix M of the moments method in* (1.18) *is nonsingular and the moments solution λ_M of* (3.3) *is admissible as x varies in Σ.*

Proof. Since Q is non-degenerate, the origin is not a vertex, and $0 \in \overline{Q}$, then the quadrilateral $\widehat{Q}$ on the unit circle obtained by using the polar representation of (3.4) is non-degenerate and the origin is either internal to $\widehat{Q}$ or on just one edge. Recall that $\widehat{Q}$ is the quadrilateral $\widehat{w}_1, \widehat{w}_2, \widehat{w}_4, \widehat{w}_3$.

From Proposition 3.3, the quadrilateral obtained from $\widehat{Q}$ reflecting with respect to the origin the diagonal joining $\widehat{w}_2$ and $\widehat{w}_3$ is convex. That is, the quadrilateral of vertices $\widehat{w}_1, -\widehat{w}_2, \widehat{w}_4, -\widehat{w}_3$, is convex. This means that the signed areas of the triangles $(\widehat{w}_1, -\widehat{w}_2, -\widehat{w}_3)$, $(\widehat{w}_1, -\widehat{w}_2, \widehat{w}_4)$, $(\widehat{w}_1, \widehat{w}_4, -\widehat{w}_3)$, and $(-\widehat{w}_2, \widehat{w}_4, -\widehat{w}_3)$, all have the same sign. [Since $\widehat{Q}$ is non-degenerate, some but not all of these areas may be 0].

By looking at the determinants appearing in (3.5), we recognize them exactly as the areas of the aforementioned triangles, and therefore all the components of $M_{\text{adj}}(3,:)$ have the same sign, and then, by (3.3), λ_{M} is the unique solution of (3.1), further admissible since $\sum_{j=1}^{4} M_{\text{adj}}(3, j) = d^\top v$.

The fact that λ_{M} varies smoothly with $x \in \Sigma$ is a consequence of the smoothness of the determinant with respect to the matrix entries. ☐

Corollary 3.5. *If the quadrilateral Q is non-degenerate, and the origin is internal to Q, then $\lambda_M > 0$; i.e., all components of λ_M are positive.*

Proof. Let the origin be in the interior of Q. By Theorem 3.4, λ_{M} is therefore admissible. Let us assume, by contradiction, that for some $i = 1, 2, 3, 4$, $(\lambda_{\text{M}})_i = 0$: without loss of generality, let $(\lambda_{\text{M}})_1 = 0$. Looking at (3.5), this happens if and only if the area of the triangle on the unit circle with vertices $-\widehat{w}_2$, $\widehat{w}_4$, $-\widehat{w}_3$ is zero; but this is equivalent to say that either $-\widehat{w}_2 = \widehat{w}_4$ or $\widehat{w}_4 = -\widehat{w}_3$, which in turn is true if and only if the origin belongs to either w_2w_4 or w_4w_3, that

is to the boundary of Q, which contradicts the assumption.
A similar argument holds for the other cases. ☐

Remark 3.6. *Suppose that the origin is on the segment w_1w_2 and it is not a vertex (similarly, for any other side of the quadrilateral). Then, the unique solution of* (3.1)*, under the assumptions of Theorem 3.4, is*

$$\lambda := \begin{bmatrix} \frac{d_2}{d_1+d_2} \\ \frac{d_1}{d_1+d_2} \\ 0 \\ 0 \end{bmatrix}.$$

Remark 3.7. *As we said, our motivation was in validating the moments method under the conditions of partial nodal attractivity. Theorem 3.4 does achieve this. But in fact, it does more, only needing nondegeneracy of Q and that the origin be either inside Q or on at most one edge. In particular, Theorem 3.4 validates the moments method also in the case of Σ being spirally attractive, see* [4]*. This is simply because, when Σ is spirally attractive, the origin is inside Q, see Lemma 2.3.*

As a consequence of Theorem 3.4, we have the following result, which will be useful in Section 5.

Theorem 3.8. *Let $x \in \Sigma$, let w_i, $i = 1, \dots, 4$, be given in* (1.11) *(these vertices of course depend on x), let Q be the quadrilateral of Definition 2.2, and let M be given in* (1.18)*. Assume that Q is non-degenerate and that $w_i \neq 0$, for all $i = 1, \dots, 4$.*

Then, for each $\epsilon > 0$ sufficiently small, if $dist(0, Q) := \min_{y\in\overline{Q}} \|y\| < \epsilon$, the matrix M in (1.18) *is invertible. Moreover, if $0 \notin \overline{Q}$, then the unique solution of* (1.18) *is not admissible.*

Proof. Since the determinant function is continuous as a function of the entries of W, and $\det(M) \neq 0$ as $0 \in \overline{Q}$, then $\det(M) \neq 0$ if 0 is sufficiently close to $\overline{Q}$.
If $0 \notin \overline{Q}$, then since M is nonsingular the unique solution $\lambda_{\rm M}$ of (1.18) is still given by (3.3). But, looking at the signed areas in (3.5), we see that two of them are negative, making $\lambda_{\rm M}$ not admissible. ☐

3.1. **One vertex of Q at the origin.** Our results, particularly the construction of the quadrilateral $\widehat{Q}$ and therefore Theorem 3.4, have relied on the assumption that $w_i \neq 0$, for every $i = 1, 2, 3, 4$. As we will clarify below, this is a very mild and natural assumption, both in terms of the problem dynamics and of the geometrical interpretation of the same. At the same time, let us consider here the case when this assumption is violated, and what it implies.

First of all, *if two or more* of the w_i's were zero, then the quadrilateral Q would be degenerate, and as a consequence (see (1.15) and (3.1)) W would be of rank 2, and M would be singular; so, the moments regularization would not be of any use. Moreover, the problem dynamics would be inherently ambiguous since two of the w_i's being 0 (say $w_1 = w_2 = 0$), implies that there are two admissible exit vector fields in two different regions R_i's (say, in R_1 and R_2). Finally, note that this case of two w_i's equal to 0 is a co-dimension 4 phenomenon.

Suppose now that *there is just one* index $i = 1, 2, 3, 4$, for which $w_i = 0$. In this case, something more can be said. Without loss of generality, suppose that we are at a point $x \in \Sigma$ where $w_1 = 0$, and $w_i \neq 0$, $i = 2, 3, 4$.

(a) In terms of the problem's dynamics, $w_1 = 0$ means that the vector field f_1 is itself tangent to Σ, and therefore f_1 is an exit vector field. Clearly, this is not a first order exit condition (which is a co-dimension 1 phenomenon), and it is a co-dimension 2 phenomenon. Moreover, it is not clear that we can *predict* the dynamics after this situation occurs. See Example 3.11 below.

(b) In terms of the quadrilateral Q, if Q is non-degenerate, then there is still a unique solution to (1.18), as we show below.

Lemma 3.9. *If $w_1 = 0$, and Q is non-degenerate, then the matrix*

$$N = \begin{bmatrix} w_2 & w_3 & w_4 \\ -d_2 & -d_3 & d_4 \end{bmatrix}$$

is invertible.

Proof. Suppose not. Then, without loss of generality we have $\begin{bmatrix} w_2 \\ -d_2 \end{bmatrix} = \alpha \begin{bmatrix} w_3 \\ -d_3 \end{bmatrix} + \beta \begin{bmatrix} w_4 \\ d_4 \end{bmatrix}$ for some α, β, not both 0. Then, we have $w_2 = \alpha w_3 + \beta w_4$ and $-d_2 = \alpha d_3 + \beta d_4$. From the first relation, we get

$$d_2^2 = \alpha^2 d_3^2 + \beta^2 d_4^2 + 2\alpha\beta w_3^\top w_4$$

and from the second one we get

$$d_2^2 = \alpha^2 d_3^2 + \beta^2 d_4^2 - 2\alpha\beta d_3 d_4\,.$$

Comparing these two expressions for d_2^2, we get the following.

