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MINIMUM VARIATION SOLUTIONS FOR SLIDING VECTOR FIELDS ON THE INTERSECTION OF TWO SURFACES IN $\mathbb{R}^3$

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Abstract. In this work, we consider model problems of piecewise smooth systems in $\mathbb{R}^3$, for which we propose minimum variation approaches to find a Filippov sliding vector field on the intersection Σ of two discontinuity surfaces. Our idea is to look at the minimum variation solution in the H^1-norm, among either all admissible sets of coefficients for a Filippov vector field, or among all Filippov vector fields. We compare the resulting solutions to other possible Filippov sliding vector fields (including the bilinear and moments solutions). We further show that –in the absence of equilibria– also these other techniques select a minimum variation solution, for an appropriately weighted H^1-norm, and we relate this weight to the change of time variable giving orbital equivalence among the different vector fields. Finally, we give details of how to build a minimum variation solution for a general piecewise smooth system in $\mathbb{R}^3$.

1. The problem

In this work, we explore model problems in $\mathbb{R}^3$ in order to understand how to properly define a smooth *minimum variation sliding vector field* in the case of sliding on a co-dimension 2 discontinuity manifold Σ, intersection of two co-dimension 1 discontinuity surfaces. Whereas our model problems are sufficiently simple to allow explicit computations, the process we propose is rather general, as it will be clarified in this work.

We restrict attention to smooth sliding vector fields of Filippov type, in which case there is an inherent algebraic ambiguity in the construction of a sliding vector field. Indeed, the general concern of defining suitable Filippov sliding vector fields on a co-dimension 2 discontinuity manifold Σ has received considerable attention in recent times (e.g., see [10, 4, 8] and references therein).

Our idea in this paper is to select a smooth Filippov sliding vector field as solution of a minimum variation problem. As far as we know, in this context, this idea is new. [In [1, 6] the authors discussed selection of a vector field so to minimize the 2-norm of either the vector field itself or of the coefficients entering in the Filippov convexification, but these attempt produced non-smoothly varying sliding vector fields]. At the same time, minimum variation techniques have proven quite powerful in Mathematics and Engineering studies, notably in Optimal Control applications (see [14, 11]), and in studying stick-slip motion phenomena for solid/solid interactions (see [3]).

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We will be interested in the situation in which Σ is an arc which attracts the dynamics of the given piecewise smooth system, with endpoints corresponding to isolated values where Σ ceases to be attractive (generic first order exit points). This way, we will be able to set up the boundary value problem corresponding to the minimality conditions of a minimum variation solution (Euler-Lagrange equation).

A plan of this paper is as follows. In the remainder of this Introduction, we set notation and review some of the basic theory, in particular insofar as the ambiguity in selecting the coefficients of a Filippov sliding vector field. We express this ambiguity in terms of selecting the function c in (1.10) and (1.11). In Section 2, we consider a model problem for which we propose minimum variation techniques in order to select the above function c. In Section 3 we generalize the construction we made on the model problem of Section 2 to the case of weighted minimum variation techniques. Under suitable assumptions, chiefly the absence of equilibria on Σ, we relate different weights to the orbital equivalence factors of different Filippov sliding vector fields. We also generalize the construction to the case of equilibria on Σ, obtaining singular weights. Conclusions are in Section 4.

Notation. In this work, the norm on vectors, $\|\cdot\|$, is always the Euclidean norm.

1.1. **Background.** For the material in this section, we refer to the recent works [8] and [7] where the concepts and results below are introduced and justified.

Consider the following piecewise smooth system in $\mathbb{R}^n$:

$$\dot{x} = f(x)\ ,\ \ f(x) = f_i(x)\ ,\ \ x \in R_i\ ,\ i = 1, \ldots, 4, \tag{1.1}$$

with initial condition $x(0) = x_0 \in R_i$, for some i. Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and are (locally) separated by two intersecting smooth surfaces of co-dimension 1. That is, we have

$$\Sigma_1 = \{x :\ h_1(x) = 0\}\ ,\ \ \Sigma_2 = \{x :\ h_2(x) = 0\}\ ,\ \ \Sigma = \Sigma_1 \cap \Sigma_2\ , \tag{1.2}$$

and we will label the region R_i's as

$$\begin{array}{llllll} R_1 & : & \text{when} & h_1 < 0\ , h_2 < 0\ , & R_2 & : \quad \text{when} \quad h_1 < 0\ , h_2 > 0\ , \\ R_3 & : & \text{when} & h_1 > 0\ , h_2 < 0\ , & R_4 & : \quad \text{when} \quad h_1 > 0\ , h_2 > 0\ . \end{array} \tag{1.3}$$

We will always assume that $\nabla h_1(x) \neq 0, x \in \Sigma_1$, $\nabla h_2(x) \neq 0, x \in \Sigma_2$, that $h_{1,2}$ are $\mathcal{C}^k$ functions, with $k \geq 2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for x on (and in a neighborhood of) Σ.

Finally, we will use the following notation for the projections of the vector fields in the directions normal to the discontinuity surfaces:

$$\begin{array}{l} w_1^1 = \nabla h_1^\top f_1\ ,\ w_2^1 = \nabla h_1^\top f_2\ ,\ w_3^1 = \nabla h_1^\top f_3\ ,\ w_4^1 = \nabla h_1^\top f_4\ , \\ w_1^2 = \nabla h_2^\top f_1\ ,\ w_2^2 = \nabla h_2^\top f_2\ ,\ w_3^2 = \nabla h_2^\top f_3\ ,\ w_4^2 = \nabla h_2^\top f_4\ , \\ W\ =\ \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix}\ . \end{array} \tag{1.4}$$

We are interested is the case when (portion of) Σ *attracts* nearby trajectories: solution trajectories of (1.1) starting near Σ will reach it in finite time and will not leave it, giving rise to so-called *sliding motion* on Σ. Since trajectories cannot leave Σ, sliding motion must take place with a vector field in the tangent plane to Σ, hence orthogonal to ∇h_1

and ∇h_2 (see (1.5)). According to first order theories, there are two mechanisms by which Σ can be attractive: *through sliding* or *by spiraling* (see [8, 5]). Furthermore, when Σ loses attractivity, a trajectory sliding on Σ may leave Σ; typically, this will happen with sliding motion on one of Σ_1 or Σ_2 (this is what one expects to happen at *generic first order exit points*), though trajectories may also leave Σ to enter directly into one of the regions R_i's (e.g., this is what one might expect to happen when Σ loses attractivity in a spiraling regime).

On Σ, we restrict consideration to the class of *smooth Filippov sliding vector fields*, that is, smooth vector fields of the form

$$
\begin{aligned}
f_{\mathrm{F}} \;&=\; \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \;,\;\; \lambda_i \geq 0 \;,\, i = 1, \ldots, 4 \;,\;\; \sum_{i=1}^{4} \lambda_1 = 1 \;, \\
\nabla h_1^{\top} f_{\mathrm{F}} \;&=\; \nabla h_2^{\top} f_{\mathrm{F}} \;=\; 0 \,.
\end{aligned}
\tag{1.5}
$$

Thus, we have to solve the problem (for $x \in \Sigma$):

$$
\begin{bmatrix} W \\ \mathbb{1}^{\top} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;,\; \text{where } \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \;,\; \text{and } W = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix} \;,\; \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \;,
\tag{1.6}
$$

and a solution λ of (1.6) will be called *admissible* if $\lambda \geq 0$ and λ depends smoothly on $x \in \Sigma$.

As amply discussed elsewhere (e.g., [8, 7]), and as it is plainly seen from (1.6), there is an algebraic ambiguity in the selection of a Filippov vector field, and one needs to further regularize the problem in order to obtain a unique solution. Two ways to do this have been studied in greater details.

- The *moments method* (see [6, 7]), whereby one solves the regularized system

$$
M \lambda_{\mathrm{M}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \;, \qquad \text{where} \qquad M := \begin{bmatrix} W \\ \mathbb{1}^{\top} \\ d^{\top} \end{bmatrix} \;,
\tag{1.7}
$$

  with W defined in (1.6) and

$$
d := \begin{bmatrix} d_1 \\ -d_2 \\ -d_3 \\ d_4 \end{bmatrix} \;, \qquad \text{where} \qquad d_i := \|w_i\| \;,\;\; i = 1, \ldots, 4 \;,
\tag{1.8}
$$

  and then uses λ_{M} in (1.5).
- The *bilinear interpolant* method (e.g., see [2, 4, 8]) whereby one restricts to the following special convex combination and needs to solve a nonlinear system:

$$
\begin{aligned}
& f_{\mathrm{B}} := (1-\alpha)\left((1-\beta)f_1+\beta f_2\right)+\alpha\left((1-\beta)f_3+\beta f_4\right)\ , \\
& (\alpha,\beta)\in(0,1)^2\ :\ W\lambda_{\mathrm{B}}=0 \quad \text{with } \lambda_{\mathrm{B}} := \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix}\ .
\end{aligned} \tag{1.9}
$$

As proven in [7] and [8], the moments and bilinear methods give well defined choices of coefficients λ_{M} and λ_{B}, **when Σ is attractive**. Moreover, the moments method is further guaranteed to smoothly exit at generic first order exit points, that is to produce coefficients λ_{M} that render an exiting vector field, whereas, in general, the bilinear method does not lead to smooth exits.

