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ON THE INVERSE OF SOME SIGN MATRICES AND ON THE MOMENTS SLIDING VECTOR FIELD ON THE INTERSECTION OF SEVERAL MANIFOLDS: NODALLY ATTRACTIVE CASE

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Abstract. In this paper, we consider selection of a sliding vector field of Filippov type on a discontinuity manifold Σ of co-dimension 3 (intersection of three co-dimension 1 manifolds). We propose an extension of the *moments vector field* to this case, and –under the assumption that Σ is nodally attractive– we prove that our extension delivers a uniquely defined Filippov vector field. As it turns out, the justification of our proposed extension requires establishing invertibility of certain sign matrices. Finally, we also propose the extension of the moments vector field to discontinuity manifolds of co-dimension 4 and higher.

1. Introduction

An outstanding difficulty in the study of piecewise smooth (PWS) systems is the selection of a sliding vector field of Filippov type ([9]) on a discontinuity manifold Σ of co-dimension 2 or higher. In recent times, this problem has received considerable attention in case Σ has co-dimension 2; e.g., see [1, 4, 5, 11, 14] for studies of the so-called *bilinear vector field* (see below). Instead, in [6], we proposed an alternative way to define a smooth sliding vector field on a co-dimension 2 discontinuity manifold, which we called *moments vector field*, and that presents some advantages with respect to the bilinear vector field, specifically when Σ loses attractivity at first order exit points.

Our purpose in this work is to propose the extension of the moments vector field to the case of a discontinuity manifold Σ of co-dimension 3 (and higher), intersection of three co-dimension 1 manifolds. In this case, and unlike the extension of the bilinear vector field (which suffers from severe lack of uniqueness), our proposed moments vector field is uniquely defined under appropriate attractivity assumptions on Σ.

As it turns out, the justification of our proposal rests on the following two theorems. The first theorem is about invertibility of a $(8,8)$ matrix M whose entries have some specified sign-pattern (and specific relations to one another). The second theorem is about a particular solution of the system with matrix M and right-hand side given by the unit vector e_8.

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Theorem 1.1. *Let* $w_i = \begin{bmatrix} w_i^1 \\ w_i^2 \\ w_i^3 \end{bmatrix}$, $i = 1, \dots, 8$, *be eight vectors in* $\mathbb{R}^3$, *and consider the matrix* $W \in \mathbb{R}^{3\times 8}$ *given by*

$$W := \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix} . \tag{1.1}$$

Assume that the entries of W *are nonzero and have the following signs*

$$\begin{bmatrix} + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{bmatrix} \tag{1.2}$$

Let $\Delta \in \mathbb{R}^{3\times 8}$ *be the following matrix of "signed" partial distances*

$$\Delta := \begin{bmatrix} \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} & \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23} \\ \delta_1^{13} & -\delta_2^{13} & \delta_3^{13} & -\delta_4^{13} & -\delta_5^{13} & \delta_6^{13} & -\delta_7^{13} & \delta_8^{13} \\ \delta_1^{12} & \delta_2^{12} & -\delta_3^{12} & -\delta_4^{12} & -\delta_5^{12} & -\delta_6^{12} & \delta_7^{12} & \delta_8^{12} \end{bmatrix} , \tag{1.3}$$

where, for each $i = 1, \dots, 8$,

$$\delta_i^{23} := \sqrt{(w_i^2)^2 + (w_i^3)^2} \ , \ \ \delta_i^{13} := \sqrt{(w_i^1)^2 + (w_i^3)^2} \ , \ \ \delta_i^{12} := \sqrt{(w_i^1)^2 + (w_i^2)^2} \ .$$

Finally, let

$$d^\top := \begin{bmatrix} d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} , \quad d_i := \|w_i\|_2, \ i = 1, \dots, 8 \ , \tag{1.4}$$

and let $\mathbb{1} \in \mathbb{R}^8$ *be the vector of all* 1*'s.*

Then, the matrix

$$M := \begin{bmatrix} W \\ \Delta \\ d^\top \\ \mathbb{1}^\top \end{bmatrix} \tag{1.5}$$

is invertible. □

Theorem 1.2. *With* M *as in Theorem 1.1, consider the linear system*

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \tag{1.6}$$

Call λ_M *the solution of* (1.6)*. Then,* λ_M *has all positive components:* $\lambda_{M,i} > 0$, $i = 1 \dots, 8$. □

Remark 1.3. *There are several works in linear algebra about sign-invertibility of a matrix, that is relying solely on the signs of the entries of the given matrix; see the works of Thomassen,* [15]*, Klee and others,* [12, 13]*, and the comprehensive treatment in* [3]*. For example, if the matrix* M *in* (1.5) *were an* L*-matrix, then it would be possible to establish its invertibility and signs of the entries of the inverse by appealing to these results. Unfortunately, however, these results are quite general and an* L *matrix is really an equivalence class of matrices (with respect to the sign of their entries), and our special matrix* M *in* (1.5) *does not fit in the* L*-matrix class. As a consequence, the existing results on sign-invertibility of matrices cannot be used to establish that* M *is invertible*

(Theorem 1.1) nor of course that the solution of the system in Theorem 1.2 is positive. For this reason, and motivated by the specific geometric structure of our problem, we will resort to a direct proof which uses tools from convex geometry. See Sections 2 and 3.

As we clarify below, the two theorems above validate our proposed extension of the moments' method for selecting a sliding vector field on a discontinuity manifold Σ of co-dimension 3, under the assumption that Σ is nodally attractive.

In the remainder of this Introduction we give background information needed to set forth our proposal for the moments' method on a co-dimension 3 discontinuity manifold. In Section 2 we give preliminary results that clarify our construction and lay the groundwork to prove Theorems 1.1 and 1.2, which we do in Section 3. In Section 4 we propose the extension of the moments' method to the case of discontinuity manifolds of co-dimension 4 and higher. Conclusions are in Section 5.

Notation 1.4. *We will write 0_k for the origin in $\mathbb{R}^k$ and $\mathbb{1}_k$ for the vector of all 1's in $\mathbb{R}^k$, or simply 0 and $\mathbb{1}$ when it is clear from the context. Unless otherwise stated, the norm is always the 2-norm.*

1.1. **Motivation.** Consider the following piecewise smooth system,

$$x'(t) = f_i(x),\ x \in R_i,\ \ i = 1, \dots, 8, \tag{1.7}$$

where the regions R_i's are open, disjoint and connected sets of $\mathbb{R}^n$, so that $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region R_i the function f_i is smooth.

Moreover, the regions R_i's are separated by manifolds defined as 0-sets of smooth (at least $\mathcal{C}^2$) scalar functions h_i: $\Sigma_i := \{x \in \mathbb{R}^n\ :\ h_i(x) = 0\}$, $i = 1, 2, 3$, which intersect pairwise and all three of them. For notational convenience, we use

$$\Sigma_{1,2} := \Sigma_1 \cap \Sigma_2\ ,\ \ \Sigma_{1,3} := \Sigma_1 \cap \Sigma_3\ ,\ \ \Sigma_{2,3} := \Sigma_2 \cap \Sigma_3\ ,$$

to describe the three possible co-dimension 2 discontinuity manifolds, and further

$$\Sigma_{1,2}^{\pm} := \{x \in \Sigma_1 \cap \Sigma_2\ \ :\ h_3(x) \gtrless 0\}$$

and similarly for $\Sigma_{1,3}^{\pm}$ and $\Sigma_{2,3}^{\pm}$. Finally,

$$\Sigma := \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$$

will be the co-dimension 3 manifold of interest to us.

Without loss of generality, we label the regions R_i's as follows (see Figure 1 for an illustration of the situation)

$$\begin{aligned}
R_1 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) < 0, h_2(x) < 0, h_3(x) < 0\},\\
R_2 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) < 0, h_2(x) < 0, h_3(x) > 0\},\\
R_3 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) < 0, h_2(x) > 0, h_3(x) < 0\},\\
R_4 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) < 0, h_2(x) > 0, h_3(x) > 0\},\\
R_5 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) > 0, h_2(x) < 0, h_3(x) < 0\},\\
R_6 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) > 0, h_2(x) < 0, h_3(x) > 0\},\\
R_7 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) > 0, h_2(x) > 0, h_3(x) < 0\},\\
R_8 &:= \{x \in \mathbb{R}^n\ :\ h_1(x) > 0, h_2(x) > 0, h_3(x) > 0\}\ .
\end{aligned}$$

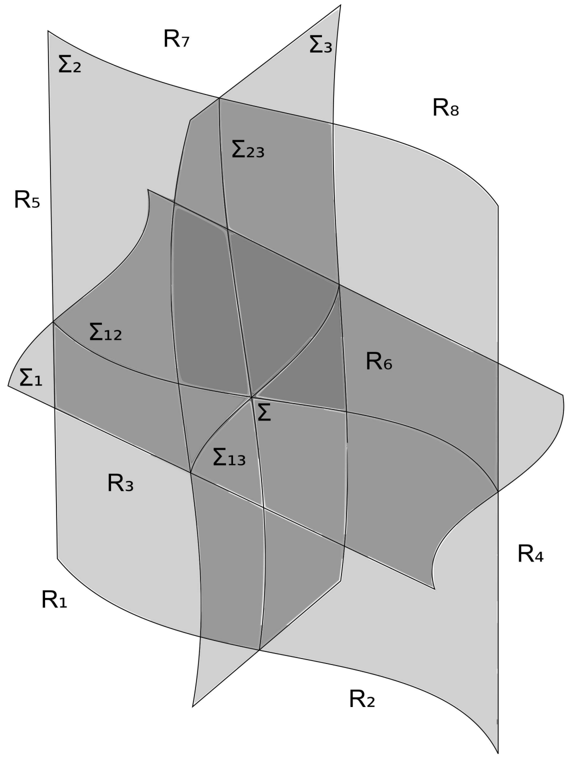

FIGURE 1. Regions and discontinuity surfaces.