(i) If both α and β are nonzero, then we must have $w_3^\top w_4 = -d_3 d_4$. From the Cauchy-Schwartz inequality, this implies that w_3 and w_4 are aligned with 0 and so Q would be degenerate, which is a contradiction.
(ii) Now suppose just one of α or β is 0. If $\alpha = 0$, then w_2 and w_4 would need to be aligned with the origin. If $\beta = 0$, then w_2 and w_3 would need to be aligned with the origin. Either way, Q would be degenerate and we reach a contradiction.

□

As a consequence of Lemma 3.9, we have the following result.

Theorem 3.10. *Let $x \in \Sigma$ and w_i, $i = 1, 2, 3, 4$, and Q be defined as usual. Suppose that, at such x, $w_i = 0$ for an index i, and $w_j \neq 0$, $j \neq i$, and let Q be non-degenerate.*

Then, the moments matrix M is invertible, and (1.18) has a unique admissible solution λ_M:

$$(\lambda_M)_j = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}, \quad j = 1, 2, 3, 4.$$

Moreover, as long as Q remains non-degenerate, the solution λ_M is continuous, but not differentiable, in $x \in \Sigma$.

Proof. Without loss of generality, let $w_1 = 0$, so that (1.18) rewrites as:

$$\begin{bmatrix} 0 & w_2 & w_3 & w_4 \\ 1 & 1 & 1 & 1 \\ 0 & -d_2 & -d_3 & d_4 \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

Clearly, $\lambda_{\mathrm{M}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ solves this system. The solution is further unique since N (defined as in Lemma 3.9) is invertible.

Continuity of λ_{M} is a consequence of continuity and invertibility of M with respect to x. Lack of differentiability is due to lack of smoothness at the origin for the square root function (viz., for $\|\cdot\|$). □

The above lack of smoothness is responsible for the difficulties one may have in locating an exit point where $w_1 = 0$, and hence to properly predict the dynamics past such an exit point.

Example 3.11. *Consider the dynamics on Σ embodied by the points*

$$w_1 = \begin{bmatrix} -t \\ t \end{bmatrix}, \quad w_2 = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad w_4 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \text{and } -1 \le t \le 0.5 . \tag{3.6}$$

As long as $t < 0$, Σ is attractive and we have well defined sliding motion on Σ. At $t = 0$, $w_1 = 0$ and for $t > 0$ the origin exits the quadrilateral Q: attractivity is violated, and a co-dimension 2 exit phenomenon from Σ into R_1 is taking place. However, suppose we continue following the trajectory on Σ (this can be done because of Theorem 3.8). The components of the moments solution λ_M behave as in Figure 3, and we observe that two of them (here, λ_3 and λ_4) change of sign through this non-generic exit point. A naive application of Theorem 5.7 below may lead us to believe that exiting and sliding on Σ_2^- with $f_{\Sigma_2^-}$ should be taking place past the exit point, rather than exiting onto R_1.

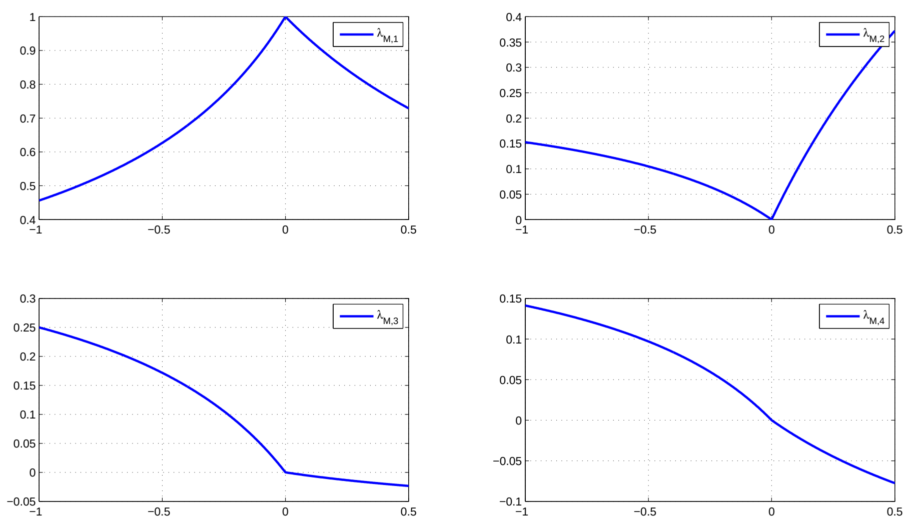

FIGURE 3. Solution components of λ_M for the dynamics given by (3.6).

□

4. EXAMPLES: COMPARING BILINEAR AND MOMENTS SOLUTIONS

Our purpose in this section is to show some numerical experiments with the moments method and compare it (qualitatively) to the bilinear interpolation technique (see Remark 1.8) insofar as sliding on Σ.

The basic numerical integration scheme is a 4^{th} order embedded Runge-Kutta pair based on the $\frac{3}{8}$-th Runge-Kutta method, with Butcher's tableau

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 & 0 & 0 \\
2/3 & -1/3 & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 \\
\hline
b & 1/8 & 3/8 & 3/8 & 1/8 & 0 \\
1 & 1/8 & 3/8 & 3/8 & 3/8 & 0 \\
\hline
\widehat{b} & 1/12 & 1/2 & 1/4 & 0 & 1/6
\end{array} .
$$

Adaptive step size control is done as suggested in [12]:

$$h_{\text{new}} := h \cdot \min\left\{ \mathit{facmax}, \max\left\{ \mathit{facmin}, \mathit{fac} \cdot \left(\frac{1}{\mathit{err}}\right)^{\frac{1}{q+1}} \right\}\right\},$$

where h is the current step size, $q := \min\{p, \widehat{p}\}$, being p the order of the Runge-Kutta scheme and $\widehat{p}$ the order of the error estimator, and we have chosen

$$\mathit{facmax} = 5 \ , \ \ \mathit{facmin} = \text{eps} \ \ (\approx 10^{-16}) \ , \ \ \mathit{fac} = 0.8 \ ,$$

and

$$\mathit{Err} := \left(\frac{|x_i - \widehat{x}_i|}{1 + |x_i| \cdot \mathit{tol}}\right)_{i=1:n} \quad , \quad \mathit{err} := \|\mathit{Err}\|_\infty \ ,$$

where tol is a given error tolerance (below, $\mathit{tol} = 10^{-6}$).

The overall method is an *event driven method* (according to the naming in [1]), whereby different regimes (entering and exiting from the discontinuity manifolds) are monitored, and the appropriate vector fields are integrated. Integration in the regions R_i's ($i = 1, 2, 3, 4$) is standard. Integration during sliding motion is done with a projected version of the basic integration scheme to guarantee that the stage values and the computed approximations remain on the discontinuity manifold(s).

In all problems below, integration on $\Sigma = \Sigma_1 \cap \Sigma_2$ will proceed according to two different choices of convex combination coefficients, and the associated vector fields: the coefficients λ_{B} used to form the bilinear vector field in (1.16)-(a), and the moments coefficients λ_{M} used to form the moments vector field. Let us stress that λ_{B} is found by solving the nonlinear system (1.16)-(b) for α and β; as the bilinear trajectory evolves on Σ, the coefficients α, β, are updated by continuation with respect to the value at the previous integration step.