1.2. **General form of coefficients.** The following result is helpful in order to write the general form of an admissible solution λ in (1.5).

Lemma 1.1 ([7])**.** *When Σ is attractive, or we are at a generic first order exit point, the matrix $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ in* (1.6) *has full rank* 3*. Furthermore, there is a nontrivial vector v, as smooth as W, spanning* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$*, and v can be chosen as the eigenvector relative to the* 0*-eigenvalue of* $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}^\top \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$*.* □

In light of Lemma 1.1, clearly any admissible solution of $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ can be written as

$$
\lambda \ = \ \mu + cv\,, \tag{1.10}
$$

where μ is any (smooth) particular solution of $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}\mu = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and v (smoothly) spans $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. We note that, since $\mathbb{1}^\top v = 0$, then v cannot have all components of the same sign. In particular, in order for λ to be admissible, we must have that the function c satisfies

$$
\alpha \le c \le \beta\ ,\ \alpha := \max\left\{-\frac{\mu_i}{v_i} : v_i > 0\right\}\ ,\ \beta := \min\left\{-\frac{\mu_i}{v_i} : v_i < 0\right\}\ , \tag{1.11}
$$

for each x in (the sliding portion of) Σ. Note that $\alpha \le 0$ and $\beta \ge 0$. Of course, α and β are functions of x (since so are μ and v), and in general are only continuous functions (even if μ and v are smooth). Finally, we note that, by the nature of the solution set in (1.10), although the admissible region for c in (1.11) depends on the specific choices of μ and v, the admissible set of coefficients λ does not. Further, the topological properties (say, connectedness) of the admissibility region in (1.11) are preserved by choosing different μ and v.

To sum up, in our present context, all possible admissible smooth sliding vector fields of Filippov type (i.e., with smooth and positive coefficients) arise from (1.10), for given smooth μ and v as above, and selecting a smooth function c satisfying (1.11).

2. An example: Minimum variation solutions

Here we consider a model problem in $\mathbb{R}^3$, and give details of the construction of a *minimum variation* Filippov solution for it. This is a model used in [6] to illustrate different possibilities in forming a smooth sliding vector field. Later, we will consider a different model, and give a new interpretation of other admissible Filippov solutions also as minimum variation solutions, but with respect to a different minimization task and ultimately with respect to a different parametrization of time.

Example 2.1 (A model problem from [6])**.** We have f_i, $i = 1, 2, 3, 4$, taking values in $\mathbb{R}^3$:

$$f_1(x) := \begin{bmatrix} 2x_1 + 1 \\ -x_1 + x_2x_3 + 1 \\ x_1 + x_2 + 1 \end{bmatrix}, \, x \in R_1 \ , \quad f_2(x) := \begin{bmatrix} 2x_1 - 1 \\ -x_1 + x_3 - 1 \\ x_1 + x_2x_3 + 2 \end{bmatrix}, \, x \in R_2 \ ,$$
$$f_3(x) := \begin{bmatrix} 2x_1 - 3 \\ -x_1 + x_2 + 2 \\ x_1 + x_2x_3 - 1 \end{bmatrix}, \, x \in R_3 \ , \quad f_4(x) := \begin{bmatrix} 2x_1 + 2 \\ -x_1 + x_3 - 2 \\ x_1 + x_3 - 2 \end{bmatrix}, \, x \in R_4 \ ,$$

where $\Sigma_1 = \{x : \ x_3 = 0\}$, $\Sigma_2 = \{x : \ x_2 = 0\}$, so that $\Sigma = \Sigma_1 \cap \Sigma_2$ is the x_1-axis.

Here, the matrix W of (1.4) is

$$W(x) = \begin{bmatrix} x_1 + 1 & x_1 + 2 & x_1 - 1 & x_1 - 2 \\ -x_1 + 1 & -x_1 - 1 & -x_1 + 2 & -x_1 - 2 \end{bmatrix} , \tag{2.1}$$

and it is simple to verify that Σ is attractive in the segment $|x_1| < 1.2$ and the values $x_1 = \pm 1.2$ are generic first order exit points, at which point Σ is no longer attractive. Since $W(-1.2) = \begin{bmatrix} -0.2 & 0.8 & -2.2 & -3.2 \\ 2.2 & 0.2 & 3.2 & -0.8 \end{bmatrix}$ then one should expect to exit Σ at $x = -1.2$ by sliding on Σ_1^+; similarly, since $W(1.2) = \begin{bmatrix} 2.2 & 3.2 & 0.2 & -0.8 \\ -0.2 & -2.2 & 0.8 & -3.2 \end{bmatrix}$ then one should expect to exit Σ at $x = 1.2$ by sliding on Σ_2^+.

The general form of the solution λ to $W\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ can be written as

$$\lambda = \mu + cv \qquad \text{or, explicitly:} \qquad \lambda = \begin{bmatrix} \frac{2}{3} - \frac{5}{9}x_1 \\ 0 \\ \frac{2}{3}x_1 \\ \frac{1}{3} - \frac{1}{9}x_1 \end{bmatrix} + c \begin{bmatrix} -\frac{5}{3} \\ 1 \\ 1 \\ -\frac{1}{3} \end{bmatrix} , \tag{2.2}$$

which is admissible for (x_1, c) in the triangular region in Figure 1. Note that, in particular, we must have $c(-1.2) = 0.8$ and $c(1.2) = 0$. For any admissible λ, we will get a

Filippov sliding vector field of the form:

$$f_{\rm F} \;=\; \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \;, \quad \text{or}$$

$$f_{\rm F} = \begin{bmatrix} \lambda_1(2x_1+1) + \lambda_2(2x_1-1) + \lambda_3(2x_1-3) + \lambda_4(2x_1+2) \\ 0 \\ 0 \end{bmatrix}. \tag{2.3}$$

Hence, on Σ, the differential equation to solve is simply:

$$\dot{x}_1 = \frac{4}{3} - \frac{7}{9}x_1 - \frac{19}{3}c \;, \tag{2.4}$$

and we observe that there is an equilibrium on Σ at the value x_1 for which $c(x_1) = \frac{4}{19} - \frac{7}{57}x_1$. Given the admissibility region of Figure 1, any smooth selection of c will give an equilibrium, which will be unstable. Different ways to select c, in general will give a different location for the equilibrium.

Both the moments and bilinear solutions of (1.7), (1.9), are well defined for this problem, exit smoothly at $x = \pm 1.2$, and select (similar) c-curves; see Figure 1 below. For this problem, there is also another obvious solution, the so-called *triangular solution*, namely the solution obtained choosing for c the straight line segment $c_{\rm tr}(x_1) = \frac{8}{20} - \frac{x_1}{3}$, $-1.2 \le x_1 \le 1.2$, joining the boundary values, that is the longest side of the triangle in Figure 1.

Next, we consider new types of solutions, still on Example 2.1, obtained via a variational formulation.

2.1. **Minimum variation solutions for model problem.** Recall that we want to have c (hence λ) smooth functions of x_1. Further, recall that we have a family of solutions, depending on how we select an admissible function c. The choice of an admissible c impacts the choice of the coefficients λ_i's, and clearly the resulting sliding vector field in (2.4).

So, a natural idea is to seek an admissible function c that, for $-1.2 \le x_1 \le 1.2$, minimizes the H^1-norm of either λ or of the sliding vector field itself.

Remark 2.2. A version of Weierstrass' Theorem (e.g., see [12]) states that, if $A \subset \mathbb{R}^n$ is closed and $f : A \longrightarrow \mathbb{R}$ is continuous and coercive, then f has a minimum in A. This justifies all the minimization problems we examine below. In particular, the well posedness of Problems (2.5) and (2.8) below, as well as (3.5) and (3.9) in Section 3. This is because all of these problems amount to minimization of the functional given by $\|\cdot\|_{H^1}$ over the compact set of $\lambda \in \mathbb{R}^4$ with nonnegative components adding to 1.

