The singular value decomposition to approximate spectra of dynamical systems. Theoretical aspects ⋆

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Abstract

In this paper we consider the singular value decomposition (SVD) of a fundamental matrix solution in order to approximate the Lyapunov and Exponential Dichotomy spectra of a given system. One of our main results is to prove that SVD techniques are sound approaches for systems with stable and distinct Lyapunov exponents. We also show how the information which emerges with the SVD techniques can be used to obtain information on the growth directions associated to given spectral intervals.

Key words: Lyapunov exponents, exponential dichotomy, singular value decomposition
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1 Introduction

Our goal in this work is to examine the feasibility of techniques based on the Singular Value Decomposition (SVD) to approximate spectra of dynamical systems, and at the same time to explore what additional information becomes available when using SVD techniques, and how it can be used. The spectra of

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interest to us are the so-called Lyapunov and Exponential Dichotomy spectra $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$, respectively, to be defined in Sections 2 and 3. As amply documented elsewhere, these spectra address different concerns: In essence, whereas $\Sigma_{\rm L}$ is of use in analyzing variations of (1) below with respect to the initial condition, $\Sigma_{\rm ED}$ is especially useful when the vector field f in (1) depends on parameters, and we want to analyze variations with respect to the parameters. Finally, it must be appreciated that, although from a theoretical point of view Lyapunov exponents and exponential dichotomy spectrum have been studied for many years (e.g., see [1], [7], [18], [25]), it is generally impossible to obtain these quantities analytically, and numerical methods are required for their approximation. At the same time, to understand what a numerical method can offer (and when it works) it is important to distinguish between assumptions needed for the methods to work and assumptions needed for the spectra to be robust in the first place. These considerations have guided us in the present study.

To set the stage, consider the following initial value problem

$$\dot{x} = f(x) \ , \qquad t \geq 0 \ , \qquad x(0) = x_0 \ , \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and continuously differentiable, and let $\phi^t(x_0)$ be the solution of (1). Some of the most powerful tools to analyze the nonlinear system (1) rely on the spectral information associated to the linearized system along $\phi^t(x_0)$:

$$\dot{\Phi}(t, x_0) = Df(\phi^t(x_0))\Phi(t, x_0) \ , \ \ \Phi(0, x_0) = I \ . \tag{2}$$

Indeed, one of the most celebrated and powerful results we have at our disposal to study (1) is the Oseledec's Multiplicative Ergodic Theorem (MET), [22]. The idea behind this remarkable theorem is well known, but worth stressing once more. Roughly speaking, the MET is based on the realization that for many systems of physical interest nearly all trajectories settle on a (low dimensional) bounded attractor, and thus linearized analysis is feasible. Moreover, it is often the case that almost all trajectories on the attractor fill the attractor densely. Thus, each trajectory is typical of the general behavior, and linearized analysis along any of these typical trajectories will be conducive to identical information. This is the idea of the MET. To be precise, one assumes that the phase space of (1), call it M, is compact, and considers an invariant measure ρ on M (the existence of ρ is ensured by the Krylov Bogoliubov Theorem, see [27]). Then, the MET states that there is a measurable set B, with $\rho(B) = 1$, such that for all $x_0 \in B$, the following limit exists

$$\lim_{t\to\infty} (\Phi(t, x_0)^T \Phi(t, x_0))^{1/2t} = \Lambda(x_0). \tag{3}$$

If we let $e^{\lambda_1(x_0)}, \ldots, e^{\lambda_s(x_0)}$ be the eigenvalues of $\Lambda(x_0)$, then their logarithms are called *Lyapunov exponents*, LEs for short, of (2). If ρ is ergodic, then the limit in (3) is the same for all $x_0 \in B$ and it makes sense to talk about the LEs of the non linear system (1) with respect to the measure ρ; in [18], this is called the *measurable spectrum*. The MET has found wide ranging and far reaching applications, in finite and infinite dimensional settings, as a glance at the extensive bibliography in [2] attests. As a result, Lyapunov exponents are routinely used in applications, not only to measure stability of the given trajectory, but also to estimate dimension of an attractor, entropy of a system, to establish chaotic behavior, and they are also of use in studies of nonautonomous bifurcations, as well as to assess continuability and/or bifurcations of invariant manifolds; e.g., see [2,3,4,14,23].

At the same time, the MET suggests a way for the numerical approximation of the LEs: If one could compute the singular value decomposition of the fundamental matrix solution $\Phi(t,x_0)$, $\Phi(t,x_0) = U(t)\Sigma(t)V^T(t)$, then, for large t, the time average of the logarithms of the singular values will approach the LEs. Indeed, methods based on both a continuous and a discrete version of the SVD of the fundamental matrix $\Phi(t,x_0)$ have been used in the literature (e.g., see [16], [15], and [26]), though theoretical justification on the use of these methods is largely absent. Two exceptions are the work [21], which deals with the linear algebraic aspects of a discrete SVD technique, and [12], which is an initial attempt to justify SVD methods for approximation of Σ_{L}. One of our goals in this work is to put SVD techniques on safer ground, by continuing and adapting the approach that the authors used in [12] for so-called QR methods. Indeed, ever since the work [5], most people have favored a different class of methods to approximate the LEs, based on the QR decomposition of an appropriate fundamental matrix solution of the linearized problem. To appreciate the issues involved, we need to quickly review the classical theory of LEs.

Consider the linear system

$$\dot{X} = A(t)X, \qquad t \geq 0, \tag{4}$$

where $A : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ is continuous and bounded, and X is some fundamental matrix solution (i.e., $X(0)$ is invertible). For later reference, we will reserve the notation Φ to indicate the principal matrix solution of (4): $\Phi(0) = I$. Define the following quantities

$$\lambda_j^s = \limsup_{t\to+\infty} \frac{1}{t} \log \|X(t)e_j\|, \qquad \lambda_j^i = \liminf_{t\to+\infty} \frac{1}{t} \log \|X(t)e_j\| \ . \tag{5}$$

[Here, and elsewhere in this work, the vector norm is always the 2-norm, and the matrix norm is the induced norm. Also, the vectors e_j, $j = 1,\dots,n$, are the standard unit vectors.] When $\sum_{j=1}^n \lambda_j^s$ ($\sum_{j=1}^n \lambda_j^i$) is minimized with respect to all fundamental matrix solutions of the system, the λ_j^s (λ_j^i) are called *upper* (*lower*) *Lyapunov exponents* and the corresponding X (i.e., $X(0)$)

is said to form a *normal basis*. Furthermore, for so-called *regular* systems, one has $\lambda_j^s = \lambda_j^i$, for all $j = 1, \dots, n$. (In particular, under the assumptions of the MET, (2) is regular, though one cannot say that $\Phi(0)$ is normal.) Now, if we had a regular, and normal, fundamental matrix solution X, and further had its unique QR decomposition $X(t) = Q(t)R(t)$ for all $t \geq 0$, with Q orthogonal and R upper triangular with positive diagonal, then we could obtain the LEs as

$$\lambda_j = \lim_{t \to +\infty} \frac{1}{t} \log R_{jj}(t) \ , \quad j = 1 \dots, n \, .$$

This is the basic idea of the QR methods, though to obtain viable computational procedures, one must be careful. See [5] for the first algorithmic description of QR methods, and see [11,12] for more recent algorithmic developments also for non-regular systems, and for approximation of both Σ_{L} and Σ_{ED}.

Notice: As it turns out, regularity is not sufficient to guarantee stability of the LEs, and this is the main reason that in this work we will instead assume that (4) is integrally separated, a fact which is conducive to stability of Σ_{L}.

Regardless of regularity, the main conceptual caveat of QR techniques is that one needs to work with a normal fundamental matrix solution. Lyapunov himself had shown more than 100 years ago in his thesis, reprinted in [19], that there always is a normal fundamental matrix solution. But, in practice, how do we know if we are working with a normal fundamental matrix solution? In particular, is the principal matrix solution Φ normal? To get around this impasse, in [5] the authors argued that any randomly chosen initial condition $X(0, x_0)$ will almost surely give a normal fundamental matrix solution for (2). Still, we believe that the entire concept of normal fundamental matrix solution is a somewhat artificial complication, since the LEs are intrinsic quantities of the system we have, and we should be able to extract them from any fundamental matrix solution, regardless of it being normal. With SVD techniques, we will prove that we can. This is one potential advantage of SVD over QR techniques. Another important consequence of SVD methods is that –as we will show– we are able to approximate not only the LEs, but also the set of directions leading to specific LEs, and more generally the directions associated to spectral intervals. Oversimplifying, the situation is similar to being able to approximate not just the eigenvalues of a matrix, but also the eigenspaces. Of course, these potential advantages come with a couple of price tags. The first, and most relevant from the practical point of view, is that SVD techniques are harder to implement than QR techniques. We refer to [8] for a discussion of the delicate computational issues one has to face with SVD techniques, and for a particular implementation. The other is that, at least so far, we are only able to fully justify SVD techniques in case the original system has stable and distinct LEs, and this is true also in case we want to obtain Σ_{ED}. [In some cases, we can relax the assumption of distinct LEs, when the system has some special symmetries, see [9], but not in general.] QR methods, instead, do not seem to suffer from the theoretical need of having distinct LEs, and have been fully justified as techniques to approximate $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$, see [12] and especially [13], for systems with stable LEs.