Our goal is to describe a Filippov sliding vector field on Σ, which extends the moments vector field we proposed in [6] in the co-dimension 2 case.

1.2. **Sliding vector field.** We assume that $\{\nabla h_i(x)\}_{i=1,2,3}$ is a linearly independent set at any $x \in \Sigma$ and in a neighborhood of Σ.

For $x \in \Sigma$, define the projections of the vector fields f_i, $i = 1, \dots, 8$, onto the normal directions to the three manifolds:

$$w_i = \begin{bmatrix} w_i^1 \\ w_i^2 \\ w_i^3 \end{bmatrix} := \begin{bmatrix} \nabla h_1^\top f_i \\ \nabla h_2^\top f_i \\ \nabla h_3^\top f_i \end{bmatrix} , \quad i = 1, \dots, 8 . \tag{1.8}$$

Consider the matrix $W \in \mathbb{R}^{3\times 8}$ (which depends smoothly on x) as in (1.1):

$$W = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix} . \tag{1.9}$$

Next, we assume that the manifold Σ is **nodally attractive**, which we characterize by the following first order condition, that of course depends on the regions' labeling.

Definition 1.5. *We say that Σ is nodally attractive if the matrix W has the sign pattern of* (1.2). □

On Σ, we are interested in Filippov solutions of (1.7). In particular, we seek a sliding vector field of the form

$$f_F = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 + \lambda_5 f_5 + \lambda_6 f_6 + \lambda_7 f_7 + \lambda_8 f_8 \tag{1.10}$$

with nonnegative coefficients λ_i's adding to 1. Imposing that $f_{\rm F}$ is tangent to Σ, gives the following underdetermined linear system

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{1.11}$$

where $W \in \mathbb{R}^{3\times 8}$ is defined as in (1.9). It is evident that (1.11) is an underdetermined system. In Corollary 2.3 below we will show that the matrix $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ has a four dimensional kernel; hence, to select a unique Filippov sliding vector field on Σ, the issue is how "to fix" the four available degrees of freedom. Again, we stress that we are specifically interested in smooth vector fields on Σ; for this reason, we seek solutions of (1.11) with positive components, and with the λ_i's smoothly varying with $x \in \Sigma$, which we will call *admissible solutions*.

1.2.1. *Trilinear (interpolant) vector field.* A possible choice to determine an admissible solution of (1.11), and a vector field as in (1.10), is to select $\lambda \in \mathbb{R}^8$ of the form

$$\lambda = \begin{bmatrix} (1-\alpha)(1-\beta)(1-\gamma) \\ (1-\alpha)(1-\beta)\gamma \\ (1-\alpha)\beta(1-\gamma) \\ 1-\alpha)\beta\gamma \\ \alpha(1-\beta)(1-\gamma) \\ \alpha(1-\beta)\gamma \\ \alpha\beta(1-\gamma) \\ \alpha\beta\gamma \end{bmatrix}, \tag{1.12}$$

where $\alpha,\ \beta,\ \gamma \in (0,1)$. Since the choice (1.12) clearly gives $\sum_i \lambda_i = 1$, one would need that $\alpha, \beta, \gamma \in [0,1]$ to have an admissible solution. Now, the relation (1.11) gives a nonlinear system of three equations in the three unknowns α, β, γ. As proven in [8], when Σ is nodally attractive, this nonlinear system always has a solution α, β, $\gamma \in (0,1)$. The choice (1.12) is the "natural" extension to the co-dimension 3 case of the bilinear interpolant method, and it is important to observe that the choice (1.12) is consistent with the bilinear interpolant technique on the lower co-dimension manifolds; indeed, alternately setting one of α, β, γ, to be 0 or 1, gives the 6 possible combinations needed for a sliding vector field on the relevant co-dimension 2 manifolds (namely, on $\Sigma^{\pm}_{1,2}, \Sigma^{\pm}_{1,3}, \Sigma^{\pm}_{2,3}$). For example, when $\gamma = 0$, one obtains the bilinear vector field on Σ^{-}_{12}, namely

$$(1-\alpha)\left[(1-\beta)f_1 + \beta f_3\right] + \alpha\left[(1-\beta)f_5 + \beta f_7\right]\ . \tag{1.13}$$

However, there is a difficulty with the formulation (1.12): even when Σ is nodally attractive, in general there is more than one admissible solution of the nonlinear system; see Example 1.6 below.

Example 1.6. *Consider the following matrix W, which corresponds to a nodally attractive discontinuity surface Σ (see Definition 1.5 and the sign pattern of* (1.2)*)*

$$W := \begin{bmatrix} 1 & 3 & 1 & 11 & -7 & -1 & -3 & -5 \\ 1 & 1 & -11 & -3 & 3 & 11 & -1 & -1 \\ 1 & -9 & 5 & -1 & 1 & -5 & 9 & -1 \end{bmatrix}.$$

As reported in [7]*, searching for the trilinear solution* (1.12) *relative to the system*

$$W\lambda = 0_3 \ , \tag{1.14}$$

gives two distinct solutions, associated to $(\alpha,\beta,\gamma) = (1/2, 1/2, 1/2)$ and to $(\alpha,\beta,\gamma) \approx (0.3316, 0.2913, 0.3080)$, namely

$$\lambda = \begin{bmatrix} 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \end{bmatrix} \quad \text{and} \quad \lambda \approx \begin{bmatrix} 0.3268 \\ 0.1459 \\ 0.1347 \\ 0.06 \\ 0.1626 \\ 0.0724 \\ 0.0668 \\ 0.0298 \end{bmatrix}.$$

(The Jacobian of the nonlinear system in (α,β,γ) associated to the first root is singular, as that root is double).

1.3. **Moments method.** In case of a discontinuity manifold of co-dimension 2 (intersection of two co-dimension 1 manifolds), in [6] we proposed a methodology to select a uniquely defined sliding vector field of Filippov type, and we called the resulting method the *moments' method*. Here we propose an extension of the *moments' method* as a mean to provide a sliding vector field in case Σ is of co-dimension 3.

Let us recall that, if Σ from (1.7) is a co-dimension 2 manifold, intersection of two co-dimension 1 manifolds Σ_1, Σ_2, then computing the moments' solution amounts to solving the linear system

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{1.15}$$

where

$$M := \begin{bmatrix} W \\ \mathbb{1}^\top \\ d^\top \end{bmatrix}, \quad W := \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \quad d := \begin{bmatrix} \|w_1\| \\ -\|w_2\| \\ -\|w_3\| \\ \|w_4\| \end{bmatrix}, \tag{1.16}$$

with

$$w_i^j := \nabla h_j^\top f_i, \quad i = 1,2,3,4, \quad j = 1,2,$$

being h_1 and h_2 the event functions of which Σ_1, and Σ_2, are the 0-sets.

In [6] it is proven that M is invertible whenever Σ is attractive by subsliding, in particular when Σ is nodally attractive, and that (1.15) provides a unique admissible solution λ_{M}. For later reference, we summarize this special case in the following theorem.

Theorem 1.7 ([6]). *Let* $W = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix} \in \mathbb{R}^{2\times 4}$ *have the following sign pattern:*

$$\begin{bmatrix} + & + & - & - \\ + & - & + & - \end{bmatrix}, \tag{1.17}$$

and let M *be defined as in* (1.16)*. Then the linear system* (1.15) *is nonsingular and has a unique admissible solution.* □

At this point, the key to understand how to provide the extension of the moments' method is to realize that –alongside the co-dimension 3 manifold Σ– there are also several lower co-dimension manifolds where solution trajectories can slide, approaching Σ. Specifically, in a neighborhood of Σ, there are three co-dimension 1 manifolds (namely, Σ_1, Σ_2, Σ_3), and three co-dimension 2 manifolds, namely $\Sigma_{1,2}$, $\Sigma_{1,3}$, $\Sigma_{2,3}$. Now, under the assumption of nodal attractivity of Σ, there is a unique Filippov sliding vector field on the co-dimension 1 manifolds, but there is an ambiguity of how to select a Filippov sliding vector field on the co-dimension 2 manifolds. Therefore, to arrive at an appropriate extension of the moments' method, we will need to insist that on the co-dimension 2 manifolds we are using the moments' vector field as sliding vector field. We will need to further make sure that an appropriate distinction is made between the cases of $\Sigma_{1,2}^+$ and $\Sigma_{1,2}^-$, since different vector fields enter in the convex combination defining the moments sliding vector field in these cases (and similarly for Σ_{13}^+ and $\Sigma_{1,3}^-$, and $\Sigma_{2,3}^+$ and $\Sigma_{2,3}^-$).