2.1.1. *Minimum variation for λ.* Accounting for the fact that we want the solution to be defined from $x_1 = -1.2$ to $x_1 = 1.2$, we seek the value of the function c such that the following functional is minimized:

$$\min_{\lambda \in \mathcal{C}^1, \lambda \ge 0} \int_a^b \left[\|\lambda(x_1)\|^2 + \|\lambda'(x_1)\|^2 \right] {\rm d}x_1 \;, \quad a = -1.2 \;,\; b = 1.2 \,. \tag{2.5}$$

With the Lagrangian given by the integrand, next we write down the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial c} - \frac{\rm d}{{\rm d}x_1}\frac{\partial \mathcal{L}}{\partial c'} \;=\; 0 \;.$$

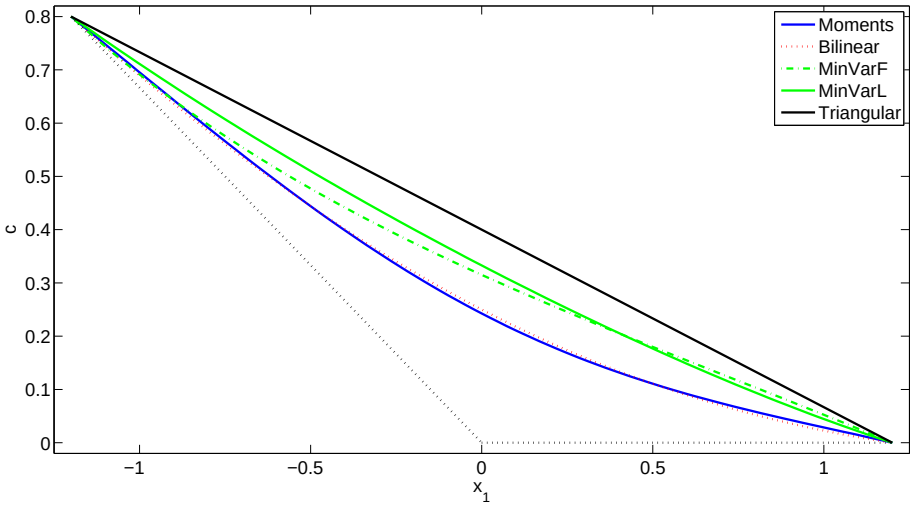

FIGURE 1. Admissible region (x_1, c) in (2.2), and moments, bilinear, triangular, and minimum variations values of c.

With a little algebra, and using the exit conditions, this gives the boundary value problem for c:

$$c'' - c \;=\; x_1/3 - 1/4\;, \quad c(-1.2) = 0.8\;,\; c(1.2) = 0\;, \tag{2.6}$$

which has the solution

$$c \equiv c_{\mathrm{MV},\lambda}(x_1) \;=\; \frac{0.15}{e^{1.2} + e^{-1.2}}\left(e^{x_1} + e^{-x_1}\right) - \frac{x_1}{3} + \frac{1}{4}\;. \tag{2.7}$$

With this value of $c_{\mathrm{MV},\lambda}$, we obtain what we call *minimum variation* solution with respect to λ. See Figure 1 for a plot of $c_{\mathrm{MV},\lambda}$.

2.1.2. *Minimum variation for f_F.* Now we consider the general form of the smooth sliding vector field f_{F} and seek the function c in order to minimize the H^1 norm of f_{F}, still considering the model problem of Example 2.1.

In other words, we seek the (smooth) function c such that the following functional is minimized among smooth admissible functions c:

$$\min_c \int_a^b \left[\|f_{\mathrm{F}}(x_1)\|^2 + \|f_{\mathrm{F}}{}'(x_1)\|^2\right] \mathrm{d}x_1\,, \quad a = -1.2\;,\, b = 1.2\,. \tag{2.8}$$

Given the simple expression (2.4), this reduces to minimizing

$$\int_a^b \mathcal{L}(x_1, c, c')\mathrm{d}x_1\;,\;\; \mathcal{L} = \left(\frac{4}{3} - \frac{7}{9}x_1 - \frac{19}{3}c\right)^2 + \left(\frac{7}{9} + \frac{19}{3}c'\right)^2\;.$$

The Euler-Lagrange equation gives the following boundary value problem for c:

$$c'' - c \;=\; \frac{7}{57}x_1 - \frac{12}{57}\;, \quad c(-1.2) = 0.8\;,\; c(1.2) = 0\;, \tag{2.9}$$

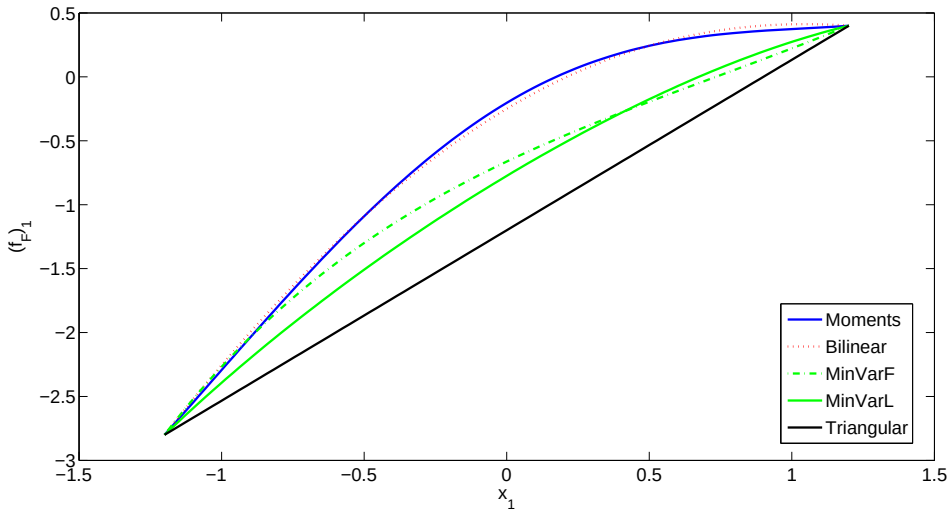

FIGURE 2. Sliding vector fields for moments, bilinear, triangular, and minimum variations solutions. All have an equilibrium.

which has the solution

$$
\begin{aligned}
c \equiv c_{\mathrm{MV},f_{\mathrm{F}}}(x_1) &= A_1 e^{x_1} + B_1 e^{-x_1} + \frac{12 - 7x_1}{57}\,, \\
\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} &= \frac{6}{95(e^{-2.4} - e^{2.4})} \begin{bmatrix} 7e^{-1.2} + e^{1.2} \\ -e^{-1.2} - 7e^{1.2} \end{bmatrix}\,.
\end{aligned} \tag{2.10}
$$

With this value of $c_{\mathrm{MV},f_{\mathrm{F}}}$, we obtain what we call *minimum variation* solution with respect to the H^1-variation of f_{F}. See Figure 1 for a plot of $c_{\mathrm{MV},f_{\mathrm{F}}}$.

Remark 2.3. It is a simple computation to verify that the minimum variation solutions we obtained, both with respect to λ and with respect to the vector field f_{F}, in the end give parameters values λ, in an independent way of how we chose μ and v in (2.2).

Questions 2.4. The above example suggests several questions, which we will address in the next section.

(i) In Example 2.1, in spite of the different expressions for the functions c we obtained, in the end all sliding vector fields have a similar behavior: there is an equilibrium on Σ, and –depending on where one enters Σ– motion goes to the right/left until an exit point is reached. Different choices of admissible functions c determine the position of the equilibrium. See Figure 2.

(ii) Below, we will consider a similar model, for which no smooth Filippov vector field has an equilibrium on Σ. In this case, according to the results in [9], we know that all possible smooth Filippov sliding motions are orbitally equivalent. Are there functionals, related to the change of time variable in the aforementioned orbital equivalency, whose minimizers give –say– the moments, or the bilinear solutions?

(iii) Finally, how can one extend our construction to a broader class of problems?

3. Orbital equivalence and weighted minimum variation

In this section, we consider another pattern of sliding motion, which has the key features outlined below.

Conditions 3.1.

(a) The state space is $\mathbb{R}^3$.
(b) The sliding manifold Σ is a smooth arc: $\Sigma = \{x \in \mathbb{R}^3 : \ x = \gamma(s) \ , \ a \leq s \leq b\}$.
(c) For $a < s < b$, Σ is attractive, there are no equilibria on Σ for any smooth Filippov sliding vector field, and motion on Σ proceeds from $x_a := \gamma(a)$ to $x_b := \gamma(b)$.
(d) The point x_b is a generic first order exit point, and the point x_a is a generic first order exit point for the time reversed problem.