Example 4.1. This is a problem in $\mathbb{R}^3$. We have (1.9) with $x(0) = \begin{bmatrix} -0.1 \\ -0.1 \\ -0.1 \end{bmatrix}$, $\Sigma_1 := \{x \in \mathbb{R}^3 : h_1(x) := x_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^3 : h_2(x) := x_2 = 0\}$, and $\Sigma := \Sigma_1 \cap \Sigma_2$ is just the x_3-axis. The vector fields are given by

$$f_1(x) := \begin{bmatrix} \frac{\sqrt{2}}{8} \sin\left(\frac{\pi}{4} - x_3^2\right) \\ \frac{\sqrt{2}}{8} \cos\left(\frac{\pi}{4} - x_3^2\right) \\ x_1^2 + x_2^2 + 1 \end{bmatrix} \ , \ f_2(x) := \begin{bmatrix} 2\sqrt{2} \sin\left(\frac{3}{4}\pi - x_3^2\right) \\ \sqrt{2} \cos\left(\frac{3}{4}\pi - x_3^2\right) \\ x_1^2 + x_2^2 + 1 \end{bmatrix} \ ,$$

$$f_3(x) := \begin{bmatrix} \sqrt{2} \sin\left(\frac{\pi}{4} - 2x_3^2\right) \\ \sqrt{2} \cos\left(\frac{\pi}{4} - 2x_3^2\right) \\ x_1^2 + x_2^2 + 1 \end{bmatrix} \ , \ f_4(x) := \begin{bmatrix} -2 \\ -1 \\ x_1^2 + x_2^2 + 1 \end{bmatrix} \ .$$

Since $x(0) \in R_1$, we integrate $\dot{x} = f_1(x)$, until we hit Σ_2^- transversally at $\xi_1 \approx \begin{bmatrix} -0.0208 \\ 0 \\ 0.6320 \end{bmatrix}$.
Notice that Σ_2^- is attractive, since (see (1.5))

$$f_1(\xi_1) \approx \begin{bmatrix} 0.0665 \\ 0.1638 \\ 1.0004 \end{bmatrix}, \quad f_2(\xi_1) \approx \begin{bmatrix} 2.6204 \\ -0.5324 \\ 1.0004 \end{bmatrix} .$$

Thus, from ξ_1, the trajectory starts sliding on Σ_2^- directed towards Σ with vector field

$$f_{\Sigma_2^-}(x) := (1 - \alpha_{\Sigma_2^-}(x)) f_1(x) + \alpha_{\Sigma_2^-}(x) f_2(x) \ , \ \ \alpha_{\Sigma_2^-}(x) := \frac{w_1^2(x)}{w_1^2(x) - w_2^2(x)} .$$

At $\xi_2 \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.6619 \end{bmatrix}$, the trajectory reaches Σ transversally. At this point, Σ is nodally attractive, since

$$W(\xi_2) \approx \begin{bmatrix} 0.0602 & 2.6596 & -0.1285 & -2 \\ 0.1662 & -0.4812 & 1.4084 & -1 \end{bmatrix} .$$

Observe that there is a unique Filippov sliding vector field (1.14) on Σ, namely $\dot{x}_3 = 1$; however, λ_{B} and λ_{M} are different.

With both the bilinear and moments methods the solution trajectory eventually reaches the first order exit point

$$\xi_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{\pi}{2}} \end{bmatrix} , \quad \text{where } W(\xi_3) = \begin{bmatrix} -\frac{1}{8} & 2 & -1 & -2 \\ \frac{1}{8} & 1 & -1 & -1 \end{bmatrix} .$$

For values of x_3 greater than $\sqrt{\frac{\pi}{2}}$, Σ looses attractivity, and thus, at this value we consider it to be desirable that also sliding motion on Σ ceases, and –for this reason– at ξ_3 we would like to leave Σ sliding on Σ_1^+. Depending on whether we have λ_{B} or λ_{M}, however, we witness very different behaviors as we reach ξ_3.

As Figure 4 shows, at ξ_3 the bilinear solution λ_{B} has all positive components. Instead, the moments solution λ_{M} at ξ_3 has its first and third components equal to zero: these are exactly the components of λ that do not play a role when sliding on Σ_1^+ starts; indeed, at ξ_3, λ_{M} provides the exit vector field on the sub-manifold Σ_1^+, that is $f_{\mathrm{F}_1^+}$ (see (1.12)). Moreover, we note that if we force integration on Σ past ξ_3 for the moments trajectory (note that the moments' matrix remains invertible, at least near ξ_3, because of Theorem 3.8), then the first and third components become negative past the exit point, hence the moments solution is not admissible. [This fact provides a powerful characterization of first order exit points, and a very useful criterion to detect them numerically.]

As far as the bilinear solution, at ξ_3 (1.16)-(b) must have two admissible solutions (see [7]): the solution (α^*, β^*) we had been following (which gives λ_{B} in Figure 4), and a new one, necessarily being $(\alpha^+, 1)$ which has "entered" the admissible region. As shown in [7], the nonlinear system (1.16)-(b) reduces to a quadratic equation in β, which at ξ_3 has the two roots β^* and 1. We stress that, by solving the nonlinear system (1.16)-(b) by continuation, the "new entering" root goes unnoticed. To sum up, assuming that, somehow, all roots of the nonlinear system (1.16)-(b) are monitored, one could force the trajectory to exit at ξ_3, but following the solution (α^*, β^*) we have been continuing gives no indication that a first order exit point has been reached; all components of λ_{B} remain positive past ξ_3, even though Σ is no longer attracting. Moreover, as Figure 5 shows, if we do not exit Σ at ξ_3 and continue integrating on Σ with f_{B} (using the continuation

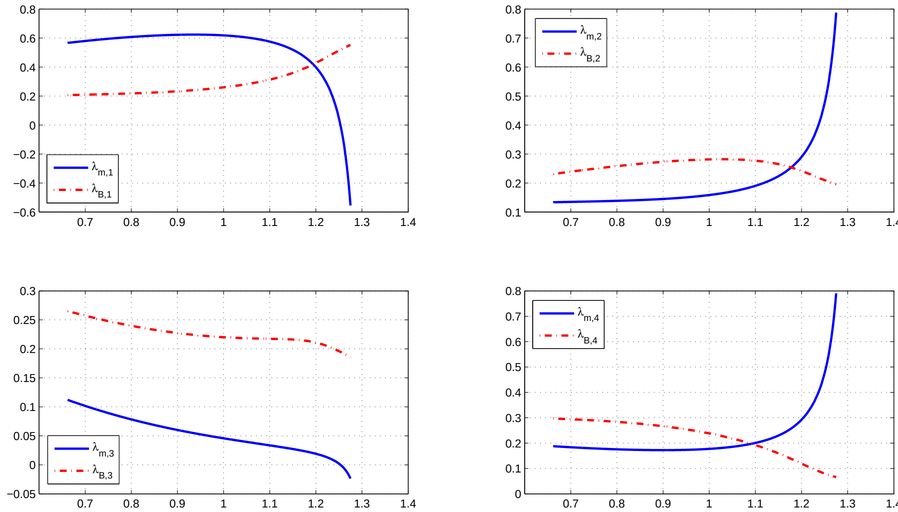

FIGURE 4. Moments and bilinear solutions for $x_3 \in [0.6619..., 1.275]$.