Guided by the above consideration, our idea is to normalize (1.11) in the same fashion of co-dimension 2 which leads to consider precisely the matrix of "signed" partial distances (1.3). To witness, consider the sub-surface $\Sigma_{2,3}$, that is the subset of $x \in \mathbb{R}^3$ for which $h_2(x) = 0$ and $h_3(x) = 0$. Looking at the sign pattern of W in (1.2), we notice that two natural sets of vertices w_i's arise, namely $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6, w_7, w_8\}$, according to the sign of their first component: the first four vertices have $w_i^1 > 0$, $i = 1, 2, 3, 4$; the last four vertices have $w_i^1 < 0$, $i = 5, 6, 7, 8$. Moreover, the sign pattern of $\begin{bmatrix} w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \end{bmatrix}$ and $\begin{bmatrix} w_5^2 & w_6^2 & w_7^2 & w_8^2 \\ w_5^3 & w_6^3 & w_7^3 & w_8^3 \end{bmatrix}$ is the same as that in (1.17), that is the nodal attractivity sign pattern in co-dimension 2. This implies that the two sets $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6, w_7, w_8\}$ are determining subsliding towards Σ, on $\Sigma_{2,3}^-$ and $\Sigma_{2,3}^+$ respectively. From Theorem 1.7, we know that the moments vector fields on $\Sigma_{2,3}^\pm$ is well defined. This means that, on $\Sigma_{2,3}^+$, there are unique admissible solutions of

$$\begin{bmatrix} w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \\ 1 & 1 & 1 & 1 \\ \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and of

$$\begin{bmatrix} w_5^2 & w_6^2 & w_7^2 & w_8^2 \\ w_5^3 & w_6^3 & w_7^3 & w_8^3 \\ 1 & 1 & 1 & 1 \\ \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

where $\delta_i^{23} = \sqrt{(w_i^2)^2 + (w_i^3)^2}$, $i = 1, \dots, 8$. This implies that –within the moments' method framework– we must regularize those two blocks with the corresponding partial distance vector relative to $\Sigma_{2,3}$: we then choose to append the row

$$\begin{bmatrix} \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} & \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23} \end{bmatrix}$$

to $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ in order to obtain consistency with the moments solution on $\Sigma_{2,3}$. Analogous reasoning relative to $\Sigma_{1,2}^{\pm}$ and $\Sigma_{1,3}^{\pm}$ leads us to regularize $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ by appending to it the matrix Δ as defined in (1.3).

Notice that, when $\delta_i^{jk} \neq 0$ for all $i = 1, \dots, 8$ and $j, k = 1, 2, 3$ (e.g., this is guaranteed when (1.2) holds for the signs of the entries of W), the sign pattern of Δ is

$$\begin{bmatrix} + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \end{bmatrix}. \tag{1.18}$$

Finally, we assemble the matrix (1.5), where the row of 1's is merely reflecting the convexity requirement, and the last row, the vector d in (1.4), formally expresses our proposal of *weights* to place on the vertices w_i's, $i = 1, \dots, 8$, to maintain the geometrical flavor of *moments* (so to make the origin the barycenter of the polytope).

Definition 1.8. *The matrix M* (1.5) *is called the* moments matrix*, and the* moments' method *(on Σ) consists in solving* (1.6) *for λ, and then using this λ in the construction of the sliding vector field* (1.10)*, which will be called* moments vector field*.*

As a consequence of Theorems 1.1 and 1.2, and under the assumptions therein, we thus have that the moments' method selects a unique solution λ with positive entries, and a unique sliding vector field (further, varying smoothly, since so do the entries of the matrix M).

Example 1.9. *With the matrix W as in Example 1.6, and forming M as in* (1.5)*, the unique moments solution λ_M, computed according to Theorem 1.2 and relative to* (1.14)*, is given by*

$$\lambda_M \approx \begin{bmatrix} 0.4492 \\ 0.0502 \\ 0.0327 \\ 0.1019 \\ 0.0492 \\ 0.0279 \\ 0.0321 \\ 0.2569 \end{bmatrix}.$$

Remark 1.10. *In the present case of Σ of co-dimension 3, to prove our results on the feasibility of the moments method, we are assuming that Σ is nodally attractive. Extensive computational evidence indicates that the method proposed herein continues to provide a unique solution with nonnegative entries also under more general attractivity configurations of Σ. Although we have not attempted a complete proof to include all other possible cases, we note that the proof of Theorem 1.1 (and thus also Theorem 1.2)* *holds under more generous assumptions that those of nodal attractivity only; see Remark 3.1.*

2. Preliminary results

In this section, we show that the system (1.11) has a 4-parameter family of solutions, as well as other results which we will use to prove Theorem 1.1.

First, we have the following handy linear algebra result.

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times m}$, $n < m$, be full rank, and let $b \in \mathbb{R}^n$. Consider the system*

$$Ax = b \ , \tag{2.1}$$

and let $d \in \mathbb{R}^m$ be a nonzero vector.

If there exist x and y solutions of (2.1), such that

$$d^\top x = \xi \ , \quad \text{and} \quad d^\top y = \eta \ ,$$

with $\xi \neq \eta$, then $\begin{bmatrix} A \\ d^\top \end{bmatrix}$ has rank $n+1$.

Proof. By hypothesis, $\dim\ker(A) = m - n$. Let then $V \in \mathbb{R}^{m\times(m-n)}$ be such that $\text{range}(V) = \ker(A)$, and by contradiction suppose that $d \in \text{range}(A^\top)$. Then we must have

$$d^\top Vc = 0,$$

for all $c \in \mathbb{R}^{m-n}$. Since both x and y are solutions of (2.1), then there exists $\overline{c} \in \mathbb{R}^{m-n}$ such that

$$y = x + V\overline{c}.$$

Therefore

$$\eta = d^\top y = d^\top x + d^\top V\overline{c} = \xi \ ,$$

and this contradicts the assumption $\xi \neq \eta$. Hence, $\begin{bmatrix} A \\ d^\top \end{bmatrix}$ has full rank $n+1$. $\square$

Next, we have the following simple result.

Lemma 2.2. *Let W satisfy the sign pattern of (1.2). Then*

$$\text{rank}\, W = 3.$$

Proof. By the sign pattern of $W(2:3, 1:2)$, $\text{rank}\, W \geq 2$. If, by contradiction, $\text{rank}\, W = 2$, then $w_i \in \text{span}\{w_1, w_2\}$ for all $i = 3, \dots, 8$; nonetheless, no linear combination of w_1, w_2 can match the signs of all w_i, $i = 3, \dots, 8$, at once. $\square$

Finally, we have the anticipated result.

Corollary 2.3. *Let $\widetilde{W} := \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. Then $\text{rank}\, \widetilde{W} = 4$, hence $\ker(\widetilde{W})$ is 4-dimensional.*

Proof. Because of Theorem 1.7, the matrix $\begin{bmatrix} w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix}$ contains a non-singular submatrix, hence it must have rank 3. Let us then consider the system

$$\begin{bmatrix} w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By Theorem 1.7, considering the first four columns and the last four columns of $\begin{bmatrix} w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix}$, there exist the two corresponding moments solutions λ and μ to these system with the following structures:

$$\lambda = \begin{bmatrix} * \\ * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ * \end{bmatrix},$$

and all their entries are nonnegative. Therefore, considering the extended matrix $\widetilde{W} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix}$ and exploiting the sign pattern of W in (1.2), we obtain that

$$\widetilde{W}\lambda = \begin{bmatrix} > 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \widetilde{W}\mu = \begin{bmatrix} < 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, using Lemma 2.1, we get that $\widetilde{W}$ has rank 4. □

In order to prove our main theorem, Theorem 1.1, we will show that the 8 columns of M, without the last entry equal to 1, give 8 affinely independent vectors. To achieve this, we will use a convex geometry construction, and the following objects will simplify our presentation.

Let

$$v_i := \begin{bmatrix} w_i \\ \Delta_i \\ d_i \end{bmatrix}, \quad i = 1, \dots, 8, \tag{2.2}$$

$$\widehat{v}_i := \operatorname{sign}(v_i), \quad i = 1, \dots, 8, \tag{2.3}$$

$$\widetilde{v}_i := \frac{\widehat{v}_i}{\|v_i\|_1}, \quad i = 1, \dots, 8, \tag{2.4}$$

and

$$\widetilde{V} := \operatorname{conv}\{\widetilde{v}_i \,:\, i = 1, \dots, 8\}, \tag{2.5}$$

and

$$V := \operatorname{conv}\{v_i \,:\, i = 1, \dots, 8\}. \tag{2.6}$$

We will further consider the polar of V:

$$V^\circ = \left\{ x \in \mathbb{R}^7 \, : \, v_i^\top x \le 1, \; i = 1, \dots, 8 \right\} \; . \tag{2.7}$$

Note that V° is a closed convex set containing the origin in its interior, and the boundary of V°, ∂V°, is given by $\partial V^\circ = \{x \in V^\circ \, : \, v_i^\top x = 1 \;, \;\text{ for some } \; i = 1, \dots, 8\}$.