When Conditions 3.1 hold (in particular Σ is attractive), the function W (which depends solely on the parameter s), is of full rank. Therefore, the general form of an admissible solution λ in (1.5), can be written as (see Section 1.2)

$$\lambda(s) = \mu(s) + c(s)v(s) \, , \quad a \leq s \leq b \, , \tag{3.1}$$

where μ is any given (smooth) particular solution, v is a given (smooth) vector spanning $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, and the function c is subject to restrictions as in (1.11).

Note. We will want to select an admissible function $c(s)$, $a \leq s \leq b$, so that the resulting $\lambda(s)$ in (3.1) at the endpoints $s = a$ and $s = b$ gives the respective "exiting" vector fields. We know that this is possible, since it is achieved, for example, by the moments method. Indeed, as proved elsewhere (see [7] and [8]), both moments and bilinear solutions give well defined Filippov sliding vector fields, the moments vector field further being guaranteed to give coefficients that render the exit vector field at first order exit points. Below, we show how to formally define a minimum variation solution in this general case.

Now, in light of the results in [9], for a problem with the above characteristics, all smooth sliding vector fields on Σ are orbitally equivalent. That is, if we have two different smooth sliding vector fields, say $f_{\mathrm{F}1}$ and $f_{\mathrm{F}2}$, then the solutions associated to these vector fields are tracing the same orbit, but at different speeds. In other words, we must have

$$f_{\mathrm{F}1} \ = \ \omega(x) f_{\mathrm{F}2} \ , \ \ \omega \in \mathcal{C}^1 \ , \ \ \omega > 0 \, , \tag{3.2}$$

and therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} \ = \ f_{\mathrm{F}1} \iff \frac{\mathrm{d}x}{\mathrm{d}\tau} \ = \ f_{\mathrm{F}2} \text{ and } \omega(x) = \frac{\mathrm{d}t}{\mathrm{d}\tau} \, .$$

This being the case, and the system being autonomous, it means that we can interpret the two distinct vector fields above as follows:

$$\begin{aligned} \text{If } \ f_{\mathrm{F}1} \ &= \ \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \ , \\ \text{then } \ f_{\mathrm{F}2} \ &= \ \lambda_1(\omega f_1) + \lambda_2(\omega f_2) + \lambda_3(\omega f_3) + \lambda_4(\omega f_4) \ , \end{aligned} \tag{3.3}$$

which means that "*Any sliding vector field can be interpreted as having modified* **all** *vector fields f_i, $i = 1, 2, 3, 4$, through the reparametrization of time*". Observe that – under Conditions 3.1– we can assume that ω is parametrized by s. Therefore, for all

orbitally equivalent smooth vector fields, further smoothly aligning at the exit points with the exit vector fields, we must have $\omega|_{s=a} = \omega|_{s=b} = 1$.

3.1. **Weighted Minimum Variation.** Motivated by the above, we are thus lead to consider a generalization of the approach in Section 2.1.2, and seek minimization of functionals more general than those in Section 2.1.2. Namely, we will seek the function c so that in the end we will minimize either

(i) the H^1-variation of $w\lambda$, or
(ii) the H^1-variation of the sliding vector field wf_{F}.

Above, the function w –which we will call *weight function*– is required to satisfy these properties:

$$\begin{aligned} &\text{(i)} \quad w \text{ is smooth (at least } \mathcal{C}^2) \ \forall s \in (a,b) \\ &\text{(ii)} \quad w > 0 \ \forall s \in [a,b] \ , \ \text{and } \ w|_{s=a} = w|_{s=b} = 1 \ . \end{aligned} \tag{3.4}$$

Each of the above H^1-minimization tasks has its merits, though minimization of $\|wf_{\mathrm{F}}\|_{H^1}$ is more in tune with the previously mentioned reparametrization of time.

Remark 3.2. In all cases, the value of c will be **required** to take the values $c(a) = c_a$, and $c(b) = c_b$, specified so that $\lambda(a)$ and $\lambda(b)$ give the exiting vector fields at $\gamma(a)$ and $\gamma(b)$. Therefore, we emphasize that, with the choices we made for the weight function w and the values of $c(a)$ and $c(b)$, the solutions of our minimization problems (when solvable) will give smoothly exiting solutions.

3.1.1. *Minimum variation for λ.* With the function w as in (3.4), we seek c such that

$$\min_c \int_a^b \left[\|w\lambda\|^2 + \|(w\lambda)'\|^2\right] \mathrm{d}s \ , \quad c(a) = c_a \ , \, c(b) = c_b \, . \tag{3.5}$$

Consider the Lagrangian associated to (3.5), that is

$$\mathcal{L}(s, c, c') = \|w\lambda\|^2 + \left\|(w\lambda)'\right\|^2 = w^2 \|\lambda\|^2 + (w')^2 \|\lambda\|^2 + w^2 \left\|\lambda'\right\|^2 + 2ww'\lambda^\top\lambda' \ .$$

The Euler-Lagrange equation on this functional (with some algebra), gives the following boundary value problem to be solved for c (note that $\|v\| \neq 0$)

$$\begin{cases} \big[c''w\,\|v\|^2 + 2c'\big(w'\,\|v\|^2 + w(v^\top v')\big) - \\ \qquad c\big((w - w'')\,\|v\|^2 - w(v^\top v'') - 2w'(v^\top v')\big)\big] \ = \\ \qquad (w - w'')(v^\top\mu) - wv^\top\mu'' - 2w'(v^\top\mu') \ , \\ \qquad c(a) = c_a \ , \ \ c(b) = c_b \ . \end{cases} \tag{3.6}$$

Remark 3.3. In general, it is not clear how to obtain the exact solution of the boundary value problem (3.6). However, there is an important special case where (3.6) can be solved exactly. This is when the null vector $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ is constant. In fact, in this case (3.6) becomes

$$\begin{aligned} &c''w\,\|v\|^2 + 2c'w'\,\|v\|^2 - c(w - w'')\,\|v\|^2 \ = \\ &\qquad (w - w'')v^\top\mu - wv^\top\mu'' - 2w'v^\top\mu' \ , \qquad c(a) = c_a \ , \ \ c(b) = c_b \ . \end{aligned} \tag{3.7}$$

The differential equation in (3.11) rewrites as

$$y'' \;=\; y + g(s)\;, \quad \text{where} \quad y = cw\,\|v\|^2 + wv^\top\mu\;,\ \text{and}\ g(s) = 2v^\top\mu'(w' - w'')\;.$$

For this, letting $y_1(s) = e^s$ and $y_2(s) = e^{-s}$, the solution can be written as

$$y(s) = Ay_1(s) + By_2(s) + y_p(s)\;.$$

The associated Wronskian is $\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = -2$, and using the variation of constants formula gives

$$y_p(s) = \frac{1}{2}\left[e^s\int e^{-s}g(s)ds - e^{-s}\int e^s g(s)ds\right]\;,$$

from which one can obtain the solution of (3.7):

$$\text{(3.8)}\qquad \begin{cases} c(s) \;=\; \dfrac{Ae^s + Be^{-s} + y_p(s) - w(s)v^\top(s)\mu(s)}{w(s)\,\|v(s)\|^2}\;,\ a \le s \le b\;, \\ A, B : c(a) = c_a\;,\ \ c(b) = c_b\;. \end{cases}$$

Observe that since $w(a) = w(b) = 1$, the values of A and B in (3.8) are independent of the weight function w.

3.1.2. *Minimum variation for f_F.* Now, with the weight function w as above, we seek c such that

$$\text{(3.9)}\qquad \min_c \int_a^b \left[\|wf_{\rm F}\|^2 + \|(wf_{\rm F})'\|^2\right]\mathrm{d}s\;, \quad c(a) = c_a\;,\, c(b) = c_b\,.$$

Again, $c(a) = c_a$, and $c(b) = c_b$, must be assigned to make sure that $\lambda(a)$ and $\lambda(b)$ give the exiting vector fields at $\gamma(a)$ and $\gamma(b)$.

For a general sliding vector field $f_{\rm F}$, given the form of λ (3.1), we will use the notation

$$f_{\rm F} = F_\mu + cF_v\;,$$

where $F_\mu = \mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3 + \mu_4 f_4$, and $F_v = v_1 f_1 + v_2 f_2 + v_3 f_3 + v_4 f_4$.

We will **assume** that $F_v \neq 0$, for all $s \in [a, b]$ (see Remark 3.9 below when this is violated).