A plan of the paper is as follows. In Section 2, we review some of the classical theory of LEs with particular attention to their stability. Most results in this section are known, and our contributions are essentially related to the lower exponents, in particular Theorem 2.6 and Proposition 2.10 are new. In Section 3 we discuss $\Sigma_{\rm ED}$. Our presentation is influenced by the seminal work of Sacker and Sell, [25]. Our setup is different, though, since we need to work on the half-line rather than the entire line, because many problems of practical interest just cannot be integrated backward in time (e.g., the famous Lorenz system). As a consequence, we need to work with the adjoint problem as well. Moreover, Theorem 3.2 and especially Theorems 3.9 and 3.11 and Corollary 3.12 are new and quite useful to identify growth subspaces associated to the interval making up $\Sigma_{\rm ED}$, in a way which will be conducive to their approximation via SVD techniques. In Section 4 we discuss SVD methods. We emphasize a continuous SVD framework, though the results apply equally well to so-called discrete SVD techniques. The main result in this section is Theorem 4.2, showing that SVD methods are capable to approximate $\Sigma_{\rm L}$ as long as the LEs are stable and distinct, regardless of which fundamental matrix solution we consider. Analogously, Theorem 4.6 shows that we can get $\Sigma_{\rm ED}$ from SVD techniques. Section 5 deals with the directional information associated to $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$. We first show that the factor V in the SVD of X converges exponentially fast to a constant matrix $\overline{V}$, this is the content of Lemma 5.2 and Theorem 5.4. Proposition 5.7 gives a constructive way to obtain the integral separation constants which we need in our analysis. Also, we further identify, see Theorems 5.8, 5.12, and 5.14, the columns of $\overline{V}$ with appropriate growth directions for $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$. These are some of our main results, and are all obtained under the assumption of stable and distinct LEs. We finally give some concluding remarks.

2 Classical theory of Lyapunov exponents

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a non-vanishing function. The following quantities

$$\chi^s(f) = \limsup_{t\to+\infty} \frac{1}{t} \log|f(t)|, \qquad \chi^i(f) = \liminf_{t\to+\infty} \frac{1}{t} \log|f(t)| \ , \tag{6}$$

are called *upper*, respectively *lower*, *characteristic numbers* of f, or equivalently *upper*, *lower*, *Lyapunov exponents* of f. For short, we will write LE to mean Lyapunov exponent. In a similar way, one defines upper and lower LEs for vector valued functions, where the absolute values will be replaced by norms. All norms lead to the same LE, and we will always consider the 2-norm.

To elucidate the meaning of (6), the following characterization is useful. The characterization for $\chi^s(f)$ is in [1], the one for $\chi^i(f)$ is immediate.

Lemma 2.1 *Given a non-vanishing function $f : \mathbb{R}^+ \to \mathbb{R}$, we have*

- $\chi^s(f) = \alpha \iff \forall \epsilon > 0$, *we have both* $\lim_{t\to\infty} \frac{|f(t)|}{\exp((\alpha+\epsilon)t)} = 0$ *and* $\limsup_{t\to\infty} \frac{|f(t)|}{\exp((\alpha-\epsilon)t)} = +\infty$.
- $\chi^i(f) = \beta \iff \forall \epsilon > 0$, *we have both* $\lim_{t\to\infty} \frac{|f(t)|}{\exp((\beta-\epsilon)t)} = +\infty$ *and* $\liminf_{t\to\infty} \frac{|f(t)|}{\exp((\beta+\epsilon)t)} = 0$.

Also the following properties will be handy later.

Property 2.2

(a) *Let $f_1, \dots f_n$ be n non vanishing scalar functions, then*

$$\chi^s(\sum_{j=1}^{n} f_j) \leq \max_{j=1,\dots,n} \chi^s(f_j)$$

where equality holds when the maximum characteristic exponent is attained by only one function.

(b) *Let $\chi^s(f) < 0$ and define $F(t) = \int_t^{+\infty} f(s)ds$. Then, $\chi^s(F) \leq \chi^s(f)$.*

(c) *Let $\chi^s(f) < 0$ and define $F(t) = \sum_{n=0}^{+\infty} f(t + n\tau)$, with $\tau > 0$, fixed and finite. Then, $\chi^s(F) \leq \chi^s(f)$.*

Proof The proofs of (a) and (b) are in [1, pp. 27 and 30]. We prove (c). Let $\chi^s(f) = \lambda$, then for all $\epsilon > 0$ there exists $\overline{T} > 0$ such that for all $t \geq \overline{T}$, $|f(t)| \leq e^{(\lambda+\epsilon)t}$. For $t \geq \overline{T}$, $F(t) \leq \sum_{n=0}^{+\infty} |f(t + n\tau)| \leq \sum_{n=0}^{+\infty} e^{(\lambda+\epsilon)(t+n\tau)} = e^{(\lambda+\epsilon)t} \sum_{n=0}^{+\infty} (e^{(\lambda+\epsilon)\tau})^n$, and if we choose ϵ such that $(\lambda + \epsilon) < 0$, it follows $|F(t)| \leq Ce^{(\lambda+\epsilon)t}$ with $C = \frac{1}{1-e^{(\lambda+\epsilon)\tau}}$. The statement now follows upon observing that χ^s does not depend on the chosen initial time. $\square$

2.1 Stability and integral separation

Now consider the linear system (4). In general, the LEs of (4) are not stable (i.e., continuous) with respect to perturbation of the coefficients, though their stability is essential for the success of any numerical method used for their approximation. The standard definition of stability is the following.

Let $\lambda_1^s \geq \ldots \geq \lambda_n^s$ be the upper LEs of system (4) and let $\gamma_1^s \geq \ldots \geq \gamma_n^s$ be the upper LEs of the (perturbed) system $\dot{y} = [A(t) + B(t)]y$. Then, (4) has stable upper LEs if for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\|B(t)\| \leq \delta_\epsilon \Longrightarrow |\lambda_j^s - \gamma_j^s| < \epsilon, \quad j = 1, \dots n \ .$$

An analogous definition can be given for stability of the lower LEs, see also Proposition 2.7 below.

An important property of stable upper LEs is that they do not change when the perturbation B is vanishing as $t \to +\infty$:

Property 2.3 [1]. *If system* (4) *has stable upper (lower) LEs and* $B(t) \to 0$ *for* $t \to +\infty$, *then* $\lambda_j^s = \gamma_j^s$ ($\lambda_j^i = \gamma_j^i$) *for* $j = 1, \dots, n$.

As we said, not all systems admit stable LEs. The most important class of systems having stable LEs is that of **integrally separated** systems.

Definition 2.4 *The system* (4) *is called integrally separated, if there exists a fundamental matrix solution* X *of* (4) *and constants* $a > 0$ *and* $0 < d \leq 1$ *such that*

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq d e^{a(t-s)}, \qquad \forall t, s, \quad t \geq s \geq 0, \tag{7}$$

and for all $i = 1, \dots, n-1$.

It is important to notice that integral separation is an intrinsic property of a given system, just like stability of the LEs.

As we will see shortly, integral separation is a fundamental property for the justification of our technique. We will need to use several properties related to integral separation, which we now recall.

Properties 2.5

1. [1, p.148]. *Integrally separated systems have distinct and stable upper LEs.*
2. [1, p. 172, 149]. *If* (4) *has different upper LEs* $\lambda_1^s > \dots > \lambda_n^s$ *then they are stable if and only if* (4) *is integrally separated. Moreover, an integrally separated fundamental matrix solution,* X, *is normal, and thus one has* $\lambda_j^s = \limsup_{t\to+\infty} \frac{1}{t} \log \|X(t)e_j\|$, $j = 1, \dots, n$.
3. [1, p.149]. *Integral separation is preserved under Lyapunov transformations. That is, under a smooth invertible change of variables* $Z = LX$, *with* L, L^{-1} *and* $\dot{L}$ *bounded. In the new variables, the coefficient matrix becomes* $(LAL^{-1} + \dot{L}L^{-1})$.
4. [1, p.149]. *The diagonal system*

$$\dot{z} = \text{diag}(d_1(t), \dots, d_n(t))z, \tag{8}$$

is integrally separated if and only if its diagonal is integrally separated. That *is, if the functions $d_1, \dots, d_n$ are integrally separated, which means that there exist $a > 0$ and $d \geq 0$ such that*

$$\int_s^t [d_i(\tau) - d_{i+1}(\tau)]d\tau \geq a(t-s) - d, \qquad \forall t, s, \quad t \geq s \geq 0, \tag{9}$$

for $i = 1, \dots, n-1$.

5. [1, p.172]. *System (4) has distinct and stable upper LEs $\lambda_1^s > \lambda_2^s > \dots > \lambda_n^s$, if and only if there exists a Lyapunov transformation L that reduces the given system to a diagonal one with integrally separated diagonal.*
6. [12]. *An upper triangular system $\dot{R} = BR$, where B is bounded and has integrally separated diagonal, has an integrally separated fundamental matrix solution.*
7. [24]. *Integral separation is a generic property in the Banach space of continuous and bounded coefficients' functions with the* sup*-norm: $d(A, B) = \sup_t \|A - B\|_\infty$.*

We will also need a property of the lower characteristic number of the sum of functions.