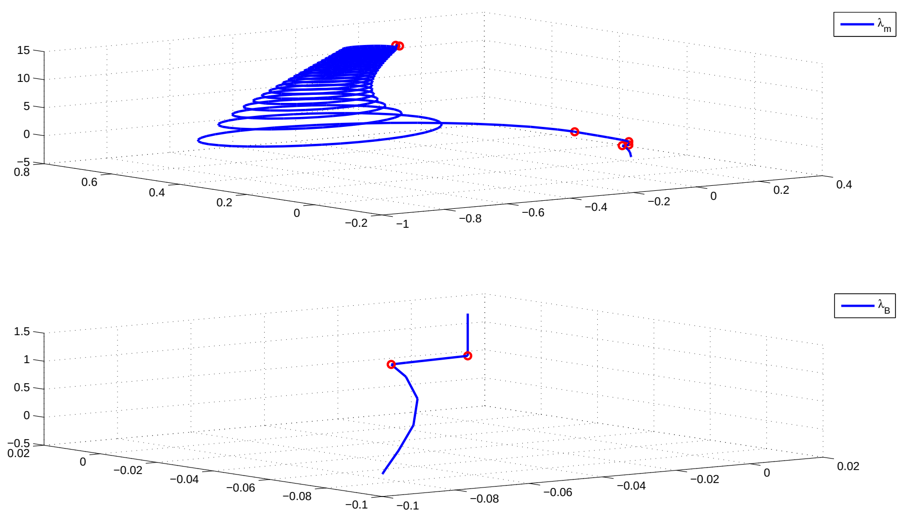

FIGURE 5. Moments and bilinear trajectories for Example 4.1.

of (α^*, β^*)), then the bilinear solution develops a singularity. Namely, at $x_s \approx \begin{bmatrix} 0 \\ 0 \\ 1.4163 \end{bmatrix}$, λ_B becomes complex valued, and motion on Σ with f_B ceases to make sense. [This last fact is easy to explain, since the roots of the above parabola in β collide at x_s and become complex valued.]

□

Remark 4.2. *In Example 4.1, we have a system in $\mathbb{R}^3$, Σ is a straight line, and all sliding trajectories satisfy $\dot{x}_3 = 1$. In particular, using either λ_B or λ_M, a sliding trajectory must reach the point ξ_3 above. Although, in principle, both bilinear and moments trajectories could exit at ξ_3, there is a major difference in what happens to λ_B or λ_M if we let the trajectory continue on Σ past ξ_3. At first, λ_B has all components positive and seemingly well behaved, and it does not betray that the origin has gone outside of the quadrilateral Q. On the other hand, λ_M has two components going to 0 at ξ_3, and then becoming negative. This is an important fact, which betrays that the origin has exited the quadrilateral Q, and that allows automatic detection of exit points, as we will elaborate in Section 5.*

In the next example, we show that, in general (that is, when the phase space is not $\mathbb{R}^3$, nor $\mathbb{R}^2$), even when they seemingly are both well defined and exit smoothly, the moments and bilinear methods lead to different dynamics, and the bilinear solution may again eventually develops a singularity, similarly to Example 4.1.

Example 4.3. We have (1.9) with $x(0) = \begin{bmatrix} -0.1 \\ -0.1 \\ -0.1 \\ 0.1 \end{bmatrix}$, $\Sigma_1 := \{x \in \mathbb{R}^4 \,:\, h_1(x) := x_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^4 \,:\, h_2(x) := x_2 = 0\}$, and $\Sigma := \Sigma_1 \cap \Sigma_2$ is the (x_3, x_4) plane. The vector fields are given by

$$f_1(x) := \begin{bmatrix} \frac{\sqrt{2}}{8}\sin\left(\frac{\pi}{4} - x_4 x_3^2\right) \\ \frac{\sqrt{2}}{8}\cos\left(\frac{\pi}{4} - x_3^2\right) \\ x_1^2 + x_2^2 + 1 \\ x_1 + x_2 + x_3 + x_4^2 \end{bmatrix} \ , \ f_2(x) := \begin{bmatrix} 2\sqrt{2}\sin\left(\frac{3}{4}\pi - x_3^2\right) \\ \sqrt{2}\cos\left(\frac{3}{4}\pi - x_4 x_3^2\right) \\ x_4 x_1^2 + x_2^2 + 1 \\ e^{x_1} + x_2 + x_3^2 + x_4 \end{bmatrix} \ ,$$

$$f_3(x) := \begin{bmatrix} \sqrt{2}\sin\left(\frac{\pi}{4} - 2x_3^2\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4} - 2x_3^2\right) \\ x_1^2 + x_2^2 + 1 \\ x_1 + e^{x_2} + x_3 x_4 \end{bmatrix} \ , \ f_4(x) := \begin{bmatrix} -2 \\ -1 \\ x_1 + x_2 \\ x_3 x_4^2 + 1 \end{bmatrix} \ .$$

We integrate $\dot{x} = f_1(x)$ until the trajectory reaches Σ_2^- transversally at $\xi_1 \approx \begin{bmatrix} -0.0108 \\ 0 \\ 0.6319 \\ 0.2256 \end{bmatrix}$, where Σ_2^- is attractive, since (see (1.5))

$$f_1(\xi_1) \approx \begin{bmatrix} 0.1132 \\ 0.1638 \\ 1.0001 \\ 0.6721 \end{bmatrix} \ , \quad f_2(\xi_1) \approx \begin{bmatrix} 2.6203 \\ -0.9060 \\ 1.0000 \\ 1.6142 \end{bmatrix} \ .$$

There is sliding motion on Σ_2^- directed towards Σ, with vector field

$$f_{\Sigma_2^-}(x) := (1 - \alpha_{\Sigma_2^-}(x)) f_1(x) + \alpha_{\Sigma_2^-}(x) f_2(x) \ , \ \alpha_{\Sigma_2^-}(x) := \frac{w_1^2(x)}{w_1^2(x) - w_2^2(x)} \ .$$

At $\xi_2 \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.6533 \\ 0.2436 \end{bmatrix}$, the trajectory reaches Σ transversally (see Figure 6). Since at ξ_2 we have

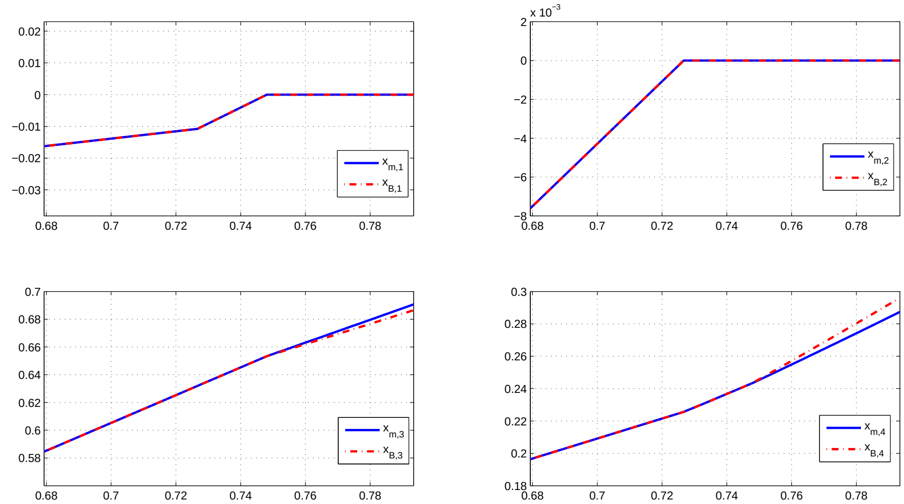

FIGURE 6. First enter on Σ of moments and bilinear trajectories.

$$W(\xi_2) = \begin{bmatrix} 0.1114 & 2.6486 & -0.0966 & -2 \\ 0.1655 & -0.8908 & 1.4110 & -1 \end{bmatrix},$$

then Σ is (at least, near ξ_2) nodally attractive. We slide on Σ using either f_{B} or f_{M}. The respective solution trajectories now follow different paths on Σ, but eventually both reach the *curve of first order exit points* given by

$$x_4 = -1 + \frac{\pi}{x_3^2} \ .$$

Remark 4.4. *For this problem, exit curves on Σ are directly computable, and are given by:*

$$x_4 = 1 - \frac{\pi + 4k\pi}{2x_3^2} \ , \ k \in \mathbb{Z} \ , \quad \text{and } x_4 = -1 + \frac{\pi + 2k\pi}{x_3^2} \ , \ k \in \mathbb{Z} \ .$$