The Lagrangian associated to (3.9) is

$$\mathcal{L}(s, c, c') = \|wf_{\rm F}\|^2 + \left\|(wf_{\rm F})'\right\|^2 = w^2\,\|f_{\rm F}\|^2 + (w')^2\,\|f_{\rm F}\|^2 + w^2\,\left\|f_{\rm F}'\right\|^2 + 2ww'f_{\rm F}{}^\top f_{\rm F}'\;.$$

The Euler-Lagrange equation on this functional (with some algebra), gives the following boundary value problem to be solved for c:

$$\text{(3.10)}\qquad \begin{cases} \big[c''w\,\|F_v\|^2 + 2c'\big(w'\,\|F_v\|^2 + w(F_v^\top F_v')\big) - \\ \qquad c\big((w - w'')\,\|F_v\|^2 - w(F_v^\top F_v'') - 2w'(F_v^\top F_v')\big)\big] \;= \\ \qquad (w - w'')(F_v^\top F_\mu) - wF_v^\top F_\mu'' - 2w'(F_v^\top F_\mu')\;, \\ \qquad c(a) = c_a\;,\ \ c(b) = c_b\;. \end{cases}$$

Remark 3.4. Again, in general, it is not clear how to obtain the exact solution of the boundary value problem (3.10). However, there is an important special case when in fact it can be solved exactly, that is when the discontinuity surfaces Σ_1 and Σ_2 are given by coordinates' planes[1].

So, without loss of generality, in this case we can take $\Sigma_1 = \{x : \ x_2 = 0\}$ and $\Sigma_2 = \{x : \ x_3 = 0\}$. Then, Σ is (a segment on) the x_1-axis, and one has that both F_v and F_μ have only the first components different from 0, on Σ:

$$F_\mu = \begin{bmatrix} f_\mu \\ 0 \\ 0 \end{bmatrix} \ , \quad F_v = \begin{bmatrix} f_v \\ 0 \\ 0 \end{bmatrix} \ ,$$

and we are requiring that $f_v \neq 0$ for all $x_1 \in [a, b]$.

Using this in (3.10), and dividing by f_v, we get the boundary value problem (differentiation is with respect to x_1):

$$\begin{aligned} & c'' w f_v + 2c'\big(w' f_v + w f_v'\big) - c\big((w - w'')f_v - w f_v'' - 2w' f_v'\big) \ = \\ & \qquad (w - w'')f_\mu - w f_\mu'' - 2w' f_\mu' \ , \qquad c(a) = c_a \ , \ \ c(b) = c_b \ . \end{aligned} \tag{3.11}$$

The point is that now the differential equation in (3.11) rewrites as

$$\big[(cwf_v) + (wf_\mu)\big]'' \ = \ (cwf_v) + (wf_\mu) \ ,$$

from which we get the solution of (3.11):

$$\begin{cases} c(x_1) \ = \ \dfrac{Ae^{x_1} + Be^{-x_1} - w(x_1)f_\mu(x_1)}{w(x_1)f_v(x_1)} \ , \ \ a \leq x_1 \leq b \ , \\ A, B : c(a) = c_a \ , \ \ c(b) = c_b \ . \end{cases} \tag{3.12}$$

Note that since $w(a) = w(b) = 1$, the values of A and B in (3.12) are independent of the weight function w. Also, note that, as long as the value of c in (3.12) is admissible, and hence λ as in (3.1) gives an admissible Filippov sliding vector field, then we must have

$$Ae^{x_1} + Be^{-x_1} \neq 0, \ \text{ for all } \ x_1 \in [a, b] \ , \tag{3.13}$$

as otherwise the resulting vector field would be 0 at some point, giving an equilibrium, which is excluded.

Now, with respect to either of the above minimization tasks (that is, minimizing either the H^1 norm of $w\lambda$ or of wf_{F}), the following questions are natural.

Questions 3.5.

(i) Can we choose w so that the solution of (3.5)-(3.9) gives us the bilinear and moments solutions? More generally, can we interpret a given admissible solution as the minimum variation solution of (3.5)-(3.9) for some w?
(ii) Can we relate to each other the weight w and the reparametrization of time performed by ω?

[1]In fact, through a simple change of variable, the same reasoning holds true whenever $\Sigma_{1,2}$ are planes.

As already remarked, in general, the boundary value problems (3.6) and (3.10) do not appear to be easy to solve exactly, and probably one would need to solve them numerically. However, in the important special cases of Remarks 3.3 and 3.4 they can be solved exactly. We clarify in Example 3.6 below how we use these exact solutions to derive minimum variation solutions, and answer the above questions on a concrete Example. Then, we will generalize our construction.

Example 3.6 (Another model problem)**.** This is very similar to Example 2.1, except for the first component of the vector fields, chosen so that there are no equilibria on the sliding segment. We have f_i, $i = 1, 2, 3, 4$, taking values in $\mathbb{R}^3$:

$$f_1(x) := \begin{bmatrix} e^{-x_1} + 1 \\ -x_1 + x_2 x_3 + 1 \\ x_1 + x_2 + 1 \end{bmatrix}, x \in R_1 \ , \quad f_2(x) := \begin{bmatrix} e^{-x_1} - 1 \\ -x_1 + x_3 - 1 \\ x_1 + x_2 x_3 + 2 \end{bmatrix}, x \in R_2 \ ,$$

$$f_3(x) := \begin{bmatrix} -e^{-x_1} + 1 \\ -x_1 + x_2 + 2 \\ x_1 + x_2 x_3 - 1 \end{bmatrix}, x \in R_3 \ , \quad f_4(x) := \begin{bmatrix} -e^{-x_1} + 2 \\ -x_1 + x_3 - 2 \\ x_1 + x_3 - 2 \end{bmatrix}, x \in R_4 \ ,$$

where $\Sigma_1 = \{x : \ x_3 = 0\}$, $\Sigma_2 = \{x : \ x_2 = 0\}$, and so $\Sigma = \Sigma_1 \cap \Sigma_2$ is the x_1-axis. The admissible region for c is the same as in Example 2.1, that is the triangle of Figure 1, hence we have $a = -1.2$, $b = 1.2$, and $c_a = 0.8$, $c_b = 0$, and $\lambda = \mu + cv$ as in (2.2). There is sliding motion on Σ from a to b.

(a) The minimum variation solution with weight $w \equiv 1$, with respect to λ, that is the solution in (3.8), is

$$c_{MV,\lambda} \ = \ \frac{1}{\|v\|^2}\big(Ae^{x_1} + Be^{-x_1} - v^\top \mu\big) \ ,$$

with $v^\top \mu = \frac{44}{27} x_1 - \frac{11}{9}$, $\|v\|^2 = \frac{44}{9}$, and the constants A, B, so that $c_{MV,\lambda}(-1.2) = 0.8$ and $c_{MV,\lambda}(1.2) = 0$.

(b) The minimum variation solution with weight $w \equiv 1$, with respect to $f_{\rm F}$, that is the the solution in (3.12) is

$$c_{MV,f_{\rm F}} \ = \ \frac{1}{f_v}\big(Ae^{x_1} + Be^{-x_1} - f_\mu(x_1)\big) \ ,$$

with $f_v = -\frac{4}{3}e^{-x_1} - \frac{7}{3}$ and $f_\mu = -\frac{10}{9} x_1 e^{-x_1} + \frac{1}{3} e^{-x_1} - \frac{1}{9} x_1 + \frac{4}{3}$, and the constants A, B, so that $c_{MV,f_{\rm F}}(-1.2) = 0.8$ and $c_{MV,f_{\rm F}}(1.2) = 0$.

In Figure 3 we show the five functions c we discussed for this problem: moments, bilinear, triangular, and the two minimum variation solutions (with weight $w = 1$). We also show the "broken-line" solution, corresponding to the selection of c given by the path along the two other sides of the triangular region. In this case, all these solutions are admissible (all smooth, except the broken line solution), and give different Filippov sliding vector fields, all smoothly exiting. The corresponding vector fields are shown in Figure 4.

We are finally ready to answer in the positive, on this example, Questions 3.5. The reason why we can answer positively those questions is that there are no equilibria, and thus:

$$\big(Ae^{x_1} + Be^{-x_1}\big)(f_{\rm F})_1 > 0 \ ,$$

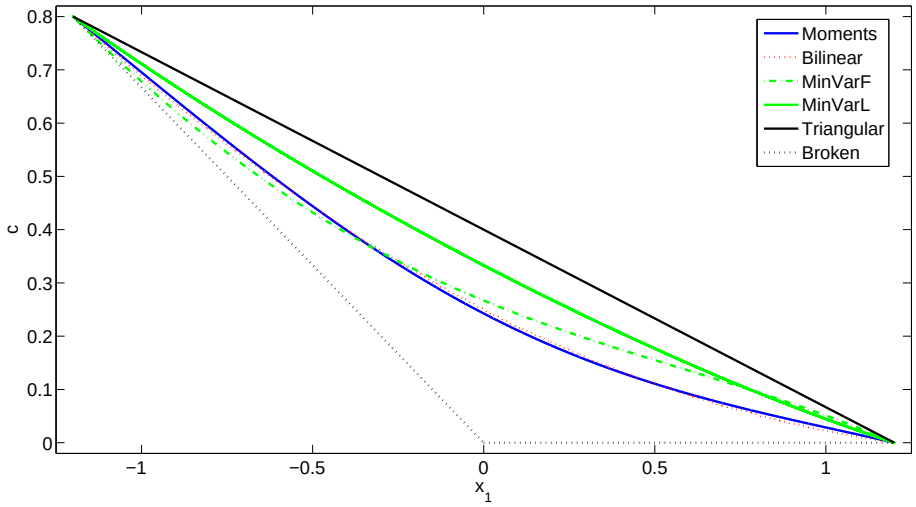

FIGURE 3. Admissible region (x_1, c) and moments, bilinear, triangular, broken-line, and minimum variation solutions of c.