Theorem 2.6 *Given non vanishing functions $f_1, f_2, \dots, f_n$, such that*

$$|f_i(t)|/|f_{i+1}(t)| \geq de^{at} \ , \ i = 1, 2, \dots, n-1 \ ,$$

for some $0 < a$ and $0 < d \leq 1$, then we have

$$\chi^i(\sum_{k=1}^n f_k) \ = \ \chi^i(f_1)\,. \tag{10}$$

Proof Under the stated assumption, $\chi^i(f_1) > \chi^i(f_2) > \dots > \chi^i(f_n)$. In particular, for all $i = 1, \dots, n-1$, $\lim_{t\to+\infty} \frac{1}{t} \log \frac{|f_{i+1}(t)|}{|f_i(t)|} = 0$. It then follows

$$\begin{aligned}
\chi^i\left(\sum_{k=1}^n f_k\right) &= \liminf_{t\to+\infty} \frac{1}{t} \log |\sum_{k=1}^n f_k(t)| \\
&= \liminf_{t\to+\infty} \frac{1}{t} \log |f_1(t)| \left|1 + \frac{f_2(t)}{f_1(t)} + \dots + \frac{f_n(t)}{f_1(t)}\right| \\
&= \liminf_{t\to+\infty} \frac{1}{t} \log |f_1(t)| + \lim_{t\to+\infty} \frac{1}{t} \log \left|1 + \frac{f_2(t)}{f_1(t)} + \dots + \frac{f_n(t)}{f_1(t)}\right| \\
&= \chi^i(f_1).
\end{aligned}$$

□

We now proceed to obtain some results on the lower LEs as well. To do this, we will use the adjoint system

$$\dot{y} = -A(t)^T y \ , \tag{11}$$

and we recall that if X is a fundamental matrix solution of (4) then X^{-T} is a fundamental matrix solution of (11).

Proposition 2.7 *Suppose* (4) *is integrally separated and let* $\mu_1^s \geq \mu_2^s \ldots \geq \mu_n^s$ *be the upper LEs of the adjoint system* (11)*. Then* $\mu_1^s = -\lambda_n^i, \ldots, \mu_n^s = -\lambda_1^i$*. Moreover, the lower LEs are stable and distinct:* $\lambda_1^i > \lambda_2^i \ldots > \lambda_n^i$*.*

Proof We know that there is an integrally separated matrix solution, X. Further, because of Properties 2.5-5, (4) is reducible to the integrally separated diagonal system $\dot{z} = \mathrm{diag}[d_1(t), \ldots, d_n(t)]z$ via a Lyapunov transformation, $Z = LX$. Consider then $X^{-T}P = L^T Z^{-1} P$ where P is the permutation $P = [e_n, e_{n-1}, \ldots, e_1]$ and further let $W = Z^{-1}P$. Thus, (11) is reducible to $\dot{w} = -[d_n(t), \ldots, d_1(t)]w$, which is integrally separated. Therefore:

$$\mu_1^s = \limsup_{t\to+\infty} \frac{1}{t} \log e^{-\int_0^t d_n(s)ds} = \limsup_{t\to+\infty} \Big(-\frac{1}{t}\int_0^t d_n(s)ds\Big)$$
$$= -\liminf_{t\to+\infty} \frac{1}{t}\int_0^t d_n(s)ds = -\lambda_n^i;$$

the equality for the other LEs can be proven analogously. Since the w-system is integrally separated, we have $\mu_1^s > \mu_2^s > \cdots > \mu_n^s$ and they are stable as well. □

Under the assumption of **distinct and stable** LEs, we now define the Lyapunov spectrum Σ_{L} of (4) as

$$\Sigma_{\mathrm{L}} = \bigcup_{j=1}^{n} [\lambda_j^i, \lambda_j^s] \; . \tag{12}$$

Because of Proposition 2.7, this definition of Σ_{L} coincides with that given in [13].

2.2 Directions leading to the LEs

To reach a complete understanding of stability of (4), the growth information enclosed in Σ_{L} must be complemented with appropriate geometrical information on the subspaces of solutions which eventually achieve a specific growth.

For this reason, let λ_j^s, $j = 1, \ldots, p$, be the distinct upper LEs of (4). For $j = 1, \ldots, p$, define the set W_j to be the set of all initial conditions w such that for the solution $\Phi(t)w$, $t \geq 0$, we have $\chi^s(\Phi(\cdot)w) \leq \lambda_j^s$. That is:

$$W_j = \{w \in \mathbb{R}^n \;:\; \chi^s(\Phi(\cdot)w) \leq \lambda_j^s \} \, , \; j = 1, 2, \ldots, p \; . \tag{13}$$

Of course, we can equivalently identify W_j with the space of all solutions whose upper LE does not exceed λ_j^s.

Proposition 2.8 *[1]. Let n_j be the greatest number of linearly independent solutions x of* (4) *such that* $\limsup_{t\to+\infty} \frac{1}{t}\log\|x(t)\| = \lambda_j^s$. *Then W_j is an n_j-dimensional linear subspace of $\mathbb{R}^n$.*

Remark 2.9 In [1], and to prove this fact Property 2.2-(a) is essential, it is also showed that the W_j's are a filtration of $\mathbb{R}^n$. That is, if p is the number of distinct upper LEs of the system, we have

$$\mathbb{R}^n = W_1 \supset W_2 \ldots \supset W_p \supset W_{p+1} = \{0\}. \tag{14}$$

Therefore, $\limsup_{t\to+\infty} \frac{1}{t}\log\|\Phi(t)w\| = \lambda_j^s$ if and only if $w \in W_j \backslash W_{j+1}$. Notice that if we have n distinct upper LEs, then each W_j, $j = 1, \ldots, n$, has dimension $(n-j+1)$.

Let V_j be the orthogonal complement of W_{j+1} in W_j, i.e.

$$W_j = W_{j+1} \oplus V_j, \qquad V_j \perp W_{j+1}.$$

Then $\mathbb{R}^n = V_1 \oplus V_2 \oplus \ldots \oplus V_p$. Moreover, if $w \in V_j$ then $\limsup_{t\to+\infty} \frac{1}{t}\log\|\Phi(t)w\| = \lambda_j^s$. Notice that if we have distinct LEs, then $\dim(V_j) = 1$ for all $j = 1, \ldots, n$. As we will see in Section 5, by the SVD technique we will be able to approximate these V_j's.

We now show that the W_j's, for integrally separated systems, characterize the set of initial conditions leading to lower Lyapunov exponents as well.

Proposition 2.10 *Assume* (4) *is integrally separated, and let W_j, $j = 1, \ldots, n$, be defined as above. Then, similarly to Proposition 2.8, for all $j = 1, 2, \ldots, n$, we have*

$$W_j = \{ w \in \mathbb{R}^n \ : \ \chi^i(\Phi(\cdot)w) \le \lambda_j^i \ \} \,. \tag{15}$$

Proof Let X be an integrally separated fundamental matrix solution, and let $x_i(t) = X(t)e_i$, $i = 1, \ldots, n$, $\forall t \ge 0$. Then, from (7), we have that for all $i = 1, \ldots, n-1$, there is some constant c such that

$$\|x_i(t)\| \ge c e^{at} \|x_{i+1}(t)\| \ , \ \ \forall \ t \ge 0 \,.$$

This implies that the χ^s and the χ^i of the columns of X are different and hence must attain the entire set of upper and lower LEs:

$$\limsup_{t\to+\infty} \frac{1}{t}\log\|x_j(t)\| = \lambda_j^s \ , \quad \liminf_{t\to+\infty} \frac{1}{t}\log\|x_j(t)\| = \lambda_j^i \ , \ \ j = 1, 2, \ldots, n \ .$$

By what we said in Remark 2.9, $\dim(W_n) = 1$, and thus $\mathrm{span}(x_n(0)) = W_n$ and (15) follows in the case of $j = n$. Next, consider $j = n-1$. In this case, we know that $\dim(W_{n-1}) = 2$ and that $x_{n-1}(0) \in W_{n-1}$. Since the W_j's are a filtration, we must have that both vectors $x_n(0)$ and $x_{n-1}(0)$ are in W_{n-1} and thus W_{n-1} is their span (since they obviously are independent). So, if we now take a vector $w \in W_{n-1}$, $w \neq 0$, we can write $w = c_n x_n(0) + c_{n-1}x_{n-1}(0)$, and we can assume $c_{n-1} \neq 0$. So, we have

$$\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)w\| = \liminf_{t\to+\infty} \frac{1}{t} \log \|c_n x_n(t) + c_{n-1}x_{n-1}(t)\| =$$
$$\liminf_{t\to+\infty} \frac{1}{t} \log \left(\|x_{n-1}(t)\| \cdot \| \frac{c_n x_n(t)}{\|x_{n-1}(t)\|} + c_{n-1} \frac{x_{n-1}(t)}{\|x_{n-1}(t)\|} \| \right)$$

and reasoning like in Theorem 2.6 we obtain that $\chi^i(\Phi(\cdot)w) = \lambda^i_{n-1}$. The proof follows by repeating this argument. $\square$

3 Exponential dichotomy spectrum

System (4) admits an exponential dichotomy in $[0, +\infty)$ if there exists a projection P and real numbers $K \geq 1$, $\alpha > 0$ s.t.

$$\begin{aligned} \|X(t)PX(s)^{-1}\| \leq & \quad Ke^{-\alpha(t-s)}, \qquad \forall t, s, \quad t \geq s \geq 0, \\ \|X(t)(I-P)X(s)^{-1}\| \leq & \quad Ke^{\alpha(t-s)}, \qquad \forall t, s, \quad 0 \leq t \leq s, \end{aligned} \tag{16}$$

where with X we denote any fundamental matrix solution of the system. Let $X = \Phi$, the principal matrix solution, then P selects a subspace in $\mathbb{R}^n$ such that all solutions in it (respectively in the complementary subspace) are uniformly exponentially decreasing (increasing); i.e, exponential dichotomy is a generalization of the concept of hyperbolicity for autonomous systems.