Lemma 2.4. *For any $i = 1, \dots, 8$, the vectors $\widetilde{v}_i \in \partial V^\circ$. Moreover, for any $i, j = 1, \dots, 8$, $i \ne j$, $\widetilde{v}_i$, $\widetilde{v}_j$ belongs to distinct facets of the polyhedron V°, and $v_i^\top \widetilde{v}_j < 0$.*

Proof. Obviously, $v_i^\top \widetilde{v}_i = 1$ for all $i = 1, \dots, 8$.
Next, consider $\widehat{v}_1$. We notice that

$$\begin{aligned} \widehat{v}_1^\top v_2 &= w_2^1 + w_2^2 + w_2^3 - \delta_2^{23} - \delta_2^{13} + \delta_2^{12} - d_2 \\ &= (w_2^1 - \delta_2^{13}) + (w_2^2 - \delta_2^{23}) + (\delta_2^{12} - d_2) - |w_2^3| \\ &< 0 \; . \end{aligned}$$

Analogously, we have that $\widehat{v}_1^\top v_3 < 0$, $\widehat{v}_1^\top v_5 < 0$. Moreover,

$$\begin{aligned} \widehat{v}_1^\top v_4 &= w_4^1 + w_4^2 + w_4^3 + \delta_4^{23} - \delta_4^{13} - \delta_4^{12} + d_4 \\ &= (w_4^1 + d_4 - \delta_4^{13} - \delta_4^{12}) + (\delta_4^{23} - |w_4^2| - |w_4^3|) < 0 \; , \end{aligned}$$

since, in general, $\sqrt{a^2 + b^2} \le |a| + |b|$, and, for our particular setting, it holds that

$$\begin{aligned} & w_4^1 + d_4 - \delta_4^{13} - \delta_4^{12} < 0 \\ & \iff w_4^1 + d_4 < \delta_4^{13} + \delta_4^{12} \\ & \iff 2(w_4^1)^2 + (w_4^2)^2 + (w_4^3)^2 + 2w_4^1 d_4 < 2(w_4^1)^2 + (w_4^2)^2 + (w_4^3)^2 + 2\delta_4^{13}\delta_4^{12} \\ & \iff 0 < (w_4^2)^2 \cdot (w_4^3)^2, \end{aligned}$$

that is clearly true. Similarly, we have $\widehat{v}_1^\top v_6 < 0$ and $\widehat{v}_1^\top v_7 < 0$.
Finally, let us prove that $\widehat{v}_1^\top v_8 < 0$. This is the same as

$$-|w_8^1| - |w_8^2| - |w_8^3| + \sqrt{d_8^2 - \left(w_8^1\right)^2} + \sqrt{d_8^2 - \left(w_8^2\right)^2} + \sqrt{d_8^2 - \left(w_8^3\right)^2} - d_8 < 0,$$

that is, dividing both sides by d_8,

$$-\frac{|w_8^1|}{d_8} - \frac{|w_8^2|}{d_8} - \frac{|w_8^3|}{d_8} + \sqrt{1 - \left(\frac{w_8^1}{d_8}\right)^2} + \sqrt{1 - \left(\frac{w_8^2}{d_8}\right)^2} + \sqrt{1 - \left(\frac{w_8^3}{d_8}\right)^2} - 1 < 0.$$

Therefore, we need to prove that

$$f(a,b,c) < 0 \; , \quad \text{where} \quad f(a,b,c) := -a - b - c + \sqrt{1-a^2} + \sqrt{1-b^2} + \sqrt{1-c^2} - 1 \; ,$$

for $a, b, c > 0$ and such that $a^2 + b^2 + c^2 = 1$. But this last verification is a simple computation, given that the function f vanishes when one of a, b, c is equal to 0, and it has a unique critical point in the region of interest, which is a minimum[1].

Identical arguments apply for all other $\widehat{v}_i$ and v_j, $i \ne j$, and the result follows. □

Next, we have the following.

[1]the minimum value is $\frac{-\sqrt{3}-3+3\sqrt{2}}{\sqrt{3}}$ and is attained at $a = b = c = \frac{1}{\sqrt{3}}$

Lemma 2.5. *The vectors $\widetilde{v}_i$, $i = 1, \dots, 8$, are affinely independent, hence $\widetilde{V}$ is a 7-simplex.*

Proof. First, consider the matrices $\widehat{V} = \begin{bmatrix} \widehat{v}_1 & \cdots & \widehat{v}_8 \end{bmatrix} \in \mathbb{R}^{7\times 8}$, which is of rank 7, and $\widetilde{W} = \begin{bmatrix} \widetilde{v}_1 & \cdots & \widetilde{v}_8 \end{bmatrix}$. By the definition of the vectors $\widetilde{v}_i$'s, it follows that $\widetilde{W} = \widehat{V}D^{-1}$, with $D = \mathrm{diag}\left(\|v_i\|_1,\ i = 1, \dots 8\right)$, and hence $\widetilde{W}$ is also of rank 7.

Now, we know that $0 = (1/8)\sum_{i=1}^{8} \widehat{v}_i$, and therefore also $0 = \sum_{i=1}^{8} \tau_i \widetilde{v}_i$, with $\tau_i = \frac{\|v_i\|_1}{\sum_{j=1}^{8}\|v_j\|_1}$, $i = 1 \dots, 8$. Let $\tau := \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_8 \end{bmatrix}$, so that $\widetilde{W}\tau = 0_7$, and also $\sigma := 2\tau$. Since $1 = \mathbb{1}^\top \tau \neq \mathbb{1}^\top \sigma = 2$, then using Lemma 2.1, the matrix $\begin{bmatrix} \widetilde{W} \\ \mathbb{1}^\top \end{bmatrix}$ is invertible, and the claim follows. □

Remark 2.6. *From the proof of Lemma 2.5 it follows that the origin is in the interior of $\widetilde{V}$. Also, any subset of $\{\widetilde{v}_1, \dots, \widetilde{v}_8\}$ gives an affinely independent set and therefore, for any $j = 1, \dots, 8$, if $y \in \mathrm{conv}\{\widetilde{v}_1, \dots, \widetilde{v}_j\}$ then there exist unique $\alpha_1, \dots, \alpha_j \in [0, 1]$, $\sum_{i=1}^{j} \alpha_i = 1$, such that $y = \sum_{i=1}^{j} \alpha_j \widetilde{v}_j$; further, as $\begin{bmatrix} \widetilde{v}_1 & \cdots & \widetilde{v}_j \\ 1 & \cdots & 1 \end{bmatrix}$ is full rank, then $\alpha_1, \dots, \alpha_j$ depend continuously on y.*

Our goal is to show that $0 \in \mathrm{int}V$, and this will be shown by using the following result from classic convex geometry (see Grünbaum, [10, pag.48, Exercise 5.(vii)]).

Lemma 2.7. *Let $A \subseteq \mathbb{R}^d$ be any set, $d \geq 1$. Then A° is bounded if and only if $0 \in \mathrm{intconv}A$.*

Proof. Let us first prove that, for any set $C \in \mathbb{R}^d$,

$$C \text{ is bounded } \iff 0 \in \mathrm{int}C^\circ. \tag{2.8}$$

If C is bounded, then there exists $\delta > 0$ such that $C \subseteq B_\delta(0)$, and by a property of the polar mapping, $B_{\frac{1}{\delta}}(0) \subseteq C^\circ$, from which $0 \in \mathrm{int}C^\circ$. Viceversa, if $0 \in \mathrm{int}C^\circ$, then there exists $\delta > 0$ such that $B_\delta(0) \subseteq C^\circ$, from which $C \subseteq C^{\circ\circ} \subseteq B_{\frac{1}{\delta}}(0)$, meaning that C is bounded.
From this result, using A° instead of C, we can conclude that A° is bounded if and only if $0 \in \mathrm{int}A^{\circ\circ}$. Since $A^{\circ\circ} = \mathrm{clconv}(A \cup \{0\})$, then $\mathrm{int}A^{\circ\circ} = \mathrm{intconv}A$. □

The next result will be the stepping stone for our argument to show that V° is bounded. The idea behind it relies on Remark 2.6: we will prove that, given any $y \in \widetilde{V} \setminus \{0\}$, we can project y, along the linear subspace $\langle y \rangle$, onto ∂V°. This geometric property will eventually allow us to construct a ball containing V° (see the proof of Theorem 1.1).