As Figure 7 shows, the moments and the bilinear trajectories exit (both of them smoothly) at different positions on the same exit curve. Namely, the moments and bilinear trajectories exit at

$$\xi_3^{(M)} \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.1725 \\ 1.2851 \end{bmatrix} , \quad \text{respectively} \quad \xi_3^{(B)} \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.1285 \\ 1.4670 \end{bmatrix} ,$$

with coefficients

$$\lambda_{\mathrm{M}}(\xi_3^{(M)}) \approx \begin{bmatrix} 0 \\ 0.4596 \\ 0 \\ 0.5404 \end{bmatrix} , \quad \text{respectively} \quad \lambda_{\mathrm{B}}(\xi_3^{(B)}) \approx \begin{bmatrix} 0 \\ 0.4446 \\ 0 \\ 0.5554 \end{bmatrix} .$$

After they exit, as shown in Figure 8, trajectories evolve in Σ_1^+ until both of them again reach Σ transversally, but at different points: namely, the moments trajectory enters Σ at

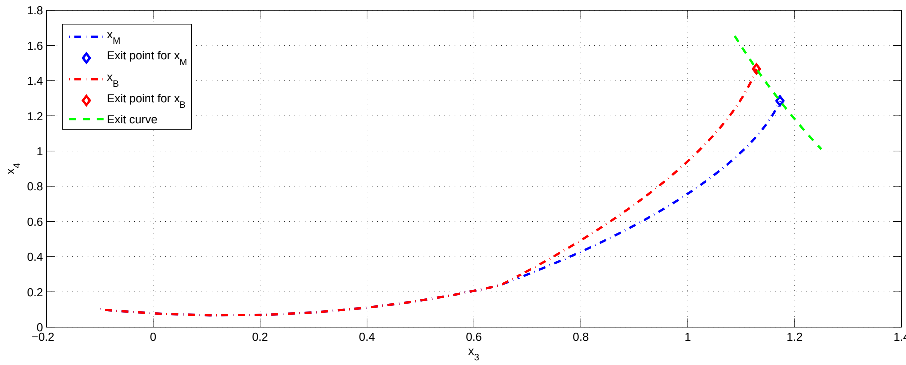

FIGURE 7. Projection of moments and bilinear trajectories in the (x_3, x_4) plane during sliding motion.

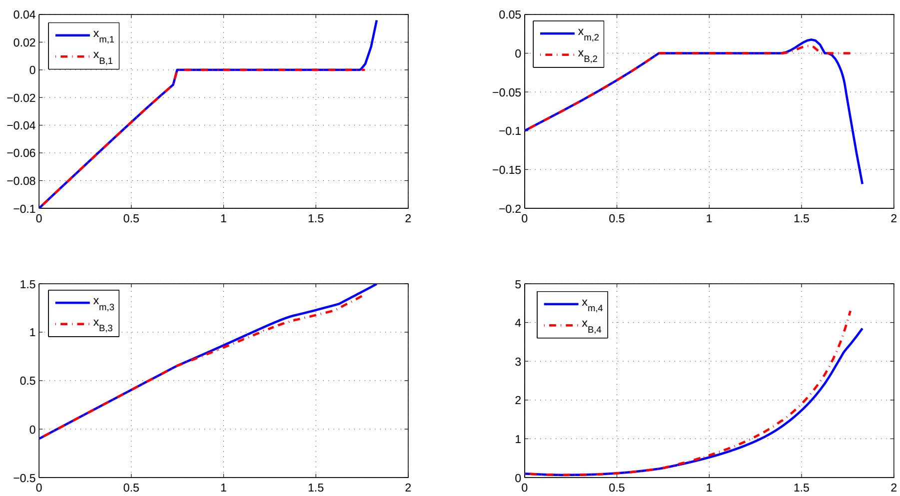

FIGURE 8. Solution components of moments and bilinear trajectories.

$\xi_4^{(M)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.2923 \\ 2.4231 \end{bmatrix}$, whereas the bilinear trajectory enters Σ at $\xi_4^{(B)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.2236 \\ 2.4714 \end{bmatrix}$. After a short sliding regime on Σ, the moments trajectory exits Σ smoothly at $\xi_5^{(M)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.3050 \\ 2.5150 \end{bmatrix}$, where

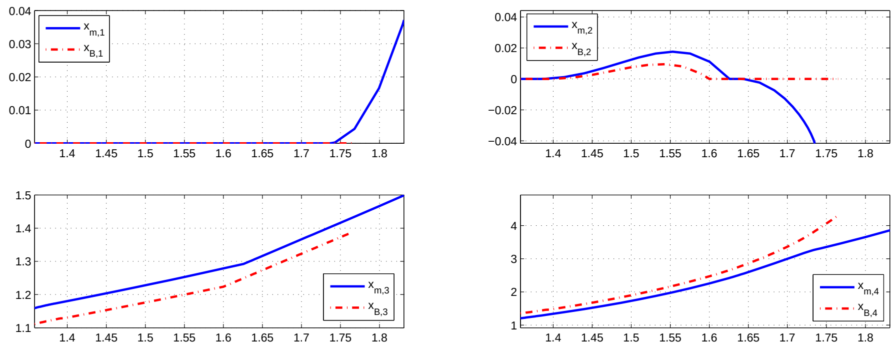

FIGURE 9. Second sliding on Σ: moments trajectory evolves properly, whereas bilinear trajectory develops a singularity after passing through a first order exit point.

$\lambda_{\mathrm{M}}\left(\xi_5^{(M)}\right) \approx \begin{bmatrix} 0.9195 \\ 0 \\ 0.0805 \\ 0 \end{bmatrix}$. On the other hand, during this second sliding motion on Σ, the bilinear trajectory passes through a first order exit point, and eventually the coefficients become complex valued[6] at $\xi_5^{(B)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.3874 \\ 4.3001 \end{bmatrix}$. See Figure 9 for a magnification of this phenomenon.

After $\xi_5^{(M)}$, the moments trajectory begins sliding on Σ_1^-, from where it exits at $\begin{bmatrix} 0 \\ -0.0427 \\ 1.4022 \\ 3.2765 \end{bmatrix}$, entering R_3; once there, the moments trajectory eventually reaches Σ_2^+ transversally at $\xi_6^{(M)} \approx \begin{bmatrix} 0.3529 \\ 0 \\ 1.9684 \\ 8.6016 \end{bmatrix}$. Then, after sliding on Σ_2^+, it exits from there at $\xi_7^{(M)} \approx \begin{bmatrix} 0.1363 \\ 0 \\ 2.0794 \\ 164.9307 \end{bmatrix}$ moving into R_3. At $\xi_8^{(M)} \approx \begin{bmatrix} 0 \\ -0.0737 \\ 2.1942 \\ 210.5975 \end{bmatrix}$ the moments trajectory reaches transversally Σ_1^-, slides on it and leaves it at $\xi_9^{(M)} \approx \begin{bmatrix} 0 \\ -0.0738 \\ 2.1953 \\ 266.7981 \end{bmatrix}$ and enters in R_3 again. □

[6]The explanation of why the bilinear trajectory we are following does not notice the generic first order exit point, and why the bilinear coefficients eventually become complex valued, is much like the explanation we provided in Example 4.1

5. Smooth exits for the moments method and extensions

In this section, we first show that –at generic first order exit points on Σ– the moments solution renders (automatically) the coefficients for the exit vector field. Then, we briefly discuss other possibilities to regularize the underdetermined system (1.15), by appending to it a linear constraint, similarly to what we did in (1.18), and ascertain when/how this will render an admissible solution λ.