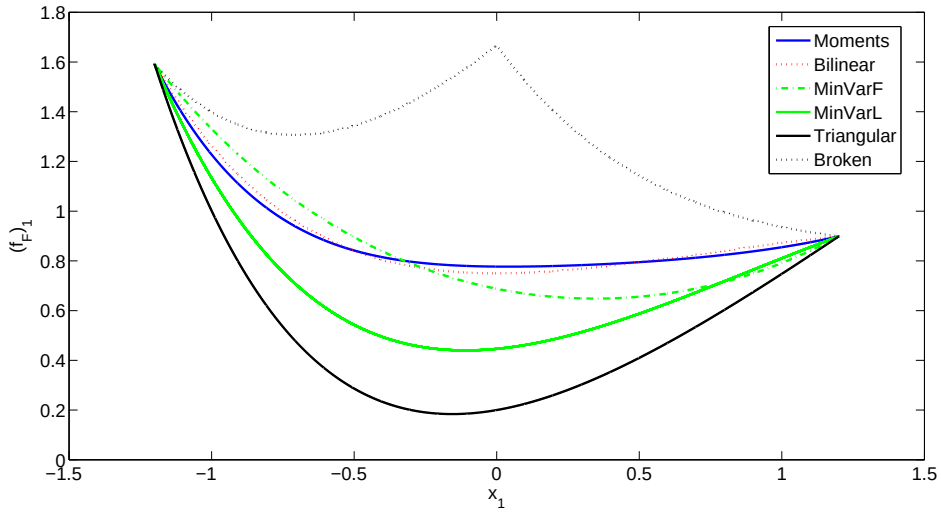

FIGURE 4. Sliding vector fields for moments, bilinear, triangular, broken-line, and minimum variation solutions.

where $(f_{\mathrm{F}})_1$ is the first component of any of the above vector fields (the second and third components being 0 in the present case).

(i) In light of the above, we can choose the weight w so that the solution of (3.9) gives us any of the above solutions. In fact, for any admissible c giving us a sliding vector field f_{F}, we **define** the weight w, which gives c as the minimum

variation of (3.9), from

$$w(x) \;=\; \frac{Ae^{x_1} + Be^{-x_1}}{(f_{\mathrm{F}})_1}\ . \tag{3.14}$$

By construction, using this weight w in the minimization of (3.9) will give us the function c which gave f_{F}. In particular, also the bilinear, triangular, and moments solutions are in fact weighted minimum variation solutions. The "broken line" solution, not being smooth, cannot be obtained as solution of (3.9) with smooth w; nonetheless, we still formally define its associated weight as above (it is attainable as the limit of smooth solutions).

(ii) As we know, the previously displayed vector fields (see Figure 4) are all orbitally equivalent. In particular, it must be true that any of the vector field is a multiple of the vector field obtained as minimum variation with respect to f_{F} with weight 1. Because of (3.14), thus we must have

$$\omega(x) \;=\; \frac{1}{w(x)}\ , \tag{3.15}$$

where $w(x)$ is the weight associated to the specific choice of f_{F} under consideration; see (3.14). (In other words, in (3.2) we are using $f_{\mathrm{F}1} = f_{\mathrm{MV}}$ -minimum variation with respect to f_{F} with weight $w = 1$- and $f_{\mathrm{F}2}$ any of the previously obtained sliding vector fields). In Figure 5 we show the values of ω for the vector fields above. We observe that the moments and bilinear solutions give quite similar functions ω. Also, observe that the broken-line solution gives (as expected) a non-smooth factor ω. Looking at Figure 5, we conclude that all possible values of ω must be within the upper and lower curves, that is in between the functions ω of the triangular and broken-line solutions.

To conclude our discussion on this example, we observe that the broken-line solution takes the least amount of "time" to travel from a to b:

$$t_{\mathrm{broken}} \approx 1.929 < t_m \approx 2.760 < t_b \approx 2.764 < t_{MV,f_{\mathrm{F}}} \approx 2.956 < t_{MV,\lambda} \approx 3.850 < t_{tr} \approx 6.554\ .$$

This was predictable, since –being all vector fields orbitally equivalent– we have that with respect to the time t given by selecting $c_{MV,f_{\mathrm{F}}}$, all other times come from $\mathrm{d}\tau = \frac{1}{\omega}\mathrm{d}t$, and therefore "the larger ω, the shorter the time" (see Figure 5). The fact that the broken-line solution gives the shortest time is also consistent with the general flavor of results in optimal control theory, whereby it is known that, for linear problems with constraints, the optimal control (here, the value of c giving the minimal time solution) lies on the boundary of the admissible region (see [13]). Likewise, the admissible solution taking the longest time is the triangular solution.

3.2. **Generalization.** Example 3.6 was sufficiently simple that we were able to answer Questions 3.5 and to explicitly give the orbital equivalency factors ω. At the same time, the process we used is fully general, and it can be used, for example, anytime the situation of Remark 3.4 applies.

With the previous notation, we summarize it in the following theorem.

Theorem 3.7. *Let Conditions 3.1 hold. Let* $\Sigma_1 = \{x :\ x_2 = 0\}$, $\Sigma_2 = \{x :\ x_3 = 0\}$. *In the notation of Conditions 3.1, let* Σ *be the segment* (a, b) *on the* x_1*-axis. Let the*

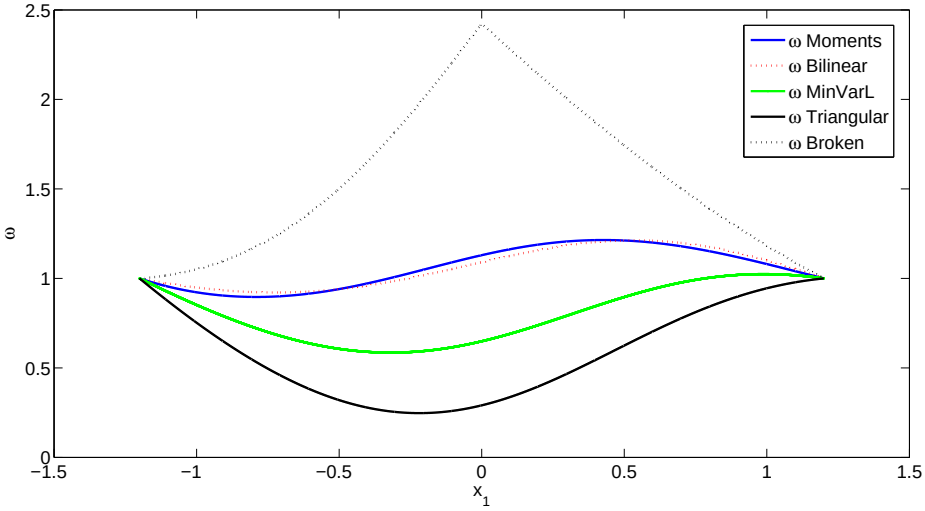

FIGURE 5. Orbital equivalence factors ω for moments, bilinear, triangular, broken-line, and minimum variation with respect to λ.

general solution for λ be as in (1.10)*, with the particular solution μ and the vector v smoothly varying in Σ (for example, μ could be the moments solution λ_M), and let the smooth function c in* (1.10) *be subject to the constraints $\alpha(x_1) \leq c(x_1) \leq \beta(x_1)$, for all $a \leq x_1 \leq b$. Let $\widehat{f_F}$ be any smooth Filippov sliding vector field on Σ, obtained from smooth, admissible coefficients (for example, the moments' vector field f_M), in particular with a smooth admissible function $\widehat{c}$ in* (1.10) *so that $\widehat{\lambda} = \mu + \widehat{c}v$ at the exit points render the coefficients of the smoothly exiting Filippov vector field.*