Several properties of exponential dichotomy are useful. Two of them follow, see [7,25].

- Let $t_0 > 0$, then if (4) admits exponential dichotomy in $[t_0, +\infty)$, it admits exponential dichotomy in $[0, +\infty)$ as well, with same α.
- Let $\widehat{P}$ be a projection matrix with same range as P and suppose (4) admits an exponential dichotomy with projection P. Then it also admits an exponential dichotomy with projection P' and same α. Therefore, one can always choose P to be an orthogonal, symmetric, projection. We will freely assume this to be the case.

Remark 3.1 By transposing the quantities in the norms in (16), and using the fact that the projection is (or can be chosen to be) symmetric, we observe that if we let $Q = I - P$, then the adjoint problem has an exponential dichotomy with same constants, but projection Q:

$$\begin{aligned} \|X^{-T}(s)QX^T(t)\| \leq Ke^{-\alpha(s-t)} \ , \quad s \geq t \geq 0 \\ \|X^{-T}(s)(I-Q)X^T(t)\| \leq Ke^{\alpha(s-t)} \ , \quad 0 \leq s \leq t \, . \end{aligned} \tag{17}$$

The **exponential dichotomy spectrum** of (4), denote it with $\Sigma_{\text{ED}}(A)$, or more simply Σ_{ED} when no ambiguity can arise, is the set of all real values λ such that the shifted system

$$\dot{x} = (A(t) - \lambda I)x, \tag{18}$$

does not admit exponential dichotomy. The complement in $\mathbb{R}$ of $\Sigma_{\text{ED}}(A)$ is called the **resolvent**, which we denote by ρ. The following properties of Σ_{ED} are well known (see [7,25]).

- Σ_{ED} is the union of m disjoint closed intervals, $m \leq n$:

$$\Sigma_{\text{ED}} = \bigcup_{i=1}^{m} [a_i, b_i] \;\; : \quad a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_m \leq b_m \, .$$

- Σ_{ED} is stable. This means that, however we take $\lambda \in \Sigma_{\text{ED}}$, and for any $\epsilon > 0$, there exists δ_ϵ such that for the perturbed system

$$\dot{z} = [A(t) + C(t)]z, \qquad \sup_{t \in [0,+\infty)} \|C(t)\| \leq \delta_\epsilon,$$

  there is $\alpha \in \Sigma_{\text{ED}}(A + C)$ with

$$|\lambda - \alpha| < \epsilon. \tag{19}$$

About the stability of Σ_{ED}, more can be said in case the given perturbation goes to zero. The following result has an analog for Σ_{L} for systems with stable LEs in [1, Theorem 5.2.1].

Theorem 3.2 *Consider the following perturbation of* (4), $\dot{x} = [A(t)+B(t)]x$, *for* $t \geq 0$, *where* $\lim_{t\to+\infty} \|B(t)\| = 0$. *Then*

$$\Sigma_{\text{ED}}(A + B) = \Sigma_{\text{ED}}(A).$$

Proof By contradiction, suppose that the two spectra are different. Then, without loss of generality, we can assume that there exists $\lambda \in \Sigma_{\text{ED}}(A)$ such that $\lambda \notin \Sigma_{\text{ED}}(A+B)$. Let $0 < a = \min_{\gamma \in \Sigma_{\text{ED}}(A+B)} |\lambda - \gamma|$ and set $\epsilon = \frac{a}{2}$. Being $\Sigma_{\text{ED}}(A)$ stable, there exists δ_ϵ such that there is an $\alpha \in \Sigma_{\text{ED}}(A + C)$ of the perturbed system

$$\dot{z} = [A(t) + C(t)]z, \qquad \sup_{t \in [0,+\infty)} \|C(t)\| \leq \delta_\epsilon,$$

with $|\lambda - \alpha| < \epsilon$. Now let $T > 0$ be such that $\|B(t)\| \leq \delta_\epsilon$ for all $t \geq T$ and set $C(t) = B(t)$ for $t \geq T$. For $t < T$, $C(t) = \overline{C}(t)$, where $\overline{C}$ is such that $\sup_{t\in[0,T]} \|\overline{C}(t)\| \leq \delta_\epsilon$ and $\overline{C}(T) = B(T)$. But, being C a perturbation finite in time of B, $\Sigma_{\text{ED}}(A + C) = \Sigma_{\text{ED}}(A + B)$ and this with (19) contradicts the hypothesis that the two spectra are different. □

The setup below is adapted from that in [25]. We will now show, as we did for the Lyapunov spectrum, that there are linear subspaces associated to Σ_{ED}. In order to do so, let us define the following sets (stable and unstable sets):

$$\mathcal{S}_\mu = \{w \in \mathbb{R}^n | \lim_{t\to+\infty} e^{-\mu t}\|\Phi(t)w\| = 0\},$$
$$\mathcal{U}_\mu = \{w \in \mathbb{R}^n | \lim_{t\to+\infty} e^{\mu t}\|\Phi^{-T}(t)w\| = 0\},$$

where Φ and Φ^{-T} are the principal matrix solutions of (4) and (11), respectively.

$\mathcal{S}_\mu$ and $\mathcal{U}_\mu$ satisfy the following

Property 3.3

(a) $\mathcal{S}_\mu \cap \mathcal{U}_\mu = \{0\}$.
(b) *If* $\mu_1 < \mu_2$ *then* $\mathcal{S}_{\mu_1} \subseteq \mathcal{S}_{\mu_2}$ *and* $\mathcal{U}_{\mu_1} \supseteq \mathcal{U}_{\mu_2}$.

Proof We just show (a), since (b) is easy to verify. For any given $t > 0$, and for any $w \in \mathbb{R}^n$, we have

$$\|w\|^2 = \left(\Phi^{-T}(t)w\right)^T \left(\Phi(t)w\right) \le \|\Phi(t)w\|\|\Phi^{-T}(t)w\| \ . \tag{20}$$

Now, if $w \in \mathcal{S}_\mu$, $w \neq 0$, then $e^{\mu t}\frac{1}{\|\Phi(t)w\|} \to +\infty$ as $t \to +\infty$. From this and (20) we have

$$\frac{1}{\|w\|^2} e^{\mu t}\|\Phi^{-T}(t)w\| \ge e^{\mu t}\frac{1}{\|\Phi(t)w\|} \to +\infty \ , \ \text{as} \ t \to +\infty \ ,$$

this implying $w \notin \mathcal{U}_\mu$. In the same way we can show that if $w \in \mathcal{U}_\mu$ then $w \notin \mathcal{S}_\mu$. □

Theorem 3.4 *Let* $\mu \in \rho(A)$*, and denote with* Φ_μ *and* Φ_μ^{-T} *respectively the principal matrix solution of system* (18) *(with* $\lambda = \mu$ *there) and of its adjoint. Let* P_μ *and* Q_μ *be the projections in* (16) *and* (17) *for* Φ_μ *and* Φ_μ^{-T}*. Then*

$$\mathrm{Range}(P_\mu) = \mathcal{S}_\mu \ , \qquad \mathrm{Range}(Q_\mu) = \mathcal{U}_\mu.$$

Proof Take $w = P_\mu c$. Then, by setting $s = 0$ in the first of (16) we have

$$\begin{aligned}\|\Phi_\mu(t)w\| &= \|\Phi_\mu(t)P_\mu\Phi_\mu(0)c\| \\ &\le \|\Phi_\mu(t)P_\mu\Phi_\mu(0)\|\|c\| \\ &\le K\|c\|e^{-\alpha t} \to 0 \quad \text{as} \quad t \to +\infty,\end{aligned} \tag{21}$$

and this implies $w \in \mathcal{S}_\mu$.