5.1. **Smooth exits.**

As shown in Figure 4 relative to the Example 4.1, when the moments trajectory reached a generic first order exit point, two components of the moments solution (i.e., of the vector λ_{M}) became zero, and the other two gave the coefficients of cthe exit vector field. In fact, more was observed to be true. Since the matrix M remained invertible (see Theorem 3.8), the solution of (3.1) could be continued past the exit point, and a trajectory sliding on Σ according to f_{M} continued to exist; however, the moments solution was no longer admissible, since the two components that had become 0 at the exit point eventually became negative. This is a general behavior, that here we are going to justify rigorously. It is also a very important and useful fact, because it allows us to **detect** that an exit point is reached, and thus to eventually leave Σ smoothly at the exit point.

First, we have the following simple result.

Lemma 5.1. *Let $T = ABC$ be a planar triangle of vertices A, B, and C, joined in this order. Then,*

$$sgn\,\mathcal{A}(AB\widehat{C}) = -sgn\,\mathcal{A}(ABC)\ ,$$

where $\widehat{C}$ is the reflection of C with respect to the origin, and $\mathcal{A}$ indicates the signed area.

Proof. The result follows from the fact that if ABC proceeds clockwise, then $AB\widehat{C}$ has counterclockwise ordering, and vice versa. □

Next, we need the following concept.

Definition 5.2 (Origin exiting along an edge)**.** *Let $x(t)$, $0 \le t \le T$, be the smooth trajectory on Σ associated to the moments vector field, where the time interval is a time interval for which the trajectory is well defined (i.e., the associated matrix M in* (1.18) *is invertible). Assume that there is a neighborhood of the trajectory, $U(x)$, such that $\Sigma \cap U(x)$ is attractive for values of t in some interval $0 \le t \le t_0$, $0 < t_0 \le T$. Let $Q(x(\cdot))$ be the quadrilateral associated to this trajectory, and let Q be non-degenerate, and such that that none of the vertices of Q be at the origin.*

Then, we say that the origin is exiting Q along *the edge w_1w_2 if and only if, by definition, the following occur:*

(i) *there exists a time $t_e > 0$ such that $x(t_e) \in \Sigma$, $\mathcal{A}_{120}(x(t_e)) = 0$, and for all $t:\ 0 \le t < t_e$, $\mathcal{A}_{120}(x(t)) \neq 0$, $\mathcal{A}_{240}(x(t)) \neq 0$, $\mathcal{A}_{430}(x(t)) \neq 0$, and $\mathcal{A}_{310}(x(t)) \neq 0$. Here, $\mathcal{A}_{120}$ is the signed area of the triangle with vertices w_1, w_2 and the origin, and similarly for $\mathcal{A}_{240}$ and so forth;*

(ii) *there exists an open interval $\mathcal{I}_e$ centered at t_e and contained in $[0, T]$, such that for all $t_1, t_2 \in \mathcal{I}_e$, with $t_1 < t_e < t_2$, then the following inequality holds:*

$$\mathcal{A}_{120}(x(t_1)) < 0 < \mathcal{A}_{120}(x(t_2))\ ; \tag{5.1}$$

(iii) *for all $t \in \mathcal{I}_e$, $\mathcal{A}_{240}(x(t)) \neq 0$, $\mathcal{A}_{430}(x(t)) \neq 0$, and $\mathcal{A}_{310}(x(t)) \neq 0$.*

Analogous definitions hold for the origin exiting along the other edges of the quadrilateral Q, that is along w_2w_4, w_4w_3, w_3w_1. The value of t_e above is called (first) exit time *for the moments trajectory.* □

Remark 5.3. *The above definition characterizes the situation when –following the moments solution trajectory on* Σ*– the origin ends up outside the quadrilateral* Q *after having encountered a first order exit point. In this case, since at* t_e *we have* $w_i(t_e) \neq 0$, $i = 1, 2, 3, 4$, *then it is meaningful to determine along which edge of* Q *the origin exited. See Lemmata 1.4 and 1.10 for motivation on the inequality* (5.1)*.*

In the Lemma below, we will use normalized barycentric coordinates of the origin with respect to a triangle. Let us recall these.

Notation 5.4. *For a given planar triangle* T_{ABC} *of distinct vertices* $A \equiv (x_A, y_A)$, $B \equiv (x_B, y_B)$, $C \equiv (x_C, y_C)$, *the normalized barycentric coordinates of the origin are given by the triplet* (τ_A, τ_B, τ_C) *satisfying the system*

$$\left\{ \begin{array}{l} \tau_A \begin{bmatrix} x_A \\ y_A \end{bmatrix} + \tau_B \begin{bmatrix} x_B \\ y_B \end{bmatrix} + \tau_c \begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \tau_A + \tau_B + \tau_C = 1 \ . \end{array} \right. \tag{5.2}$$

In particular, all coordinates are in $[0, 1]$ *whenever* $0 \in \overline{T_{ABC}}$, *and if any of them is negative then* 0 *is external to the triangle. Finally, if we need to specify the coordinates of a vertex with respect to the specific triangle* T_{ABC}, *we will write* $(\tau_A^{ABC}, \tau_B^{ABC}, \tau_C^{ABC})$*.* ☐

Lemma 5.5. *With the notation of Definition 5.2, suppose that the origin exited* Q *along* w_1w_2*. Let* $t \in \mathcal{I}_e$, $t > t_e$, *so that* $0 \notin \overline{Q(x(t))}$*. For any such* t, *let* w_i, $i = 1, 2, 3, 4$, *be the vertices of* Q, *and let* T_{ijk} *be the triangles of vertices* w_i, w_j, w_k *(in this order), for different indices* $i, j, k \in 1, 2, 3, 4$*.*

Then:

$$0 \in \overline{T}_{123^\dagger} \ , \quad or \ \ 0 \in \overline{T}_{124^\dagger} \ ,$$

where $w_3^\dagger$ *and* $w_4^\dagger$ *are, respectively, the reflections of* w_4 *and* w_3 *with respect to the origin.*

Proof. For simplicity, below we will omit writing the dependence on the point $x(t)$, and simply write Q for $Q(x(t))$, and so forth.

Since Q is not degenerate, and $0 \notin \overline{Q}$, then $0 \notin \overline{T}_{123}$ or $0 \notin \overline{T}_{124}$ (both could be true, of course). Suppose that $0 \notin \overline{T}_{123}$.

Consider the triangle $T_{124^\dagger}$ of vertices $w_1, w_2, w_4^\dagger$, and look at the normalized barycentric coordinates of the origin with respect to $T_{124^\dagger}$. Note that $T_{124^\dagger}$ cannot be degenerate. (In fact, assume it was: then $w_4^\dagger \in w_1w_2$, and hence the entire segment with extrema $w_4^\dagger$ and its transformed with respect to the origin, that is w_3, would be contained in T_{123}. In particular, this would imply that $0 \in T_{123}$, which is a contradiction.)

Therefore, from (5.2), using Cramer's rule and Lemma 5.1, we get

$$\begin{aligned} \tau_{4^\dagger}^{124^\dagger} &= \frac{\det \begin{bmatrix} w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}{\mathcal{A}(w_1w_2w_4^\dagger)} = \frac{\det \begin{bmatrix} w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}{|\mathcal{A}(w_1w_2w_4^\dagger)| \operatorname{sgn} \mathcal{A}(w_1w_2w_4^\dagger)} = \\ &= -\frac{|\mathcal{A}(w_1w_2w_3)|}{|\mathcal{A}(w_1w_2w_4^\dagger)|} \frac{\det \begin{bmatrix} w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}{|\mathcal{A}(w_1w_2w_3)| \operatorname{sgn} \mathcal{A}(w_1w_2w_3)} = -\frac{|\mathcal{A}(w_1w_2w_3)|}{|\mathcal{A}(w_1w_2w_4^\dagger)|} \tau_3^{123} > 0 \ , \end{aligned}$$

since $\tau_3^{123} < 0$, being $0 \notin \overline{T}_{123}$. Similarly for the other possibilities. ☐

Corollary 5.6. *With same notation as in Lemma 5.5, let* $0 \notin \overline{Q}$ *and assume the origin exited along* w_1w_2*. Then, the origin is in the interior of* $\widehat{Q}$, *where* $\widehat{Q}$ *has vertices* w_1, w_2, $w_4^\dagger$, $w_3^\dagger$, *and* $w_3^\dagger$, $w_4^\dagger$ *are, respectively, the reflections of* w_4, w_3 *with respect to the origin.*

Proof. This is a direct consequence of Lemma 5.5, and the fact that the origin cannot be on the edge w_1w_2. ☐

We are now ready for the anticipated result, stating that two components of $\lambda_{\rm M}$ change sign as the moments' trajectory continues on Σ past an exit point (cfr. Theorem 3.8).