Lemma 2.8. *For all $n = 2, \dots, 8$ and for any $y \in \widetilde{V}$, $y \neq 0$, there exists a unique $c_y \geq 1$ such that $c_y y \in \partial V^\circ$, and c_y depends continuously on y.*

Proof. We proceed recursively on $n = 2, \dots, 8$.
Let $n = 2$, and take, without loss of generality,

$$y = (1 - t)\widetilde{v}_1 + t\widetilde{v}_2, \quad t \in [0, 1].$$

Now, given any $y \in \widetilde{V} \setminus \{0\}$, we look for $c_t \geq 1$ such that

$$v_1^\top \left(c_t \left[(1-t)\widetilde{v}_1 + t\widetilde{v}_2\right]\right) = 1 \quad \text{or} \quad v_2^\top \left(c_t \left[(1-t)\widetilde{v}_1 + t\widetilde{v}_2\right]\right) = 1,$$

that is (see Lemma 2.4)

$$c_t(1-t) + c_t t v_1^\top \widetilde{v}_2 = 1 \quad \text{or} \quad c_t(1-t)v_2^\top \widetilde{v}_1 + c_t t = 1.$$

Let now

$$t_{\mathrm{I}} := \frac{1 - v_2^\top \widetilde{v}_1}{(1 - v_2^\top \widetilde{v}_1) + (1 - v_1^\top \widetilde{v}_2)} \ ,$$

and note (see Lemma 2.4) that $t_{\mathrm{I}} \in (0,1)$. Moreover, letting $t_{\mathrm{C}} := \frac{1}{1 - v_1^\top \widetilde{v}_2}$, we claim that $t_{\mathrm{I}} < t_{\mathrm{C}}$. It is easily seen that $t_{\mathrm{I}} < t_{\mathrm{C}}$ holds if and only if the following relation holds

$$1 - \left(v_2^\top \widetilde{v}_1\right)\left(v_1^\top \widetilde{v}_2\right) > 0. \tag{2.9}$$

But this last relation can be rewritten as

$$\left(v_2^\top \widehat{v}_1\right)\left(v_1^\top \widehat{v}_2\right) < \|v_1\|_1 \|v_2\|_1$$

which is clearly true since v_1 and v_2 have different signs. Also, let us set

$$c_t := \begin{cases} \frac{1}{1 - t(1 - v_1^\top \widetilde{v}_2)}, & t \in [0, t_{\mathrm{I}}], \\ \frac{1}{v_2^\top \widetilde{v}_1 - t(v_2^\top \widetilde{v}_1 - 1)}, & t \in (t_{\mathrm{I}}, 1]. \end{cases}$$

Now, for $t \in [0, t_{\mathrm{I}}]$ and since $t_{\mathrm{I}} < t_{\mathrm{C}}$, we have that $1 - t(1 - v_1^\top \widetilde{v}_2) > 0$, and of course $1 - t(1 - v_1^\top \widetilde{v}_2) \leq 1$, so that

$$c_t \geq 1, \quad t \in [0, t_{\mathrm{I}}].$$

If $t \in (t_{\mathrm{I}}, 1]$, then $v_2^\top \widetilde{v}_1 - t\left(v_2^\top \widetilde{v}_1 - 1\right) > 0$ if and only if $\frac{v_2^\top \widetilde{v}_1}{v_2^\top \widetilde{v}_1 - 1} < t$. But since (2.9) is equivalent to $t_{\mathrm{I}} > \frac{v_2^\top \widetilde{v}_1}{v_2^\top \widetilde{v}_1 - 1}$, and $v_2^\top \widetilde{v}_1 - t\left(v_2^\top \widetilde{v}_1 - 1\right) < 1$ if and only if $t < 1$, then it holds again that

$$c_t \geq 1, \quad t \in (t_{\mathrm{I}}, 1].$$

Also, note that, for $t = t_{\mathrm{I}}$, it holds that

$$v_1^\top \left[c_{t_{\mathrm{I}}}((1-t_{\mathrm{I}})\widetilde{v}_1 + t_{\mathrm{I}}\widetilde{v}_2)\right] = v_2^\top \left[c_{t_{\mathrm{I}}}((1-t_{\mathrm{I}})\widetilde{v}_1 + t_{\mathrm{I}}\widetilde{v}_2)\right] = 1 \ , \tag{2.10}$$

and therefore c_t is continuous for $t \in [0,1]$. Moreover, simple computations show that, for any $t \in [0, t_{\mathrm{I}}]$, we have

$$\begin{aligned} &v_1^\top \left[c_t((1-t)\widetilde{v}_1 + t\widetilde{v}_2)\right] = 1, \\ &v_j^\top \left[c_t((1-t)\widetilde{v}_1 + t\widetilde{v}_2)\right] \leq 1, \quad j = 1, \ldots, 8,\ j \neq 1, \end{aligned}$$

and, for any $t \in (t_{\mathrm{I}}, 1]$ we have

$$\begin{aligned} &v_2^\top \left[c_t((1-t)\widetilde{v}_1 + t\widetilde{v}_2)\right] = 1, \\ &v_j^\top \left[c_t((1-t)\widetilde{v}_1 + t\widetilde{v}_2)\right] \leq 1, \quad j = 1, \ldots, 8,\ j \neq 2. \end{aligned}$$

Thus, the case $n = 2$ is proven.
Let now assume $y \in \operatorname{conv}\{\widetilde{v}_i \,:\, i = 1,2,3\}$:

$$y = \sum_{i=1}^{3} \alpha_i \widetilde{v}_i, \quad \sum_{i=1}^{3} \alpha_i = 1, \quad \alpha_i \in [0,1], \quad i = 1,2,3.$$

If y is in the convex hull of two of these vertices, then there is nothing to prove. Therefore, we can assume that none of the α_i's, $i = 1,2,3$, is zero. Let us then rewrite y as

$$y = \sum_{i=1}^{2} \alpha_i \widetilde{v}_i + \alpha_3 \widetilde{v}_3.$$

Let

$$\widetilde{y} := \frac{\sum_{i=1}^{2} \alpha_i \widetilde{v}_i}{\sum_{i=1}^{2} \alpha_i},$$

so that $\widetilde{y} \in \operatorname{conv}\{\widetilde{v}_i \,:\, i = 1,2\}$. As in the case $n = 2$, there exists $\widetilde{c} \geq 1$ such that $\widetilde{c}\,\widetilde{y} \in \partial V^\circ$; also, $\widetilde{v}_3 \in \partial V^\circ$.
Moreover, note that

$$v_3^\top \left(\widetilde{c}\,\widetilde{y}\right) = \frac{\sum_{i=1}^{2} \widetilde{c}\,\alpha_i \left(v_3^\top \widetilde{v}_i\right)}{\sum_{i=1}^{2} \alpha_i} < 0,$$

and since $\widetilde{c}\,\widetilde{y} \in \partial V^\circ$, then there must exist $i \neq 3$ such that

$$v_i^\top \left(\widetilde{c}\,\widetilde{y}\right) = 1.$$

Therefore, we can leverage the same arguments as in the base case $n = 2$, having that for all $t \in [0,1]$ there exists $c_t \geq 1$ such that

$$c_t \left[(1-t)\widetilde{c}\,\widetilde{y} + t\widetilde{v}_3\right] \in \partial V^\circ.$$

Setting

$$\bar{t} := \frac{\widetilde{c}\,\alpha_3}{\sum_{i=1}^{2} \alpha_i + \widetilde{c}\,\alpha_3},$$

we have that $\overline{t} \in [0,1]$ and, further,

$$(1-\overline{t})\widetilde{c}\,\widetilde{y} + \overline{t}\,\widetilde{v}_3 = \frac{\widetilde{c}\sum_{i=1}^{2}\alpha_i}{\sum_{i=1}^{2}\alpha_i + \widetilde{c}\,\alpha_3}\frac{\sum_{i=1}^{2}\alpha_i\widetilde{v}_i}{\sum_{i=1}^{2}\alpha_i} + \frac{\widetilde{c}\,\alpha_3}{\sum_{i=1}^{2}\alpha_i + \widetilde{c}\,\alpha_3}\widetilde{v}_3$$
$$= \frac{\widetilde{c}}{\sum_{i=1}^{2}\alpha_i + \widetilde{c}\,\alpha_3} y,$$

so that

$$\frac{c_{\overline{t}}\,\widetilde{c}}{\sum_{i=1}^{2}\alpha_i + \widetilde{c}\,\alpha_3} y \in \partial V^\circ.$$

Since $\frac{\widetilde{c}}{\sum_{i=1}^{2}\alpha_i + \widetilde{c}\,\alpha_3} \geq 1$ if and only if $\widetilde{c} \geq 1$, that is true, and $c_{\overline{t}} \geq 1$, then the existence is verified; continuity follows from Remark 2.6. Thus, the claim is proved for $n = 3$.
We can proceed in the same way for all $n \geq 3$; e.g., let us prove the case of $n = 8$, assuming that the claim is true for $n \leq 7$, and that $y \in \operatorname{conv}\{\widetilde{v}_i \,:\, i = 1,\dots,8\}$:

$$y = \sum_{i=1}^{8}\alpha_i\widetilde{v}_i, \quad \sum_{i=1}^{8}\alpha_i = 1, \quad \alpha_i \in [0,1], \quad i = 1,\dots,8.$$

If some of the α_i's are 0, then y is in the convex hull of fewer than 8 of the $\widetilde{v}_i$'s and there is nothing to prove. Therefore, we can assume that all of the α_i's, $i = 1,\dots,8$, are not zero. Let us then rewrite y as

$$y = \sum_{i=1}^{7}\alpha_i\widetilde{v}_i + \alpha_8\widetilde{v}_8\ , \quad \text{and let } \widetilde{y} := \frac{\sum_{i=1}^{7}\alpha_i\widetilde{v}_i}{\sum_{i=1}^{7}\alpha_i}\ ,$$

so that $\widetilde{y} \in \operatorname{conv}\{\widetilde{v}_i \,:\, i = 1,\dots,7\}$. By previous cases, there exists $\widetilde{c} \geq 1$ such that $\widetilde{c}\,\widetilde{y} \in \partial V^\circ$; also, $\widetilde{v}_8 \in \partial V^\circ$.
Moreover, note that

$$v_8^\top\left(\widetilde{c}\,\widetilde{y}\right) = \frac{\sum_{i=1}^{7}\widetilde{c}\,\alpha_i\left(v_8^\top\widetilde{v}_i\right)}{\sum_{i=1}^{7}\alpha_i} < 0,$$

and since $\widetilde{c}\,\widetilde{y} \in \partial V^\circ$, then there must exist $i \neq 8$ such that

$$v_i^\top\left(\widetilde{c}\,\widetilde{y}\right) = 1.$$

Therefore, using the same arguments as in the case $n = 2$, we have that for all $t \in [0, 1]$ there exists $c_t \geq 1$ such that

$$c_t \left[(1-t)\widetilde{c}\,\widetilde{y} + t\widetilde{v}_8\right] \in \partial V^\circ.$$