Similarly, let $w = Q_\mu b$. Then, using $t = 0$ in the first of (17) we have

$$\|\Phi_\mu^{-T}(s)w\| = \|\Phi_\mu^{-T}(s)Q_\mu\Phi_\mu(0)b\| \leq K\|b\|e^{-\alpha t} \to 0 \quad \text{as} \quad t \to +\infty \ ,$$

and so $w \in \mathcal{U}_\mu$.
Finally, $\mathcal{S}_\mu \cap \mathcal{U}_\mu = \{0\}$ and $\dim(\text{Range}(P_\mu)) + \dim(\text{Range}(Q_\mu)) = n$, so we must have $\text{Range}(P_\mu) = \mathcal{S}_\mu$ and $\text{Range}(Q_\mu) = \mathcal{U}_\mu$. □

The next two results are immediate.

Lemma 3.5 *Let μ_1 and $\mu_2 \in \rho$, $\mu_1 < \mu_2$. If $\mathcal{S}_{\mu_1} = \mathcal{S}_{\mu_2}$ and $\mathcal{U}_{\mu_1} = \mathcal{U}_{\mu_2}$ then $[\mu_1, \mu_2] \subset \rho$ and $\mathcal{S}_\mu = \mathcal{S}_{\mu_1}$, $\mathcal{U}_\mu = \mathcal{U}_{\mu_1}$, for every $\mu \in [\mu_1, \mu_2]$.*

Corollary 3.6 *Let $\mu_1, \mu_2 \in \rho$, $\mu_1 < \mu_2$, such that $[\mu_1, \mu_2] \cap \Sigma_{\text{ED}} \neq \emptyset$. Then $\mathcal{S}_{\mu_2} \cap \mathcal{U}_{\mu_1} \neq \{0\}$.*

Choose now $\mu_0 < \mu_1 < \ldots < \mu_m$, $\mu_i \in \rho$, $i = 0, \ldots, m$, in such a way that $\Sigma_{\text{ED}} \cap (\mu_{j-1}, \mu_j) = [a_j, b_j] \neq \emptyset$. To every interval $[a_j, b_j]$ we can associate a linear subspace.

Theorem 3.7 *Let $\mathcal{N}_j = \mathcal{S}_{\mu_j} \cap \mathcal{U}_{\mu_{j-1}}$, for $j = 1, \ldots, m$. Then $\mathcal{N}_j$ is a linear subspace with $\dim \mathcal{N}_j \geq 1$. Moreover the following properties are true*

(a) $\mathcal{N}_k \cap \mathcal{N}_l = \{0\}$, *for $k \neq l$.*
(b) $\mathbb{R}^n = \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_m$.

Proof That $\mathcal{N}_j$ is a linear subspace, follows from it being the intersection of linear subspaces. That $\dim \mathcal{N}_j \geq 1$, follows from Corollary 3.6. In order to prove (a), we can assume without loss of generality that $k < l$. Then $\mathcal{N}_k \subseteq \mathcal{S}_{\mu_k}$, $\mathcal{N}_l \subseteq \mathcal{U}_{\mu_{l-1}} \subseteq \mathcal{U}_{\mu_k}$. But $\mathcal{S}_{\mu_k} \cap \mathcal{U}_{\mu_k} = \{0\}$, and the claim follows.
To prove (b), notice that $\mathcal{U}_{\mu_0} = \mathbb{R}^n$. Then

$$\begin{aligned}\mathbb{R}^n &= \mathcal{U}_{\mu_0} \cap (\mathcal{S}_{\mu_1} \oplus \mathcal{U}_{\mu_1}) = (\mathcal{U}_{\mu_0} \cap \mathcal{S}_{\mu_1}) \oplus \mathcal{U}_{\mu_1} = \\ &= \mathcal{N}_1 \oplus \mathcal{U}_{\mu_1} \cap (\mathcal{S}_{\mu_2} \oplus \mathcal{U}_{\mu_2}) = \ldots = \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_m.\end{aligned}$$

□

Next, we give a geometrical characterization of the subspaces $\mathcal{N}_j$'s. The first result is in [25] and shows the relation between the Lyapunov exponents and the intervals $[a_j, b_j]$.

Theorem 3.8 *With same notation of Theorem 3.7, let $w \in \mathcal{N}_j$ and $\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)w\| = \chi^s$, $\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)w\| = \chi^i$. Then*

$$\chi^s, \, \chi^i \in [a_j, b_j].$$

Proof For a proof, see [25]. □

The following theorem characterizes the subspaces $\mathcal{N}_j$'s as those subspaces of initial conditions giving growth behavior between a_j and b_j, uniformly.

Theorem 3.9 *With same notation of Theorem 3.7, we have $w \in \mathcal{N}_j$, $w \neq 0$, if and only if*

$$\frac{\|\Phi(t)w\|}{\|\Phi(s)w\|} \leq K_j e^{b_j(t-s)} \ , \quad and \quad \frac{1}{K_{j-1}} e^{a_j(t-s)} \leq \frac{\|\Phi^{-T}(s)w\|}{\|\Phi^{-T}(t)w\|} \tag{22}$$

for all t, s: $t \geq s \geq 0$ and where $K_j \geq 1$ and $K_{j-1} \geq 1$ are constants defined in the proof below.

Proof

($\Rightarrow$) If μ_j is in the resolvent, then the shifted system $\dot{x} = (A(t) - \mu_j I)x$ admits an exponential dichotomy. Denote with Φ_{μ_j} its principal matrix solution and let P_j be the projection in (16) for $X = \Phi_{\mu_j}$, and $\alpha_j > 0$, $K_j \geq 1$, the associated dichotomy constants. In the same way, denote with Q_{j-1} the projection for the adjoint system corresponding to $\mu = \mu_{j-1}$, and K_{j-1}, α_{j-1} be the associated dichotomy constants. Then, using Theorem 3.4, $w \in \mathcal{N}_j$, $w \neq 0$, implies $w \in \mathcal{S}_{\mu_j} = \text{Range}(P_j)$, $w = P_j c$, and for $t \geq s$,

$$\frac{\|\Phi_{\mu_j}(t)w\|}{\|\Phi_{\mu_j}(s)w\|} = \frac{\|\Phi_{\mu_j}(t)P_j\Phi_{\mu_j}^{-1}(s)\Phi_{\mu_j}(s)P_jc\|}{\|\Phi_{\mu_j}(s)P_jc\|} \leq \|\Phi_{\mu_j}(t)P_j\Phi_{\mu_j}^{-1}(s)\| \leq K_j e^{-\alpha_j(t-s)}.$$

Then $\frac{\|\Phi_{\mu_j}(t)w\|}{\|\Phi_{\mu_j}(s)w\|} \leq K_j e^{(\mu_j - \alpha_j)(t-s)} \leq K_j e^{b_j(t-s)}$, since (see Lemma 3.5) μ_j can be chosen in $(b_j, b_j + \alpha_j)$.
Analogously, $w \in \mathcal{N}_j$, $w \neq 0$, implies $w \in \mathcal{U}_{\mu_{j-1}} = \text{Range}(Q_{j-1}) = \text{Range}(I - P_{j-1})$, $w = Q_{j-1}b$. Then (see (17)) for $s \geq t \geq 0$

$$\begin{aligned}\frac{\|\Phi_{\mu_{j-1}}^{-T}(s)w\|}{\|\Phi_{\mu_{j-1}}^{-T}(t)w\|} &= \frac{\|\Phi_{\mu_{j-1}}^{-T}(s)Q_{j-1}\Phi_{\mu_{j-1}}^{T}(t)\Phi_{\mu_{j-1}}^{-T}(t)Q_{j-1}b\|}{\|\Phi_{\mu_{j-1}}^{-T}(t)Q_{j-1}b\|} \\ &\leq \|\Phi_{\mu_{j-1}}^{-T}(s)Q_{j-1}\Phi_{\mu_{j-1}}^{T}(t)\| \\ &\leq K_{j-1}e^{-\alpha_{j-1}(s-t)}.\end{aligned}$$

From the last inequality, for $s \geq t$, we get

$$\frac{\|\Phi^{-T}(t)w\|}{\|\Phi^{-T}(s)w\|} \geq \frac{1}{K_{j-1}} e^{(\mu_{j-1}+\alpha_{j-1})(s-t)} \geq \frac{1}{K_{j-1}} e^{a_j(s-t)},$$

since μ_{j-1} can be chosen in $(a_j - \alpha_{j-1}, a_j)$. So, we showed that $w \in \mathcal{N}_j$ implies (22).