Theorem 5.7. *With the notation of Definition 5.2, suppose that the origin exited Q along w_1w_2, relatively to a moments solution trajectory $x(\cdot)$.*

Let $t \in \mathcal{I}_e$, $t > t_e$, and sufficiently close to t_e, so that $0 \notin \overline{Q(x(t))}$. Then, the 3rd and 4th components of λ_M are negative at such t: $\lambda_{M,3} < 0$ and $\lambda_{M,4} < 0$.

Proof. For ease of notation, we omit writing the explicit dependence of t, but all quantities below must be understood to be relative to the value $x(t)$ of the trajectory.

We prove the result by contradiction. In particular, we assume that $\lambda_{\rm M,3} < 0$ and $\lambda_{\rm M,4} \geq 0$; the other two cases are dealt with analogously (i.e., $\lambda_{\rm M,3} \geq 0$ and $\lambda_{\rm M,4} < 0$, or $\lambda_{\rm M,3} \geq 0$ and $\lambda_{\rm M,4} \geq 0$).

As usual, below M is the matrix of the moments' method: $M = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ 1 & 1 & 1 & 1 \\ d_1 & -d_2 & -d_3 & d_4 \end{bmatrix}$, which under the stated assumptions is invertible. Therefore, there is a unique solution $\lambda_{\rm M} = \begin{bmatrix} \lambda_{\rm M,1} \\ \lambda_{\rm M,2} \\ \lambda_{\rm M,3} \\ \lambda_{\rm M,4} \end{bmatrix}$ to (1.18), for which, in particular, $\lambda_{\rm M,3} < 0$ and $\lambda_{\rm M,4} \geq 0$. Next, consider the matrix

$$\widehat{M} := \begin{bmatrix} w_1 & w_2 & -w_4 & -w_3 \\ 1 & 1 & 1 & 1 \\ d_1 & -d_2 & -d_4 & d_3 \end{bmatrix},$$

and let $\widehat{Q}$ be the quadrilateral associated to $w_1, w_2, -w_3, -w_4$ (taken in this order). By Corollary 5.6, the origin is in the interior of $\widehat{Q}$, and so (by Theorem 3.4) there exists a unique admissible moments solution $\widehat{\lambda}$ such that $\widehat{M}\widehat{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, which by Corollary 3.5 has all components strictly positive.

Now, set $\widetilde{\lambda} := \begin{bmatrix} \lambda_{\rm M,1} \\ \lambda_{\rm M,2} \\ -\lambda_{\rm M,4} \\ -\lambda_{\rm M,3} \end{bmatrix}$, and note that

$$\begin{bmatrix} w_1 & w_2 & -w_4 & -w_3 \\ d_1 & -d_2 & -d_4 & d_3 \end{bmatrix} \widetilde{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the origin is exiting along w_1w_2, and at t_e we have $\lambda_{\rm M,3} = \lambda_{\rm M,4} = 0$, by continuity of $\lambda_{\rm M}$, possibly restricting the interval $\mathcal{I}_e$, we can assume that

$$\lambda_{\rm M,3} + \lambda_{\rm M,4} < \frac{1}{2} ,$$

so that

$$\lambda_{\rm M,1} + \lambda_{\rm M,2} - \lambda_{\rm M,3} - \lambda_{\rm M,4} > 0.$$

Thus,

$$\tilde{\tilde{\lambda}} := \frac{1}{\lambda_{M,1} + \lambda_{M,2} - \lambda_{M,3} - \lambda_{M,4}} \tilde{\lambda}$$

is solution of

$$\widehat{M}\tilde{\tilde{\lambda}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

But $\widehat{M}$ is non-singular, and so we get $\tilde{\tilde{\lambda}} = \widehat{\lambda}$, which contradicts the fact that $\widehat{\lambda}$ is positive, whereas $\widehat{\lambda}_3 = \tilde{\tilde{\lambda}}_3 = -\lambda_{M,4} \leq 0$. $\square$

5.2. **Extensions.** Here we consider other possible regularizations, besides that giving the moments method, of the system (1.15), still obtained enlarging the system (1.15) by appending to it a linear constraint (as we did in (1.18)). Namely, for $x \in \Sigma$, we consider the enlarged system

$$\begin{bmatrix} W \\ \mathbb{1}^\top \\ a^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{5.3}$$

where a is a smoothly varying function of $x \in \Sigma$, taking values in $\mathbb{R}^4$.

First, we have the following result, which restricts the search for possible functions a, in order to obtain an admissible solution λ of (5.3).

Theorem 5.8. *Let the quadrilateral Q be defined as usual, let it be non-degenerate, and assume that $0 \in Q$. Define*

$$\mathcal{A} := \{a \ : \ 0_3 \in \mathcal{T}_{W_a}\},$$

where $\mathcal{T}_{W_a}$ is the tetrahedron with vertices the columns of W_a, and

$$W_a := \begin{bmatrix} W \\ a^\top \end{bmatrix}.$$

Let λ be any solution of the underdetermined system (1.15)*.*

Then, λ is admissible if and only if there exists $a \in \mathcal{A}$ such that $a^\top \lambda = 0$.

Proof. Let λ be any given solution of the underdetermined system (1.15).

If there is $a \in \mathcal{A}$ such that $a^\top \lambda = 0$, then λ is a solution of

$$M_a \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where} \quad M_a = \begin{bmatrix} W \\ \mathbb{1}^\top \\ a^\top \end{bmatrix}. \tag{5.4}$$

Looking at the third row of the adjugate of M_a, similarly to what we did in Section 3, we observe that its entries are the volumes of the tetrahedra that any three vertices of $\mathcal{T}_{W_a}$ form with the origin of $\mathbb{R}^3$. Since $0 \in \mathcal{T}_{W_a}$, these entries are all positive, hence M_a is invertible, and there is a unique solution, call it λ_a, of (5.4), which is further admissible (nonnegative entries, and smoothly varying).

Next, suppose that λ is an admissible solution of (1.15). Therefore, since $\left[\begin{smallmatrix} W \\ \mathbb{1}^\top \end{smallmatrix}\right] \lambda = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} W \\ \mathbb{1}^\top \end{smallmatrix}\right]$ has rank 3, there exists a smoothly varying function a such that $a^\top \lambda = 0$. Further, since λ is admissible, from $\lambda \geq 0$, one has that $0 \in \mathcal{T}_{W_a}$, hence $a \in \mathcal{A}$. $\square$

Below, call λ_a the solution of (5.3). In Theorem 5.9, we consider λ_a at generic first order exit points, and show that λ_a has to be the moments solution $\lambda_{\rm M}$, if this λ_a renders the exit vector field.