Setting

$$\bar{t} := \frac{\widetilde{c}\,\alpha_8}{\displaystyle\sum_{i=1}^{7} \alpha_i + \widetilde{c}\,\alpha_8},$$

it comes that $\bar{t} \in [0, 1]$ and, further,

$$\begin{aligned}(1-\bar{t})\widetilde{c}\,\widetilde{y} + \bar{t}\,\widetilde{v}_8 &= \frac{\widetilde{c}\displaystyle\sum_{i=1}^{7}\alpha_i}{\displaystyle\sum_{i=1}^{7} \alpha_i + \widetilde{c}\,\alpha_8} \frac{\displaystyle\sum_{i=1}^{7}\alpha_i\widetilde{v}_i}{\displaystyle\sum_{i=1}^{7}\alpha_i} + \frac{\widetilde{c}\,\alpha_8}{\displaystyle\sum_{i=1}^{7} \alpha_i + \widetilde{c}\,\alpha_8}\widetilde{v}_8 \\ &= \frac{\widetilde{c}}{\displaystyle\sum_{i=1}^{7} \alpha_i + \widetilde{c}\,\alpha_8} y,\end{aligned}$$

so that

$$\frac{c_{\bar{t}}\,\widetilde{c}}{\displaystyle\sum_{i=1}^{7} \alpha_i + \widetilde{c}\,\alpha_8} y \in \partial V^\circ.$$

Since $\frac{\widetilde{c}}{\sum_{i=1}^{7} \alpha_i + \widetilde{c}\,\alpha_8} \geq 1$ if and only if $\widetilde{c} \geq 1$, that is true, and $c_{\bar{t}} \geq 1$, then the existence is verified; again, continuity follows from Remark 2.6.
Finally, let us now prove that c_y is unique for any $y \in \widetilde{V}$, $y \neq 0$. In fact, arguing by contradiction, let $d \geq 1$ be such that $d \neq c_y$ – say $d > c_y$ – and $dy \in \partial V^\circ$: then there exists $j = 1, \ldots, 8$ such that

$$v_j^\top (dy) = 1.$$

Let us also assume that $v_i^\top (c_y y) = 1$ for some $i = 1, \ldots, 8$. If $i \neq j$, then

$$v_i^\top (dy) = v_i^\top \left(\frac{d}{c_y} c_y y\right) = \frac{d}{c_y} v_i^\top (c_y y) = \frac{d}{c_y} > 1,$$

implying that $dy \notin \partial V^\circ$, that is not true. Therefore, $i = j$. But then

$$d = dv_i^\top (c_y y) = c_y v_i^\top (dy) = c_y,$$

which contradicts our assumption. Thus c_y is unique, as claimed, and this concludes the proof. □

Remark 2.9. *In Lemma 2.8, we have constructed a map*

$$\begin{aligned}\widetilde{V} \setminus \{0\} &\longrightarrow [1, \infty) \\ y &\longmapsto c_y\end{aligned} \tag{2.11}$$

that is well defined, and continuous.

Now, from Remark 2.6, there exists $\delta > 0$ such that

$$B_\delta(0) \subsetneq \widetilde{V} \subseteq V^\circ.$$

We can then extend the map (2.11) to $V^\circ \setminus \{0\}$.

Proposition 2.10. *For any $y \in V^\circ \setminus \{0\}$ there exists a unique $c_y \geq 1$ such that $c_y y \in \partial V^\circ$. Moreover, this map $y \to c_y$ is continuous on $V^\circ \setminus \{0\}$.*

Proof. Let $y \in V^\circ \setminus \{0\}$, and $\delta > 0$ such that $B_\delta(0) \subsetneq \widetilde{V}$. Then, setting $z := \frac{y}{\|y\|}\delta$, we have $z \in \partial B_\delta(0)$, and so $z \in \widetilde{V}$. Then, there exists $c_z \geq 1$ such that $c_z z \in \partial V^\circ$, or in other words

$$c_y y \in \partial V^\circ,$$

where

$$c_y := c_{\frac{y}{\|y\|}\delta} \frac{\delta}{\|y\|}.$$

Since $c_y y \in \partial V^\circ$, then there exists $i = 1, \ldots, 8$ such that $v_i^\top (c_y y) = 1$. If, by contradiction, $c_y < 1$, then

$$v_i^\top y = \frac{1}{c_y} > 1,$$

which implies $y \notin V^\circ$, that is not the case. Thus $c_y \geq 1$.
Let now $d \geq 1$ be such that $dy \in \partial V^\circ$. Therefore

$$d \frac{c_z}{c_y} z \in \partial V^\circ,$$

and from an argument similar to the above one, it follows that $d\frac{c_z}{c_y} \geq 1$. Since the map (2.11) is well defined, then $\frac{dc_z}{c_y} = c_z$, and so $d = c_y$.
Therefore the map

$$\begin{aligned} V^\circ \setminus \{0\} &\longrightarrow [1, \infty) \\ y &\longmapsto c_y \end{aligned}$$

is well defined and continuous. □

Corollary 2.11. *Let $\delta > 0$ be such that $B_\delta(0) \subsetneq V^\circ$. Then it holds that*

$$\sup_{\substack{y \in V^\circ \\ y \neq 0}} c_y \frac{\|y\|}{\delta} = \max_{z \in \partial B_\delta(0)} c_z.$$

Proof. First, by construction we have that

$$c_{\frac{y}{\|y\|}\delta} = c_y \frac{\|y\|}{\delta}.$$

Let now

$$M_V := \sup_{\substack{y \in V^\circ \\ y \neq 0}} c_y \frac{\|y\|}{\delta},$$

$$M_B := \max_{z \in \partial B_\delta(0)} c_z,$$

and note that M_B is well defined because of Weierstrass' Theorem, since $\partial B_\delta(0)$ is compact. If $y \in V^\circ$, $y \neq 0$, then $\frac{y}{\|y\|}\delta \in \partial B_\delta(0)$, and thus

$$c_y \frac{\|y\|}{\delta} = c_{\frac{y}{\|y\|}\delta} \leq M_B,$$

from which

$$M_V \leq M_B.$$

If $z \in \partial B_\delta(0)$, then $z \in V^\circ$, and therefore

$$c_z = c_z \frac{\|z\|}{\delta} \leq M_V,$$

from which

$$M_B \leq M_V,$$

and this concludes the proof. □

Remark 2.12. *From Corollary 2.11, we have that*

$$\mu := \sup_{\substack{y \in V^\circ \\ y \neq 0}} c_y \frac{\|y\|}{\delta}$$

is finite, since it is equal to $\max_{z \in \partial B_\delta(0)} c_z$*, that is the maximum of a continuous function (see Remark 2.9) on a compact set. Therefore*

$$\mu = \max_{\substack{y \in V^\circ \\ y \neq 0}} c_y \frac{\|y\|}{\delta}.$$

3. Proof of main results

Proof of Theorem 1.1. To prove that $0 \in \text{int}V$, using Lemma 2.7, we will prove that V° is bounded. Let $\delta > 0$ such that $B_\delta(0) \subsetneq \widetilde{V}$, and let (see Remark 2.12)

$$\mu := \max_{\substack{y \in V^\circ \\ y \neq 0}} c_y \frac{\|y\|}{\delta} = \max_{z \in \partial B_\delta(0)} c_z.$$

Pick now $y \in V^\circ$, $y \neq 0$, but otherwise arbitrary. Therefore, since $c_y \geq 1$,

$$\|y\| \leq \|c_y y\| = c_y \frac{\|y\|}{\delta}\delta = c_{\frac{y}{\|y\|}\delta}\delta \leq \mu\delta,$$

and we conclude that

$$V^\circ \subseteq \overline{B_{\mu\delta}(0)},$$

that is V° is bounded.
Therefore V contains 0_7 in its interior: so there exists $R > 0$ such that

$$B_R(0_7) \subsetneq V.$$

Since the volume function is monotonically increasing by inclusion (see [2, 10]), and

$$\text{vol}\, V = |\det M|,$$

then

$$0 < \text{vol}\, B_R(0_7) \leq \text{vol}\, V = |\det M|.$$

Thus, the matrix (see (1.5))

$$M = \begin{bmatrix} W \\ \Delta \\ d^\top \\ \mathbb{1}^\top \end{bmatrix}$$

is invertible, which concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. From Theorem 1.1, (1.6) has a unique solution λ_M. We want to show that

$$0_7 = \sum_{i=1}^{8} \lambda_i v_i \ , \quad \lambda_i > 0 \ , \quad \sum_{i=1}^{8} \lambda_i = 1 \ . \tag{3.1}$$

From the above proof of Theorem 1.1, it follows that the vectors v_i, $i = 1, \dots, 8$, are affinely independent, and that $V = \operatorname{conv}\{v_i, \ i = 1, \dots, 8\}$ is a 7-simplex containing the origin in its interior. Therefore, for sure

$$0_7 = \sum_{i=1}^{8} \alpha_i v_i \ , \quad \alpha_i \geq 0 \ , \quad \sum_{i=1}^{8} \alpha_i = 1 \ .$$

Now, by contradiction, suppose that one of the α_i's is 0, without loss of generality say $\alpha_8 = 0$. Then

$$0_7 = \sum_{i=1}^{7} \alpha_i v_i \ , \quad \alpha_i \geq 0 \ , \quad \sum_{i=1}^{7} \alpha_i = 1 \ .$$

But then $0_7 \in \operatorname{conv}\{v_1, \dots, v_7\}$, that is $0_7 \in \partial V$, and this contradicts that $0_7 \in \operatorname{int}(V)$.