($\Leftarrow$) Suppose now that $w \in \mathbb{R}^n$ is such that (22) holds. Using the first inequality in (22) with $s = 0$, we get that $\|\Phi_{\mu_j}(t)w\| \to 0$ as $t \to +\infty$. Likewise, using the second inequality in (22), with $s = 0$, gives $\|\Phi^{-T}_{\mu_{j-1}}(t)w\| \to 0$ as $t \to +\infty$. Therefore, $w \in \mathcal{S}_{\mu_j} \cap \mathcal{U}_{\mu_{j-1}}$. □

Remark 3.10 *One implication in Theorem 3.9 can be replaced by a simpler one. In fact, if $w \in \mathcal{N}_j$, we can replace* (22) *with*

$$\frac{1}{K_{j-1}} e^{a_j(t-s)} \le \frac{\|\Phi(t)w\|}{\|\Phi(s)w\|} \le K_j e^{b_j(t-s)} \ , \ \ t \ge s \ge 0 \, .$$

This is because if $w \in \mathcal{U}_{\mu_{j-1}}$, $w = Q_{j-1}b$, we can use the first of (17) *for $s \ge t \ge 0$, after transposing its argument. So doing we get:*

$$\begin{aligned}\frac{\|\Phi_{\mu_{j-1}}(t)w\|}{\|\Phi_{\mu_{j-1}}(s)w\|} &= \frac{\|\Phi_{\mu_{j-1}}(t)Q_{j-1}\Phi^{-1}_{\mu_{j-1}}(s)\Phi_{\mu_{j-1}}(s)w\|}{\|\Phi_{\mu_{j-1}}(s)w\|} \\ &\le \|\Phi_{\mu_{j-1}}(t)Q_{j-1}\Phi^{-1}_{\mu_{j-1}}(s)\| \le K_{j-1}e^{-\alpha_{j-1}(s-t)},\end{aligned}$$

and thus $\frac{\|\Phi(s)w\|}{\|\Phi(t)w\|} \ge \frac{1}{K_{j-1}} e^{a_j(s-t)}$ for $s \ge t$.

A simple characterization of $\mathcal{N}_j$ can be given when the system is integrally separated.

Theorem 3.11 *Assume that* (4) *is integrally separated. Denote with W_k the linear space associated to λ^s_k for the system* (4)*, and with L_k the linear space associated to μ^s_k for the system* (11)*. Then*

$$\mathcal{S}_\mu = W_k, \qquad \mathcal{U}_\mu = L_{n-k+2},$$

where k is such that

$$\lambda^s_k < \mu < \lambda^i_{k-1}, \quad \textit{i.e.} \quad \mu^i_{n-k+1} > -\mu > \mu^s_{n-k+2} \ .$$

Proof If $w \in \mathcal{S}_\mu$ then $e^{-\mu t}\|\Phi(t)w\| \to 0$ for $t \to +\infty$. So,
$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)w\| < \mu$ and it follows that
$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)w\| \le \lambda^s_k$ and $w \in W_k$.
Let now $w \in W_k$. Then, $\lim_{t\to+\infty} e^{-(\lambda^s_k+\epsilon)t}\|\Phi(t)w\| = 0$ and this implies $w \in \mathcal{S}_\mu$. For $\mathcal{U}_\mu$, we proceed similarly. If $w \in \mathcal{U}_\mu$ then $e^{\mu t}\|\Phi^{-T}(t)w\| \to 0$ for $t \to +\infty$. Thus $\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi^{-T}(t)w\| \le -\mu$, so that
$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi^{-T}(t)w\| \le \mu^s_{n-k+2}$. □

Using Proposition 2.7, the following is immediate.

Corollary 3.12 *With same assumptions and notations as in Theorems 3.7 and 3.11,*

$$\mathcal{N}_j = W_k \cap L_{n-l+1} \ , \ \ j = 1, \ldots, m \ ,$$

where the indices k and l, $k < l$, are such that

$$\lambda^s_k < \mu_j < \lambda^i_{k-1}, \qquad \lambda^s_{l+1} < \mu_{j-1} < \lambda^i_l.$$

Example 3.13 Assume system (4) to be integrally separated and let X be an integrally separated fundamental matrix solution. Let $X(0) = X_0$, and so $X_0^{-T}P$, with $P = [e_n, \ldots, e_1]$ leads to an integrally separated fundamental matrix solution for the adjoint system. In this context

$$\mathcal{N}_j = \text{span}(X_0[e_k, \ldots, e_n]) \cap \text{span}(X_0^{-T}[e_1, \ldots, e_l]),$$

where k and l are chosen as in the statement of Corollary 3.12.

4 The SVD method

Next we examine a technique to approximate the spectra based upon the SVD of a fundamental matrix solution of (4). As we said, the technique of choice to approximate Lyapunov exponents (and more generally the spectra) has been one based on the QR decomposition of the matrix solution ([5,11,12]), but also methods based on the SVD of a fundamental matrix solution have been used; e.g., see [15,16,26]. In these cited works, SVD methods have appeared in two flavors: discrete and continuous. Though conceptually equivalent, each has distinct advantages/disadvantages, some of which will be reviewed in [8].

Of course, SVD methods have a sound justification for approximating Lyapunov exponents: The Multiplicative Ergodic Theorem, see (3). Indeed, in the discrete SVD method one keeps the principal matrix solution at time t, say $\Phi(t, x_0)$ in (3), as a product of transition matrices on short subintervals, and then seeks an SVD decomposition of this product without forming it (a well written algorithmic description is in [26]). In principle, in the limit, this will enable approximation of the Lyapunov exponents associated to the underlying regular system, [21]. However, there is little general justification of SVD methods for approximating Σ_{L} and Σ_{ED}. Our goal in this section is to rectify this situation, by strengthening and carrying forward the initial contribution of [12]. We will take the viewpoint of a continuous SVD method, though of course our analysis will justify use of a discrete SVD method as well.

The continuous method looks for a smooth SVD of a fundamental matrix solution X: $X(t) = U(t)\Sigma(t)V^T(t)$, $t \geq 0$, where U and V are orthogonal and $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$. The existence of an SVD of $X(t)$ at every instant t is a well known fact, but it is not at all obvious that the factors can be taken smooth in t (e.g., see [6], [10]). Still, if the singular values are distinct, an SVD exists, as smooth as X, and differential equations for U, Σ and V can be derived (e.g., see [12,28]). To be precise, given initial condition $X(0) = U(0)\Sigma(0)V^T(0)$, then –if the singular values of $X(t)$ are distinct for all t– we have:

$$
\begin{aligned}
\dot{\sigma}_i &= C_{ii}\sigma_i \ , \quad i = 1, \dots, n \ , \text{ where } C = U^T A U \ , && (23)\\
\dot{U} &= \quad UH \ , && (24)\\
\dot{V} &= \quad VK \ . && (25)
\end{aligned}
$$

Here, H and K are skew-symmetric functions whose entries for $i > j$ are

$$
h_{ij} = \frac{c_{ij}\sigma_j^2 - c_{ji}\sigma_i^2}{\sigma_j^2 - \sigma_i^2}, \qquad k_{ij} = \frac{(c_{ij} + c_{ji})\sigma_i\sigma_j}{\sigma_j^2 - \sigma_i^2} \ .
$$

Notice that if $X(0)$ has distinct singular values, then $U(0)$ and $V(0)$ are uniquely defined up to joint changes of sign for their columns.

In what follows, the precise value of $X(0)$ will be irrelevant, and our results will apply to any fundamental matrix solution. Indeed, our standing hypothesis will be the integral separation of the singular values of any fundamental matrix solution X of (4). I.e., if $\sigma_1(t) \geq \ldots \geq \sigma_n(t)$ are the ordered singular values of $X(t)$, $t \geq 0$, we will assume that there exist $a > 0$ and $0 < \kappa \leq 1$ such that

$$
\frac{\sigma_j(t)}{\sigma_j(s)} \frac{\sigma_{j+1}(s)}{\sigma_{j+1}(t)} \geq \kappa e^{a(t-s)}, \qquad \forall j = 1, \dots, n-1 \ , \text{ and } \forall t \geq s \geq 0 \, . \tag{26}
$$

This assumption might seem somehow restrictive. However, we shall see shortly that it is equivalent to require that the system admits stable and distinct LEs.

When (26) holds, parts (a) and (b) of the Proposition below are proved in [12]. Part (c) is an easy consequence of (b).

Proposition 4.1 *Let X be a fundamental matrix solution and assume that it admits a smooth SVD, $X = U\Sigma V^T$, and that the diagonal of $C = U^T A U$ is integrally separated. Then the following hold true.*

(a) *There exists a finite $\bar{t} \geq 0$, such that for all $t \geq \bar{t}$,*

$$
\sigma_j(t) > \sigma_{j+1}(t) \ , \qquad j = 1, \dots, n-1 \ . \tag{27}
$$

(b) *For $t \geq \bar{t}$ in (a), let $K = V^T \dot{V}$. Then*

$$
\lim_{t \to +\infty} K(t) = 0 \ , \tag{28}
$$

and the convergence of K to 0 is exponentially fast.

(c) *Moreover,*

$$
\lim_{t \to +\infty} V(t) = \overline{V}, \tag{29}
$$

where $\overline{V}$ is a constant orthogonal matrix.

In the next section, we shall examine the rate of convergence of V to $\overline{V}$.

Our next result is fundamental, and it justifies use of the SVD to obtain the LEs.

Theorem 4.2 *The system* (4) *has stable and distinct LEs if and only if for any fundamental matrix solution X the singular values of X are integrally separated. Moreover, if X is a fundamental matrix solution, the LEs of the system can be obtained as*

$$\lambda_j^s = \limsup_{t\to+\infty} \frac{1}{t} \log \sigma_j(t) \ , \qquad \lambda_j^i = \liminf_{t\to+\infty} \frac{1}{t} \log \sigma_j(t) \ , \tag{30}$$

where with σ_j we denote the j-th ordered singular value of X.