Assume that $f_v \neq 0$ on Σ, and consider the boundary value problem (3.11) *with solution* (3.12)*, and with A and B as there. Assume that* (3.13) *holds.*

(i) *If $\left(Ae^{x_1} + Be^{-x_1}\right)\left(\widehat{f_F}(x_1)\right)_1 > 0$, for all $x_1 \in \Sigma$, then the function*

$$\widehat{w}(x_1) \;=\; \frac{Ae^{x_1} + Be^{-x_1}}{\left(\widehat{f_F}(x_1)\right)_1}\;, \tag{3.16}$$

is the weight function associated to $\widehat{f_F}$. That is, this weight function $\widehat{w}$ is such that the H^1 minimization problem for wf_F gives the function $\widehat{c}$ as solution of (3.11)*.*

(ii) *On the other hand, let w be an arbitrary weight function as in* (3.4)*, and let c be the smooth function in* (3.12)*. This will be admissible if and only if, for all $x_1 \in \Sigma$, we have*

$$\widehat{c}(x_1) + \widehat{w}(x_1)\frac{f_\mu(x_1)}{f_v(x_1)} - \beta(x_1) \leq w(x_1)\frac{f_\mu(x_1)}{f_v(x_1)} \leq \widehat{c}(x_1) + \widehat{w}(x_1)\frac{f_\mu(x_1)}{f_v(x_1)} - \alpha(x_1)\;, \tag{3.17}$$

where $\widehat{w}$ and $\widehat{c}$ are an admissible weight and its associated solution in (3.11). *When c is admissible, the resulting vector field is orbitally equivalent to that associated to $\widehat{c}$, with orbital equivalence factor $1/w$.*

(iii) *If* (3.13) *is violated, that is* $\left(Ae^{x_1} + Be^{-x_1}\right) = 0$ *at some $x_1 \in \Sigma$, then there is no admissible sliding vector field obtained as solution of the Euler Lagrange equation, by minimization of the H^1 norm of wf_F, for any weight function w.*

Proof. Statement (i) holds by construction. Indeed, since $\left(\widehat{f_{\mathrm{F}}}(x_1)\right)_1 = f_\mu + \widehat{c} f_v$, we seek the function $\widehat{w}$ for which (3.11) holds. That is, we want $\widehat{w}$ such that

$$\widehat{c}(x_1) \;=\; \frac{Ae^{x_1} + Be^{-x_1} - \widehat{w}(x_1) f_\mu(x_1)}{\widehat{w}(x_1) f_v(x_1)} \;,$$

which gives (3.16). Note that, since $\widehat{c}$ is admissible and the resulting $\widehat{\lambda}$ at the exit points give the coefficients of the smoothly exiting vector fields, then we have $w(a) = w(b) = 1$ because of the way A and B were found. To verify (3.17), we need to check whether or not the function c one finds is admissible. Because of (3.12), we always have (for all $x_1 \in \Sigma$):

$$\begin{aligned} \left(c(x_1) f_v(x_1) + f_\mu(x_1)\right) w(x_1) \;&=\; Ae^{x_1} + Be^{-x_1} \;, \text{ and} \\ \left(\widehat{c}(x_1) f_v(x_1) + f_\mu(x_1)\right) \widehat{w}(x_1) \;&=\; Ae^{x_1} + Be^{-x_1} \;. \end{aligned}$$

from which we get

$$c(x_1) \;=\; (\widehat{w}(x_1) - w(x_1)) \frac{f_\mu(x_1)}{f_v(x_1)} + \widehat{c}(x_1) \;.$$

The constraint $\alpha(x_1) \le c(x_1) \le \beta(x_1)$ can thus be rewritten as in (3.17). The statement on orbital equivalence is obvious. Finally, validity of the statement (iii) is simply because, in case (3.13) is violated, the resulting minimum variation vector field would give an equilibrium, which is excluded. □

Remark 3.8. We note that the point (iii) of Theorem 3.7 does not contradict Remark 2.2. In fact, in order to find a minimum solution for (3.9), we have solved its associated Euler-Lagrange equation without enforcing the constraint on c (ensuring that the corresponding $\lambda = \mu + cv$ has nonnegative components adding to one). Therefore, it could happen that the unconstrained solution does not lie completely in the admissibility set, as it happens when, as proven above, (3.13) is violated. In other words, the unique solution of the constrained minimization problem would be a boundary solution with respect to the admissibility set, thus not solving the Euler-Lagrange equation associated to the unconstrained problem.

Remark 3.9. When $F_v = 0$ in (3.10), and in particular $f_v = 0$ in (3.11), the technique based on minimization of the H^1-norm of wf_{F} gives a singular differential equation. We have not explored in details this situation (which would require analyzing the nature of the singular points), but observe that in the case of $F_v \equiv 0$ for all $a \le s \le b$ in (3.10), then the minimization task for wf_{F} is surely ill-posed. The next example clarifies this statement.

Example 3.10 ([7])**.** Consider the following problem in $\mathbb{R}^3$:

$$f_1(x) := \begin{bmatrix} \frac{\sqrt{2}}{8}\sin\left(\frac{\pi}{4}-x_3^2\right) \\ \frac{\sqrt{2}}{8}\cos\left(\frac{\pi}{4}-x_3^2\right) \\ x_1^2+x_2^2+1 \end{bmatrix} \ , \ f_2(x) := \begin{bmatrix} 2\sqrt{2}\sin\left(\frac{3}{4}\pi-x_3^2\right) \\ \sqrt{2}\cos\left(\frac{3}{4}\pi-x_3^2\right) \\ x_1^2+x_2^2+1 \end{bmatrix} \ ,$$

$$f_3(x) := \begin{bmatrix} \sqrt{2}\sin\left(\frac{\pi}{4}-2x_3^2\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4}-2x_3^2\right) \\ x_1^2+x_2^2+1 \end{bmatrix} \ , \ f_4(x) := \begin{bmatrix} -2 \\ -1 \\ x_1^2+x_2^2+1 \end{bmatrix} \ ,$$

$\Sigma_1 := \{x \in \mathbb{R}^3 \ : \ x_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^3 \ : \ x_2 = 0\}$ and $\Sigma := \Sigma_1 \cap \Sigma_2$ is just the x_3-axis, which is in particular attractive in the segment $\gamma := \{-\sqrt{\pi/2} < x_3 < \sqrt{\pi/2}\}$ (the endpoints being generic first order exit points).

In this problem, we stress that $f_v(x_3) = 0$ for all $x_3 \in \gamma$:

$$\left[x_1^2+x_2^2+1 \quad x_1^2+x_2^2+1 \quad x_1^2+x_2^2+1 \quad x_1^2+x_2^2+1\right]\Big|_{x\in\Sigma} v = \mathbb{1}^\top v = 0 \ ,$$

and further –no matter what choice of coefficients we make– all sliding vector fields will always be: $f_{\mathrm{F}}(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (that is, $\dot{x}_3 = 1$). As a consequence, the minimum variation requirement in (3.9) is ill-posed, as any λ solution of (1.6) would provide the same sliding vector field. The minimum variation solution requirement in (3.5) is feasible, though, and indeed not all different choices of λ will provide sets of coefficients that give the exiting vector fields.

The admissibility region for this problem (found from (1.11) using the moments solution as particular solution and the smooth eigenvector v of Lemma 1.1), is the region comprised between the two curves in Figure 6 (these are α and β in (1.11)). Looking at Figure 6, it is clear that, when the dynamics enters or exits from sliding motion on Σ, there are intervals of admissible values for c in (1.10). At the same time, for a Filippov vector field to exit smoothly from Σ, it is necessary that its corresponding λ coefficient coincides with λ_{m} at first order exit points (see [7]). Therefore, there is only one way to enter/exit smoothly from Σ in this specific problem, and it is given by the end values of c selected by the moments solution in Figure 6. For comparison, we also show the values of c selected by the bilinear solution; since the end values do not coincide with those of the moments solution, we infer that the bilinear solution cannot be a minimum variation solution nor can be smoothly exiting. This last observation is corroborated by the results in [7].

3.3. **Revisiting Example 2.1: Singular weights.** We conclude our discussion on minimization of the H^1 variation of admissible solutions, with some considerations on the case of sliding vector fields with equilibria on Σ. In particular, we reconsider Example 2.1. That was a situation where -unlike the scenario of Conditions 3.1- **every** smooth sliding vector field of Filippov type had an equilibrium on Σ. Suppose that this is indeed the case, and thus consider the following scenario, still in $\mathbb{R}^3$, and still considering as discontinuity surfaces $\Sigma_1 = \{x : \ x_2 = 0\}$, and $\Sigma_2 = \{x : \ x_3 = 0\}$ (see Remark 3.4).