Therefore, (3.1) holds, the unique solution of $M\lambda = e_8$ has all positive components, and the proof of Theorem 1.2 is completed. □

For completeness, we notice that the solution λ_M of (1.6) can be written using M_{adj}, the adjugate of M, as

$$\lambda_M = \frac{1}{\det M} M_{\text{adj}}(8,:)^\top \ ,$$

where

$$M_{\text{adj}}(8,:)^\top = - \begin{bmatrix} \det\begin{bmatrix} w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ -d_2 & -d_3 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_3 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ \det\begin{bmatrix} w_1 & w_2 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_2 & w_3 & w_5 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & -d_3 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ \det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & -d_3 & d_4 & d_6 & d_7 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_7 & -d_8 \end{bmatrix} \\ \det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_8 \\ d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_6 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 \\ d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_6 & d_7 \end{bmatrix} \end{bmatrix} .$$

As a side result, this expression shows that any seven of the v_i's are linearly independent vectors (we knew that they were affinely independent).

Remark 3.1. *Our proof of Theorem 1.1 (from which Theorem 1.2 followed as well) hinged on the fact that the vectors $\widehat{v}_i$, $i = 1, \dots, 8$, were affinely independent, and that the associated vectors $\widetilde{v}_i$'s were so as well (see Lemma 2.5). For us, affine independence of the $\widehat{v}_i$'s and $\widetilde{v}_i$'s was a consequence of nodal attractivity of Σ, and this was the only property we have used that came from the dynamics of the differential system under study. Because of these considerations, the result (i.e., invertibility of the matrix $M = \begin{bmatrix} W \\ \Delta \\ d^\top \\ 1^\top \end{bmatrix}$) would still hold true every time one has a matrix W leading to affinely independent vectors $\widehat{v}_i$'s and $\widetilde{v}_i$'s. This includes many more cases of attractive Σ than just that of nodally attractive Σ.*

4. Extension to co-dimension 4 and higher

In this section, we propose the extension of the moments solution to any co-dimension $p \geq 1$, under nodal attractivity conditions. Before doing that, we introduce the differential problem associated to it.

Consider the piecewise smooth system

$$x'(t) = f_i(x), \; x \in R_i, \;\; i = 1, \dots, 2^p, \tag{4.1}$$

where the regions R_i's are open, disjoint and connected sets of $\mathbb{R}^n$, so that $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region R_i the function f_i is smooth.

The regions R_i's are separated by manifolds defined as 0-sets of $\mathcal{C}^2$ scalar functions h_i: $\Sigma_i := \{x \in \mathbb{R}^n \, : \, h_i(x) = 0\}$, $i = 1, \dots, p$. Assume that the normals ∇h_i's are linearly independent on (hence in a neighborhood of) Σ, and let

$$\Sigma := \bigcap_{i=1}^{p} \Sigma_i \tag{4.2}$$

be the co-dimension p manifold of interest to us. Letting

$$w_i^j := \nabla h_j(x)^\top f_i(x), \quad i = 1, \dots, 2^p, \; j = 1, \dots, p,$$

we associate the matrix $W = (w_i^j) \in \mathbb{R}^{p \times 2^p}$ to (4.1). As before, the linear system to solve in order to determine a sliding vector field on Σ is given by

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0_p \\ 1 \end{bmatrix} \; . \tag{4.3}$$

Obviously, this is an underdetermined linear system, and in Lemma 4.3 and Corollary 4.4 we will see that $\left[\begin{smallmatrix} W \\ \mathbb{1}^\top \end{smallmatrix}\right]$ has rank $p+1$, under appropriate attractivity conditions of Σ. It is this system that we will regularize according to the moments' technique. Once more, we stress that we are interested in admissible solutions of (4.3), hence positive and smoothly varying with $x \in \Sigma$.

Let us first recall the sign pattern of W characterizing nodally attractive conditions, as in [8].

Definition 4.1. *We say that* Σ *in* (4.2) *is* nodally attractive*, or equivalently that* W *satisfies* nodally attractive conditions*, if the sign pattern of* W *is given by the following recursion relations:*

$$S^{(1)} = \begin{bmatrix} 1 & -1 \end{bmatrix},$$
$$S^{(k)} = \begin{bmatrix} \mathbb{1}_{2^{k-1}}^\top & -\mathbb{1}_{2^{k-1}}^\top \\ S^{(k-1)} & S^{(k-1)} \end{bmatrix}, \quad k = 2, \dots, p.$$

In [8], the authors proved that –when Σ is nodally attractive– there always exits a multilinear (interpolant) solution λ to the system $W\lambda = 0$. For later reference, we summarize this result without proof.

Lemma 4.2 ([8])**.** *Suppose that* $W \in \mathbb{R}^{p\times 2^p}$ *satisfies nodally attractive conditions. Then, for any* $p \geq 1$*, there exist* $\alpha_1, \dots, \alpha_p$*, all in* $(0,1)$*, such that the vector* $\lambda \in \mathbb{R}^{2^p}$ *defined as*

$$\lambda = \begin{bmatrix} (1-\alpha_1)(1-\alpha_2)...(1-\alpha_{p-1})(1-\alpha_p) \\ (1-\alpha_1)(1-\alpha_2)...(1-\alpha_{p-1})\alpha_p \\ (1-\alpha_1)(1-\alpha_2)...\alpha_{p-1}(1-\alpha_p) \\ (1-\alpha_1)(1-\alpha_2)...\alpha_{p-1}\alpha_p \\ \vdots \\ (1-\alpha_1)\alpha_2...(1-\alpha_{p-1})(1-\alpha_p) \\ \vdots \\ (1-\alpha_1)\alpha_2...\alpha_{p-1}\alpha_p \\ \alpha_1(1-\alpha_2)...(1-\alpha_{p-1})(1-\alpha_p) \\ \vdots \\ \alpha_1\alpha_2...\alpha_{p-1}\alpha_p \end{bmatrix}$$

solves the system $W\lambda = 0_p$*, and* $\sum_{i=1}^{2^p} \lambda_i = 1$. □

With the help of Lemma 4.2 we can prove the following.

Lemma 4.3. *For any* $k \geq 1$*, consider* $W^{(k)} \in \mathbb{R}^{k\times 2^k}$ *satisfying the sign pattern of Definition 4.1. Then*

$$\operatorname{rank} W^{(k)} = k \ .$$

Proof. The proof is by induction on k. The case $k = 1$ is in [9] ($k = 2$ is in [6], and $k = 3$ is Corollary 2.3).

Let us assume the result true for k, and let us consider $W^{(k+1)}$ with sign pattern given as in Definition 4.1. Let us pick $w_1, \dots, w_{2^k}$, the first half of the columns of $W^{(k+1)}$. By Lemma 4.2, there exist $\lambda_1, \dots, \lambda_{2^k} \in (0,1)$, such that

$$\sum_{i=1}^{2^k} \lambda_i \begin{bmatrix} w_i^2 \\ \vdots \\ w_i^{k+1} \end{bmatrix} = 0_k \ ,$$

and since $w_i^1 > 0$ for $i = 1 \dots, 2^k$, we also have

$$\sum_{i=1}^{2^k} \lambda_i w_i^1 > 0 \ .$$

Using this linear combination to replace the $(k+1)$-st column of $W^{(k+1)}$ gives the matrix

$$\widehat{W}^{(k+1)} := \begin{bmatrix} & & & >0 \\ & & & 0 \\ w_1 & \cdots & w_k & \vdots \\ & & & 0 \end{bmatrix}.$$

Now, we have that

$$\operatorname{sign}\left(\det \widehat{W}^{(k+1)}\right) = \operatorname{sign}\left(\det \begin{bmatrix} w_1^2 & \cdots & w_k^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_k^{k+1} \end{bmatrix}\right) \neq 0 \ ,$$

where the last inference comes from the inductive hypothesis, since $\begin{bmatrix} w_1^2 & \cdots & w_k^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_k^{k+1} \end{bmatrix}$ has the sign pattern of the first k columns of $W^{(k)}$, which is supposed to be full rank. This in turn implies that $\operatorname{rank} W^{(k+1)} = k+1$. □