In order to prove the theorem, the following lemma is needed.

Lemma 4.3 *Let $A, B \in \mathbb{R}^{n\times n}$ be two $n\times n$ non singular matrices. Let $\sigma_1(A) \geq \ldots \geq \sigma_n(A) > 0$, $\sigma_1(B) \geq \ldots \geq \sigma_n(B) > 0$ and $\sigma_1(AB) \geq \ldots \geq \sigma_n(AB) > 0$ be the ordered singular values of A, B and AB respectively. Then for all $i = 1, \ldots, n$*

$$\sigma_i(A)\sigma_n(B) \leq \sigma_i(AB) \leq \sigma_i(A)\sigma_1(B).$$

Proof This is in [17]. In fact, in [17, Theorem 3.3.16] it is proven that $\sigma_i(AB) \leq \sigma_i(A)\sigma_1(B)$. Using this once more in the form $\sigma_i(A) = \sigma_i((AB)B^{-1}) \leq \sigma_i(AB)\sigma_1(B^{-1})$ gives the remaining inequality. □

Proof of Theorem 4.2. (⇒) Let the system have stable and distinct LEs. Then, it admits an integrally separated fundamental matrix solution X. Moreover, by Properties 2.5-5, there exists a Lyapunov transformation L such that $X = LZ$ where $Z = \text{diag}(z_1, \ldots, z_n)$ is the principal matrix solution of system (8). By the integral separation of the d_i's, there exist $a > 0$ and $d \geq 0$ such that

$$\frac{z_i(t)}{z_i(s)} \frac{z_{i+1}(s)}{z_{i+1}(t)} = e^{\int_s^t (d_i(s)-d_{i+1}(s))ds} \geq e^{a(t-s)} e^{-d} \ , \quad \forall t \geq s \geq 0 \ . \tag{31}$$

Denote with $\sigma_1^t(X) \geq \ldots \geq \sigma_n^t(X)$ the ordered singular values of $X(t)$ for all $t \in \mathbb{R}^+$. Then by Lemma 4.3, for all $i = 1, 2, \ldots, n$, and all $t \geq s \geq 0$, we have

$$\begin{aligned} \frac{\sigma_i^t(X)}{\sigma_i^s(X)} \frac{\sigma_{i+1}^s(X)}{\sigma_{i+1}^t(X)} &= \frac{\sigma_i^t(LL^{-1}X)}{\sigma_i^s(LL^{-1}X)} \frac{\sigma_{i+1}^s(LL^{-1}X)}{\sigma_{i+1}^t(LL^{-1}X)} \geq \frac{\sigma_n^t(L)\sigma_n^s(L)}{\sigma_1^t(L)\sigma_1^s(L)} \frac{z_i(t)}{z_i(s)} \frac{z_{i+1}(s)}{z_{i+1}(t)} \\ &\geq e^{a(t-s)} e^{-d} \frac{1}{\kappa(L(t))\kappa(L(s))}, \end{aligned}$$

where $i = 1, \ldots, n-1$ and $\kappa(L(t))$ is the condition number of the matrix $L(t)$ in the spectral norm. Similarly for $\kappa(L(s))$, and we remark that $\kappa(L(t)) \geq 1$, for all t. Now, being L a Lyapunov transformation, $1/\kappa(L)$ is bounded away from zero and the integral separation of the singular values of X follows by taking $\kappa = e^{-d}\frac{1}{\kappa(L(t))\kappa(L(s))}$ in (26). We still need to prove that for any fundamental matrix solution the singular values are integrally separated. Let X_0 be the initial condition for the given integrally separated fundamental matrix solution X, i.e. $X(0) = X_0$. Also, let $\widetilde{X}$ be another fundamental matrix solution corresponding to initial condition $\widetilde{X}_0$. Then, for all t, $\widetilde{X}(t) = X(t)X_0^{-1}\widetilde{X}_0$ and

$$\frac{\sigma_i^t(\widetilde{X})}{\sigma_i^s(\widetilde{X})}\frac{\sigma_{i+1}^s(\widetilde{X})}{\sigma_{i+1}^t(\widetilde{X})} \geq \frac{\sigma_i^t(X)}{\sigma_i^s(X)}\frac{\sigma_{i+1}^s(X)}{\sigma_{i+1}^t(X)}\frac{1}{(\kappa(X_0)\kappa(\widetilde{X}_0))^2} ,$$

and integral separation of the singular values of $\widetilde{X}$ follows by using $\widetilde{\kappa} = \kappa\frac{1}{(\kappa(X_0)\kappa(\widetilde{X}_0))^2}$ with κ as in the previous argument.

($\Leftarrow$) The argument here is similar to one used in [12, Theorem 5.1], although our assumptions are different than those in this cited result. Suppose that, for any fundamental matrix solution of (4), the singular values are integrally separated. Let X be any such fundamental matrix solution. Then, there exists a time $\bar{t} = \bar{t}(X)$ such that the singular values are distinct for all $t \geq \bar{t}$ (see Proposition 4.1-(a)). The equations for a smooth SVD of X, $X = U\Sigma V^T$ can then be given for $t \geq \bar{t}$. Let $P = U^TX = \Sigma V^T$, $t \geq \bar{t}$. Then P satisfies the following differential equation

$$\dot{P} = (\mathrm{diag}(C) - \Sigma K \Sigma^{-1})P. \tag{32}$$

Let $E = \Sigma K \Sigma^{-1}$. Then, for $i = 1, \ldots, n$, $e_{ii} = k_{ii} = 0$, while for $i < j$

$$e_{ji} = k_{ji}\frac{\sigma_j}{\sigma_i} = [c_{ij} + c_{ji}]\frac{\sigma_i\sigma_j}{\sigma_i^2 - \sigma_j^2}\frac{\sigma_j}{\sigma_i} = [c_{ij} + c_{ji}]\frac{1}{\frac{\sigma_i^2}{\sigma_j^2} - 1}.$$

By the assumption of integral separation, it follows $\sigma_i(t)/\sigma_j(t) \geq \sigma_i(\bar{t})/\sigma_j(\bar{t})e^{(j-i)a(t-\bar{t})}\kappa^{(j-i)} \to +\infty$ for $t \to +\infty$. Then if we denote with $\mathrm{low}(E)$ and $\mathrm{upp}(E)$ respectively the strictly lower and upper triangular part of E, $\mathrm{low}(E) \to 0$ for $t \to +\infty$. Consider the following system

$$\dot{\overline{P}} = (\mathrm{diag}(C) + \mathrm{upp}(E))\overline{P}. \tag{33}$$

Because of the assumption that the singular values of X are integrally separated, the diagonal of C is integrally separated and, by Properties 2.5-6 (33) has an integrally separated fundamental matrix solution, i.e. distinct and stable LEs. By Property 2.3, system (32) must have distinct and stable LEs as well, and the same is true for system (4) since U is a Lyapunov transformation.

Finally, being the LEs stable and distinct, we can consider the Lyapunov spectrum of (4), as in (12). By what we just proved, Σ_{L} of (33) and (4) are the same. In [12, Theorem 5.1], it is shown that, under the assumption of integral separation of the diagonal of C, Σ_{L} of (33) is obtained as

$$\lambda_j^s = \limsup_{t\to+\infty} \frac{1}{t}\int_0^t c_{jj}(\tau)d\tau \qquad \lambda_j^i = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t c_{jj}(\tau)d\tau.$$

Then (30) follows from $\sigma_j(t) = \sigma_j(\bar{t})e^{\int_{\bar{t}}^t c_{jj}(\tau)d\tau}$ and the irrelevance for the LEs of what happens up to the finite time $\bar{t}$. $\square$

Remark 4.4 In order to prove Theorem 4.2, we relied on the continuous SVD of the fundamental matrix solution. However, it must be stressed that this result validates SVD-based techniques for the approximation of LEs no matter which method is applied to find the SVD. As a consequence, it provides also a justification for techniques based on the discrete SVD of any fundamental matrix solution of the system.

Remark 4.5 Obviously, all fundamental matrix solutions with orthogonal initial conditions have same singular values. In particular, the constants for the integral separation a, κ, and the time $\bar{t} \geq 0$ after which the singular values are distinct (see (26) and (27)), must be the same for all fundamental matrix solutions with orthogonal initial conditions. Then, in the absence of better reasons to the contrary, and to be consistent with nonlinear problems, we may as well apply SVD techniques to the principal matrix solution Φ of system (4).

Theorem 4.2 suggests also how to evaluate Σ_{ED} under the hypothesis of stable and distinct LEs, as the following result highlights.