Conditions 3.11 (Equilibrium on Σ).

(i) The sliding manifold is the segment $\Sigma = \{x_1 : \ a \le x_1 \le b\}$.

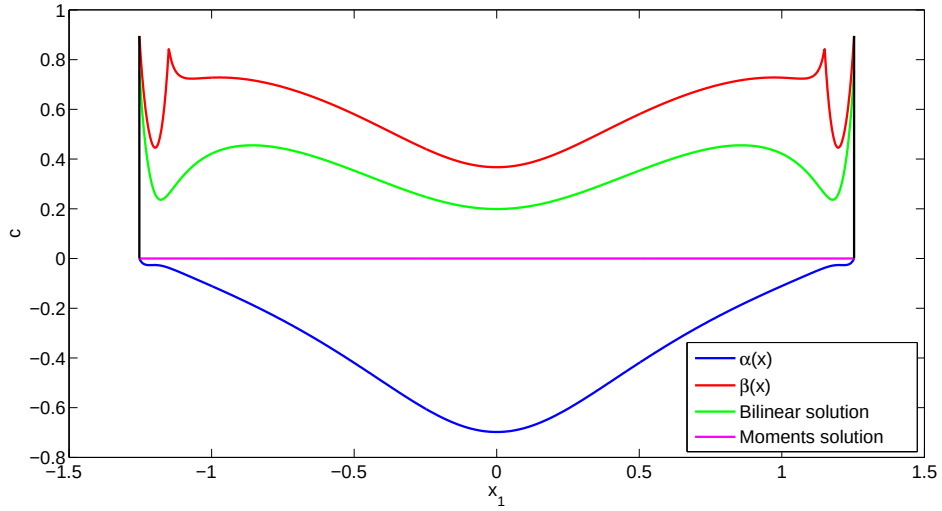

FIGURE 6. Curves of $\alpha(x_3)$, $\beta(x_3)$ defining the admissible region (1.11) for Example 3.10, and moments and bilinear solutions.

(ii) For $a < x_1 < b$, Σ is attractive, any smooth Filippov sliding vector field f_{F} has one -and just one- equilibrium $\bar{x}$ on Σ (the value of $\bar{x}$ depends on the choice of vector field), which is unstable and generic [2]. Let $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ 0 \\ 0 \end{bmatrix}$, so that motion on Σ proceeds from any left neighborhood of $\bar{x}_1$ to a (right-to-left) and from any right neighborhood of $\bar{x}_1$ to b (left-to-right).

(iii) The points $x_1 = a$ and $x_1 = b$ are generic first order exit points.

Obviously, under Conditions 3.11, different sliding vector fields cannot be orbitally equivalent, and the dynamics on Σ differ (unless all possible sliding vector field share the same equilibrium). Indeed, in the case of Conditions 3.11, and with the above notation, we have this result.

Theorem 3.12. *Assume that $f_v \neq 0$ for $x_1 \in [a, b]$, and that, for $w = 1$, the solution c_{MV,f_F} in (3.12) of the boundary value problem (3.11) is well defined and gives an admissible smooth Filippov sliding vector field f_{F1}. Then, the following holds.*

(i) *The function*

$$Ae^{x_1} + Be^{-x_1}$$

is 0 at the point $\bar{x}_1$, equilibrium of $(f_{F1})_1$ (cfr. with (3.13)).

(ii) *The only admissible weight functions w, satisfying (3.4) and giving an admissible solution c of (3.12), are those for which the resulting vector field has the equilibrium at $\bar{x}$.*

(iii) *To any other sliding vector field f_F formed from an admissible c, we can associate a* singular weight *w, namely one which goes through 0 and changes sign at the value $\bar{x}_1$, and that has a first order pole at the zero of $(f_F)_1$. As a consequence,*

[2] By this, we mean that $\frac{\mathrm{d}}{\mathrm{d}x_1}(f_{\mathrm{F}})_1\Big|_{x_1=\bar{x}_1} \neq 0$

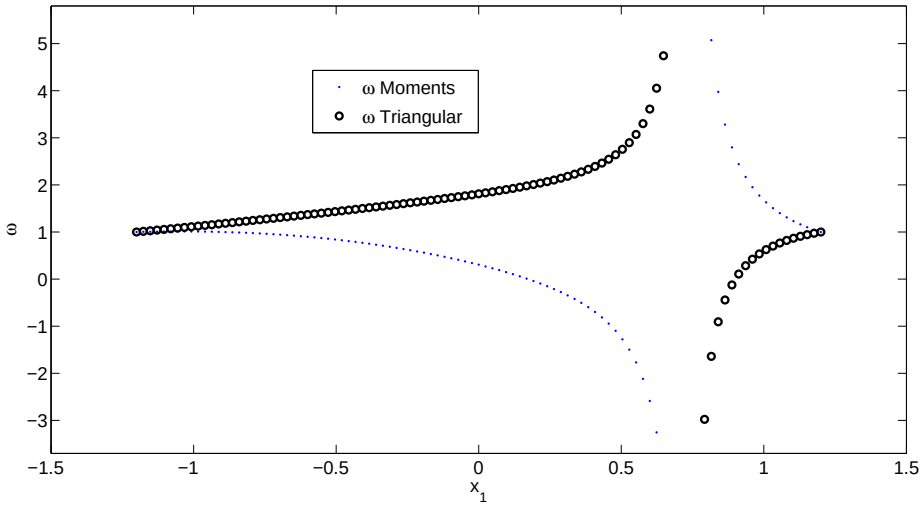

FIGURE 7. Moments and triangular orbital pseudo-equivalence factors, with respect to $f_{\mathrm{F}1}$ for Example 2.1.

there is a singular orbital pseudo-equivalence *factor* ω*, relating* f_F *and* f_{F1}*, given by* $1/w$*;* ω *is* 0 *at the equilibrium of* $(f_F)_1$ *and has a first order pole at* $\bar{x}_1$*.*

Proof. By hypothesis, we have that $c_{MV,f_{\mathrm{F}}} = \frac{Ae^{x_1}+Be^{-x_1}-f_\mu(x_1)}{f_v(x_1)}$, and therefore,

$$f_{\mathrm{F}1} = \begin{bmatrix} f_\mu(x_1) + c_{MV,f_{\mathrm{F}}}(x_1) f_v(x_1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Ae^{x_1} + Be^{-x_1} \\ 0 \\ 0 \end{bmatrix} ,$$

from which point (i) follows. To verify point (ii), suppose there were a weight function w satisfying (3.4), giving an admissible solution c_w of (3.12), and such that the resulting vector field has an equilibrium at a point different from $\bar{x}$. Then, we must have

$$w(x_1) = \frac{Ae^{x_1} + Be^{-x_1}}{f_\mu(x_1) + c_w(x_1) f_v(x_1)} \, . \tag{3.18}$$

But, the denominator of this expression vanishes at the equilibrium of the vector field $f_\mu(x_1) + c_w(x_1) f_v(x_1)$, and since -by hypothesis- this is different from $(\bar{x})_1$, we reach the contradiction that w satisfies (3.4), and the claim follows.

Finally, point (iii) follows at once from the expression (3.18). □

We illustrate Theorem 3.12, by considering the orbital pseudo-equivalence factors for the moments and the triangular solutions of Example 2.1. See Figure 7.

4. Conclusions

In this work, we considered selection of a smooth sliding vector field, in the class of Filippov sliding vector fields, on a co-dimension 2 manifolds in $\mathbb{R}^3$. It is well understood that there is a one-degree-of-freedom algebraic ambiguity in this selection process. To

resolve this amibiguity, we reformulated the problem as one in which we seek a minimum variation solution in the H^1-norm for either the coefficients entering in the convex combination, or for the sliding vector field itself. We explicitly solved the resulting Euler-Lagrange equation on some model problems, and compared the resulting minimum variation solution(s) to other sliding vector fields previously considered in the literature (most notably, the bilinear and moments solutions). Moreover, we have also proved, under suitable assumptions, that a properly weighted minimum variation solution coincides with other smoothly varying sliding vector fields (say, the moments method), the weight itself providing a time reparametrization from one vector field to the other.

Although the methodology proposed in this work does not seem to be of trivial, nor universal, applicability (already in $\mathbb{R}^3$), it provides a promising alternative to existing approaches in case the "entry" and "exit" points of sliding motion are known. In fact, it is our opinion that the present minimum-variation ideas can eventually provide insight into appropriate minimality properties of a Filippov sliding vector field.

The extension of our approach to the case of systems in $\mathbb{R}^4$ (and beyond) presents some very interesting and challenging mathematical and modeling issues. We anticipate studying these in future work.

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