Finally, we have

Corollary 4.4. *For any $k \geq 1$, consider $\widetilde{W}^{(k)} := \begin{bmatrix} W^{(k)} \\ \mathbb{1}^\top \end{bmatrix}$, where $W^{(k)} \in \mathbb{R}^{k\times 2^k}$ satisfies the sign pattern of Definition 4.1. Then $\operatorname{rank} \widetilde{W}^{(k)} = k+1$, hence $\ker\left(\widetilde{W}^{(k)}\right)$ is $(2^k - k - 1)$-dimensional.*

Proof. The case $k=1$ is elementary. So, proceeding by induction, let $k \geq 2$ be fixed and –using Lemma 4.2, and because of the nodally attractive sign pattern– consider multilinear interpolant solutions $\lambda^{(1)}$ and $\lambda^{(2)}$ associated, respectively, to the submatrices $\begin{bmatrix} w_1^2 & \cdots & w_{2^k}^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_{2^k}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix}$, and $\begin{bmatrix} w_{2^k+1}^2 & \cdots & w_{2^{k+1}}^2 \\ \vdots & & \vdots \\ w_{2^k+1}^{k+1} & \cdots & w_{2^{k+1}}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix}$, of $\widetilde{W}^{(k+1)}$. Note that $\lambda^{(1)} = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $\lambda^{(2)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{bmatrix}$. Then $\widetilde{W}^{(k+1)}\lambda^{(1)} = \begin{bmatrix} >0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, $\widetilde{W}^{(k+1)}\lambda^{(2)} = \begin{bmatrix} <0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

From inductive hypothesis, since the two submatrices

$$\begin{bmatrix} w_1^2 & \cdots & w_{2^k}^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_{2^k}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w_{2^k+1}^2 & \cdots & w_{2^{k+1}}^2 \\ \vdots & & \vdots \\ w_{2^k+1}^{k+1} & \cdots & w_{2^{k+1}}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix}$$

are full rank $k+1$ having the same sign pattern as $\widetilde{W}^{(k)} = \begin{bmatrix} W^{(k)} \\ \mathbb{1}^\top \end{bmatrix}$, using Lemma 2.1 gives

$$\text{rank} \begin{bmatrix} W^{(k+1)} \\ \mathbb{1}^\top \end{bmatrix} = k+2 \ .$$

□

Remark 4.5. *On account of Corollary 4.4, for nodally attractive* Σ*, it follows that the linear system* (4.3)*,*

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0_p \\ 1 \end{bmatrix} \ ,$$

has rank $p+1$*, therefore providing a family of solutions depending on* $(2^p - p - 1)$ *free parameters. From Lemma 4.2, one possibility to fix these is by using the* multilinear interpolant *approach. Needless to say (as already observed in Example 1.6 for the case of* $p = 3$*), there is severe lack of uniqueness of solutions in this case. Below, we will propose the* moments regularization*.*

The moments regularization requires to append a matrix Δ of signed partial distances and a row $d^\top$ of full distances of $w_1, \dots, w_{2^p}$ to $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. The matrix Δ will manage all the subslidings at lower co-dimensions: they happen from co-dimension 2 all the way to co-dimension $p-1$. Therefore, we have

$$\sum_{k=2}^{p-1} \binom{p}{k} = 2^p - p - 2$$

rows of partial distances: thus $\Delta \in \mathbb{R}^{(2^p-p-2)\times 2^p}$. Adding the row $d^\top$, gives $2^p - p - 1$ extra equations, as desired.

In order to decide the sign pattern of Δ, it is necessary to recognize the entire substructures of lower co-dimensions nested within it when a partial distance is selected: then, the sign of each entry is determined by the sign product of the components considered to compute the partial distance. This is better explained by looking at Example 4.6 below for the case of co-dimension 4, which clearly indicates how one will proceed in general. About the sign pattern of $d^\top$, our proposal is to consider the following recursion:

$$R_1 := \begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$
$$R_{k+1} := \begin{bmatrix} R_k & \\ & -R_k \end{bmatrix}, \quad k = 1, \dots, p-1,$$

and then define

$$d := R_p \begin{bmatrix} \|w_1\| \\ \vdots \\ \|w_{2^p}\| \end{bmatrix}. \tag{4.4}$$

Observe that this sign pattern is the same as considering the sign product of all the components in the vectors $\begin{bmatrix} w_i \end{bmatrix}$, $i = 1, \dots, 2^p$.

Example 4.6. *In co-dimension* 4*, the sign pattern of* W *is given by*

$$\operatorname{sign}(W) = \begin{bmatrix} + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \end{bmatrix}.$$

We split Δ*, the matrix of partial distances, as*

$$\Delta = \begin{bmatrix} \operatorname{sign}(\Delta_{\text{III}}) \odot \Delta_{\text{III}} \\ \operatorname{sign}(\Delta_{\text{II}}) \odot \Delta_{\text{II}} \end{bmatrix},$$

where $\odot$ *is the Hadamard (componentwise) product,* Δ_{III} *contains the rows of partial distances over three components of* w_i^j *at the time, and* Δ_{II} *contains the rows of partial distances over two components of* w_i^j *at the time. Therefore, choosing components* $2, 3, 4$ *for the first row,* $1, 3, 4$ *for the second row,* $1, 2, 4$ *for the third row,* $1, 2, 3$ *for the fourth row, we get that the sign pattern of* Δ_{III} *is*

$$\operatorname{sign}(\Delta_{\text{III}}) = \begin{bmatrix} + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\ + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\ + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \end{bmatrix},$$

with

$$\Delta_{\text{III}} = \begin{bmatrix} \delta_{2,3,4}(1) & \cdots & \delta_{2,3,4}(16) \\ \delta_{1,3,4}(1) & \cdots & \delta_{1,3,4}(16) \\ \delta_{1,2,4}(1) & \cdots & \delta_{1,2,4}(16) \\ \delta_{1,2,3}(1) & \cdots & \delta_{1,2,3}(16) \end{bmatrix},$$

where, for any $h = 1, \dots, 16$ *and suitably chosen* $i, j, k = 1, 2, 3, 4$*,*

$$\delta_{i,j,k}(h) := \sqrt{(w_i^h)^2 + (w_j^h)^2 + (w_k^h)^2}\,.$$

Notice that the sign pattern of the first row in Δ_{III} *is determined this way: since we are considering components* $2, 3, 4$*, then we look at second, third and fourth row of* W*; those rows present the sign pattern from co-dimension* 3 *in columns* $1, \dots, 8$ *and* $9, \dots, 16$*: we then select the sign pattern of* d *from the co-dimension* 3 *case in the corresponding columns. The same (selecting the corresponding suitable columns) has to be done for the other rows.*

The same rationale needs to be followed for determining the sign pattern of Δ_{II}*, using the sign pattern of* d *from the co-dimension* 2 *case (that is* $\begin{bmatrix} + & - & - & + \end{bmatrix}$*) in the corresponding columns giving the co-dimension* 2 *sign pattern, after we have selected the components to compute the partial distance. Therefore, the sign pattern of* Δ_{II} *is*

$$\operatorname{sign}(\Delta_{\text{II}}) = \begin{bmatrix} + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \end{bmatrix},$$

with

$$\Delta_{\mathrm{II}} = \begin{bmatrix} \delta_{1,2}(1) & \cdots & \delta_{1,2}(16) \\ \delta_{1,3}(1) & \cdots & \delta_{1,3}(16) \\ \delta_{1,4}(1) & \cdots & \delta_{1,4}(16) \\ \delta_{2,3}(1) & \cdots & \delta_{2,3}(16) \\ \delta_{2,4}(1) & \cdots & \delta_{2,4}(16) \\ \delta_{3,4}(1) & \cdots & \delta_{3,4}(16) \end{bmatrix} ,$$

where, for any $h = 1, \dots, 16$ *and suitably chosen* $i, j = 1, 2, 3, 4$,

$$\delta_{i,j}(h) := \sqrt{(w_i^h)^2 + (w_j^h)^2} \, .$$

Finally, according to (4.4),

$$\mathrm{sign}(d^\top) = [\, + - - + - + + - - + + - + - - + \,] \, .$$

Putting everything together, the sign pattern of the moments matrix M_4 *in co-dimension* 4 *is*

$$\begin{bmatrix}
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} .$$

The proof of invertibility of this matrix, and the fact that the solution of $M_4 \lambda_M = \begin{bmatrix} 0_{15} \\ 1 \end{bmatrix}$ *has all positive components, proceed precisely like the case of co-dimension 3 proved in this paper. In particular, the proof of Theorem 1.1 when* $p = 4$ *holds unchanged, aside from the obvious changes in the dimensions (we have now 16 vectors* $\widehat{v}_i'$*s, etc.).*

5. Conclusions

In this work, we have proposed an extension of the moments' method to the case of a co-dimension 3 discontinuity manifold Σ. Under the assumption that Σ is nodally attractive, we have proven that the resulting *moments matrix* is nonsingular, and further that the unique solution provided by the moments system is admissible, i.e., we recovered a unique convex combination and a unique Filippov sliding vector field on Σ. As far as we know, this is the first instance of a constructive technique providing a unique admissible sliding vector field on a nodally attractive discontinuity manifold of co-dimension 3.

We have also proposed the extension of the moments method to higher co-dimension, explicitly providing details on how to construct the moments' matrix in the co-dimension 4 case, and justification of its invertibility.

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