Theorem 5.9. *Let the quadrilateral Q be defined as usual, and let it be non-degenerate. Let v span* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, *and let a in* (5.3) *be such that $a^\top v \neq 0$. Then, considering the unique solution λ_a of* (5.3), *there holds one of the following alternatives:*

(1) *either λ_a is not admissible; or*
(2) *if λ_a is admissible, and if x_e is a generic first order exit point, then at x_e either $\lambda_a = \lambda_M$, or λ_a does not give the exit vector field, hence the trajectory associated to λ_a cannot exit Σ smoothly at x_e.*

Proof. For any given $x \in \Sigma$, since $a^\top v \neq 0$, by Lemma 3.1, (5.3) has a unique solution. Therefore, there exists a unique $c_a \in \mathbb{R}$ (of course, c_a depends on x) such that

$$\lambda_a = \lambda_M + c_a v \ ,$$

where $\lambda_{\rm M}$ is the moments' solution associated to (1.18). Denote with $[a_M, b_M]$ the admissibility interval determined by λ_M (note that $a_M < 0$ and $b_M > 0$):

$$a_M := \max \left\{ -\frac{\lambda_{M,i}}{v_i} : v_i > 0 \right\} \ , \quad b_M := \min \left\{ -\frac{\lambda_{M,i}}{v_i} : v_i < 0 \right\} \ .$$

Since $a^\top \lambda_a = 0$ and $d^\top \lambda_M = 0$, then c_a is uniquely determined as

$$c_a = \frac{d^\top \lambda_a}{d^\top v} = -\frac{a^\top \lambda_M}{a^\top v}.$$

Therefore, if $c_a \notin [a_M, b_M]$, then λ_a is not admissible.

If $c_a \in [a_M, b_M]$, and λ_a is admissible, let x_e be a generic first order exit point, and without loss of generality [7] let $f_{\rm F_2}^-$ be the associated exit vector field, that is $0 \in w_1 w_2$. Suppose by contradiction that $\lambda_a \neq \lambda_{\rm M}$ (at x_e), but that λ_a leads to the exit vector field $f_{\rm F_2}^-$ at x_e. Then, $\lambda_{a,3} = \lambda_{a,4} = 0$, and, as we know, we also have $\lambda_{M,3} = \lambda_{M,4} = 0$. By Lemma 2.8, either $v_3 \neq 0$ or $v_4 \neq 0$, and therefore $c_a = 0$, giving $\lambda_a = \lambda_M$, which is a contradiction. $\square$

Remark 5.10. *Of course, Theorem 5.9 does not say that there are no other solutions as in* (5.3) *–beside the moments solution– which enjoy the property of rendering the exit vector field at a first order generic exit point. Indeed, we regularized* (1.15) *with a vector d as in* (1.19), *using the Euclidean distance from the origin of the vertices of Q (i.e., the 2-norm), but we could have used different norms. We illustrate this in Example 5.11 below.*

Example 5.11. *With usual notation, consider f_i, $i = 1, 2, 3, 4$, below:*

$$f_1(x) := \begin{bmatrix} 2x_1 + 1 \\ -x_1 + x_2 x_3 + 1 \\ x_1 + x_2 + 1 \end{bmatrix}, x \in R_1 \ , \quad f_2(x) := \begin{bmatrix} 2x_1 - 1 \\ -x_1 + x_3 - 1 \\ x_1 + x_2 x_3 + 2 \end{bmatrix}, x \in R_2 \ ,$$

$$f_3(x) := \begin{bmatrix} 2x_1 - 3 \\ -x_1 + x_2 + 2 \\ x_1 + x_2 x_3 - 1 \end{bmatrix}, x \in R_3 \ , \quad f_4(x) := \begin{bmatrix} 2x_1 + 2 \\ -x_1 + x_3 - 2 \\ x_1 + x_3 - 2 \end{bmatrix}, x \in R_4 \ ,$$

where

$$h_1(x) := x_3, \ h_2(x) := x_2 \ .$$

[7] of course, any other choice of exit vector field is handled similarly

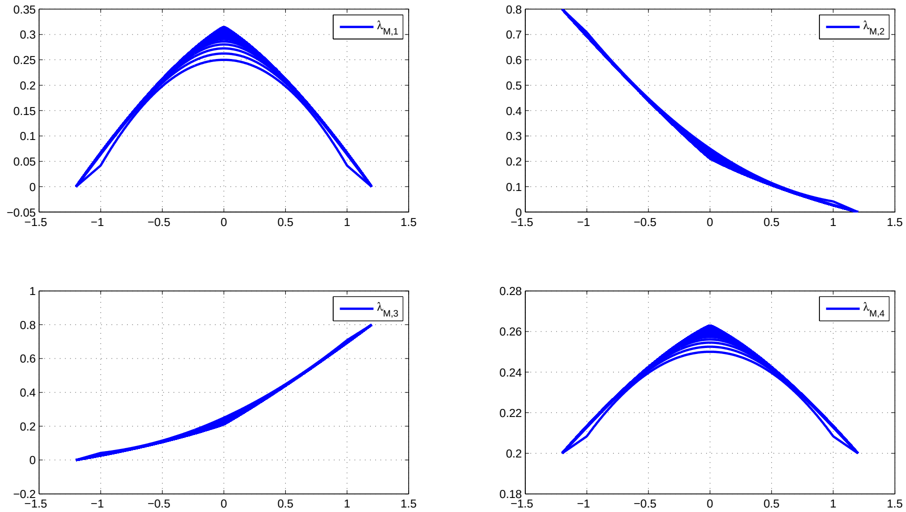

FIGURE 10. Solution components of λ_{M} for Example 5.11, using $\|\cdot\|_i$, $i = 2, \dots, 100$.

Here Σ is the x_1-axis, and the matrix W for $x \in \Sigma$ is:

$$W(x) = \begin{bmatrix} x_1+1 & x_1+2 & x_1-1 & x_1-2 \\ -x_1+1 & -x_1-1 & -x_1+2 & -x_1-2 \end{bmatrix} .$$

There is attractive sliding motion (in the direction of increasing x_1) for $|x_1| \le 1.2$. The value $x_1 = 1.2$ is a first order exit point, with exit vector field $f_{F_2^+}$.

As illustration, consider the following family of regularizations of (1.15)*:*

$$\begin{bmatrix} W \\ \mathbb{1}^\top \\ a^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{where} \quad a = \begin{bmatrix} \|w_1\|_p \\ -\|w_2\|_p \\ -\|w_3\|_p \\ \|w_4\|_p \end{bmatrix} , \; p \ge 2 .$$

In Figure 10, we show the plots of the solutions λ of this system, relative to different choices of the p-norm, for $p = 2, \dots, 100$. Clearly, the qualitative behavior of different solutions λ's relative to different norms is quite similar.

In conclusion, although there are alternatives to using the 2*-norm when forming the vector d in* (1.19)*, for the class of regularized system of the type* (5.3) *it seems natural to simply use d as we did in* (1.19)*, using $\|\cdot\|_2$, and compute λ_M. This choice allowed us to retain the geometrical flavor of "moments" for the entries of λ_M.* □

6. CONCLUSIONS

In this work, we have analyzed the moments regularization technique as a mean to select a Filippov sliding vector field on a co-dimension 2 manifold Σ (intersection of two co-dimension 1 manifolds).

We proved that –whenever Σ is attractive– the moments regularization gives a well defined, smoothly varying sets of coefficients, rendering a smooth Filippov sliding vector field on Σ, which further leads to smooth exits at generic first order exit points. In the process, we have introduced (and exploited) a quadrilateral Q which proved to be a useful tool to study sliding vector fields on a co-dimension 2 manifold. We have also shown by numerical experiments the behavior of the moments method, and the potential dangers associated to selecting a solution λ (and an associated sliding vector field) that does not smoothly render the exit vector field at a first order exit point. Finally, we discussed the case of non-generic exit points, and further generalizations of our approach.

To date (and with the exception of trivial modifications), we know of no other constructive technique that provably gives admissible (positive and smooth) coefficients, under general attractivity conditions of Σ, and that further leads to smooth exits at generic first order exit points.

In a forthcoming work, we discuss extension of the moments' technique to the case of a discontinuity manifold of co-dimension 3 (see [6]).

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