Theorem 4.6 *Assume* (4) *has stable and distinct LEs. Then it has the same exponential dichotomy spectrum as the diagonal system*

$$\dot{\Sigma} = \mathrm{diag}(C)\Sigma.$$

Proof The proof is analogous to the one given for Theorem 4.2. Indeed, let X be any fundamental matrix solution of (4). Then, after a certain time $\bar{t} \geq 0$, it admits a smooth SVD, $X = U\Sigma V^T$. Since U is a Lyapunov transformation, then (4) has same exponential dichotomy spectrum as (32), which is the same as the exponential dichotomy spectrum of (33) because of Theorem 3.2. In [12], under the hypothesis of integral separation of the diagonal of C, a fact which we now know follows from having stable and distinct LEs, it is proven that Σ_{ED} of (33) is the same as the exponential dichotomy spectrum of the diagonal system $\dot{\Sigma} = \mathrm{diag(C)}\Sigma$. $\square$

5 Leading directions

In Section 2.2 we have seen how it is possible to associate a linear space V_j to the LE λ_j^s and we defined the linear space W_j to be the set of all initial conditions leading to an exponential growth less than, or equal to, λ_j^s. Our goal in this section is to show how the spaces V_j and W_j (and thus also $\mathcal{N}_j$ associated to $\Sigma_{\rm ED}$) can be obtained from the SVD techniques. We will do this under the same assumption used in Section 4 to justify the very feasibility of the SVD technique: System (4) has stable and distinct Lyapunov exponents.

In this case, from Section 4 we know that there is a finite time $\bar{t}$ after which the singular values of any fundamental matrix solution X are distinct, and thus for $t \geq \bar{t}$ there is a smooth SVD of X: $X = U\Sigma V^T$. In what follows, without loss of generality, we will assume that $\bar{t} = 0$ and we will consider the principal matrix solution Φ.

Recall that we know, see (29), that $V \to \overline{V}$ as $t \to +\infty$. On the other hand, mere convergence would be of limited interest, since the entire approximation process takes place on the half-line. In particular, for numerical purposes it is important that convergence is rapid. For this reason, here below we will show that convergence of $V(t)$ to $\overline{V}$, as $t \to +\infty$, is exponentially fast, and we will give bounds on the exponential convergence rate. We will prove this exponential rate of convergence without the assumption of regularity, using stability of the LEs instead. In the regular case, the exponential convergence rate expressed in Corollaries 5.3 and 5.5 below is already in the work [20]; an excellent exposition, and further references, can be found in the book [2], in particular see Chapter 3 there. Although our technique is different than those used in the regular case, and it is ultimately based on the equations satisfied by the singular values, we also borrow some of the techniques used for regular problems, by adapting them to our setting. We stress once more that our motivation is dictated by using only those assumptions which are in tune with the assumptions needed to ensure the success of a numerical method to approximate the LEs.

For ease of notation, let us define $\alpha_i^j(t)$ to be the component of $\overline{v}_j$ in the direction of $v_i(t)$. That is, let

$$\alpha_i^j(t) = v_i(t)^T \overline{v}_j \ , \ i,j = 1,\dots,n \ , \quad t \geq 0 \ , \tag{34}$$

where $\overline{v}_j$ is the j-th column of $\overline{V}$ and $v_i(t)$ is the i-th column of $V(t)$, for all $t \geq 0$. Obviously, $\alpha_i^i(t) \to 1$ and $\alpha_i^j(t) \to 0$, $i \neq j$, as $t \to +\infty$. To show exponential convergence of $V(t)$ to $\overline{V}$, and to get bounds on the exponential rates, we need two preliminary Lemmas. The first, Lemma 5.1 is essentially in [20]; the result in [20] is for the regular case, but the proof for the non regular case (the case we need) is identical and therefore omitted. The second Lemma,

Lemma 5.2, is original.

Recall that a flag of type $e = (1, \dots, 1)$ in $\mathbb{R}^n$ is a filtration $\mathcal{W} = (W_i)_{i=1}^n$ such that $\mathbb{R}^n = W_1 \supset \dots \supset W_n$ and the sets V_i such that $W_i = W_{i+1} \oplus V_i$, have dimension 1. We denote the space of all these flags with $F_e(n)$.

Lemma 5.1 *([20]) Let $\Delta = \min_{i \neq j} |\lambda_i^s - \lambda_j^s|/(n-1)$. Define*

$$d(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}) = \max_{\substack{i \neq j \\ \|x\| = \|y\| = 1 \\ x \in V_i^{(1)}, y \in V_j^{(2)}}} |(x, y)|^{\Delta/|\lambda_i^s - \lambda_j^s|} \tag{35}$$

where $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}$ are two flags in $F_e(n)$ and $(x, y) = x^T y$ is the inner product in $\mathbb{R}^n$. Then (35) *is a metric in $F_e(n)$.*

We can express the metric defined in Lemma 5.1 in function of the orthogonal projections P_k into the linear subspaces V_k. Let v_k be such that $V_k = \text{span}(v_k)$, and $\|v_k\| = 1$, then $P_k = v_k v_k^T$. In the spectral norm, $\|P_i^{(1)} P_j^{(2)}\| = \max_{\|x\|=1} \|P_i^{(1)} P_j^{(2)} x\| = \max_{\|x\|=\|y\|=1} (P_i^{(1)} P_j^{(2)} x, y) = \max_{\|x\|=\|y\|=1} (P_i^{(1)} x, P_j^{(2)} y)$, so

$$\max_{\substack{\|x\| = \|y\| = 1 \\ x \in V_i^{(1)}, y \in V_j^{(2)}}} (x, y) = \|P_i^{(1)} P_j^{(2)}\|,$$

and the metric d of Lemma 5.1 can be expressed in the equivalent way:

$$d(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}) = \max_{i \neq j} \|P_i^{(1)} P_j^{(2)}\|^{\Delta/|\lambda_i^s - \lambda_j^s|}. \tag{36}$$

Lemma 5.2 *Assume system* (4) *has stable and distinct LEs, and as usual let Φ be its principal matrix solution, and $\Phi = U \Sigma V^T$ be its smooth SVD. Let τ be fixed, $0 < \tau \leq 1$. For all $i = 1, \dots, n$, and for all $t \geq 0$, let $v_i(t+\tau)$ be the $i-$th column of $V(t+\tau)$, and let*

$$\beta_i^j(t+\tau) = v_i(t+\tau)^T v_j(t), \quad \forall\, i, j = 1, \dots, n\ .$$

Then, for all $i, j = 1, \dots, n$, we have

$$\begin{aligned}
& \lim_{t \to \infty} \beta_i^i(t+\tau) = 1\ , \\
j > i: \quad & \chi^s(\beta_i^j) \leq \limsup_{t \to +\infty} \frac{1}{t} \log \frac{\sigma_j(t)}{\sigma_i(t)}, \quad \chi^i(\beta_i^j) \leq \liminf_{t \to +\infty} \frac{1}{t} \log \frac{\sigma_j(t)}{\sigma_i(t)}\ , \\
j < i: \quad & \chi^s(\beta_i^j) \leq \limsup_{t \to +\infty} \frac{1}{t} \log \frac{\sigma_i(t)}{\sigma_j(t)}, \quad \chi^i(\beta_i^j) \leq \liminf_{t \to +\infty} \frac{1}{t} \log \frac{\sigma_i(t)}{\sigma_j(t)}\ .
\end{aligned} \tag{37}$$

Furthermore, for all $i, j = 1, \dots, n$, $i \neq j$, the following bounds hold

$$\begin{aligned} \chi^s(\beta_i^j) &\le \min\left(\lambda_j^s - \lambda_i^i, -a(j-i)\right), \quad j > i, \\ \chi^s(\beta_i^j) &\le \min\left(\lambda_i^s - \lambda_j^i, -a(i-j)\right), \quad j < i, \\ \chi^i(\beta_i^j) &\le \min\left(-|\lambda_j^i - \lambda_i^i|, -|\lambda_j^s - \lambda_i^s|\right), \end{aligned} \tag{38}$$

where $a > 0$ is the a in the integral separation condition (26).

Proof For all $t \ge 0$, and $j = 1, \dots, n$, represent $v_j(t)$ in the basis $(v_1(t+\tau), \dots, v_n(t+\tau))$, as

$$v_j(t) = \sum_{i=1}^{n} \beta_i^j(t+\tau) v_i(t+\tau) \; . \tag{39}$$

From (39) and (29) it follows that $\lim_{t\to\infty} \beta_i^i(t+\tau) = \lim_{t\to\infty} v_i(t+\tau)^T v_i(t) = 1$.

In order to evaluate the upper characteristic exponent of the β_i^j's, rewrite $\Phi(t+\tau)$ as $\Phi(t+\tau) = \Phi(t+\tau, t)\Phi(t)$, where $\Phi(t+\tau, t)$ is the solution at $t+\tau$ of $\frac{\partial}{\partial t}\Phi = A(t)\Phi$, $\Phi(t,t) = I$. Then, $M\|\Phi(t+\tau)w\| \le \|\Phi(t)w\|$ and $L\|\Phi(t)w\| \le \|\Phi(t+\tau)w\|$, for all $w \in \mathbb{R}^n$, where $M^{-1} = \sup_{t\ge 0} \|\Phi(t+\tau, t)\|$ and $L^{-1} = \sup_{t \ge 0} \|\Phi(t+\tau, t)^{-1}\|$.

Case $j > i$.
Using (39) and $\Phi = U\Sigma V^T$ we get

$$\begin{aligned} \sigma_j(t) &= \|\Phi(t)v_j(t)\| \ge M\|\Phi(t+\tau)v_j(t)\| \\ &= M\|\sum_{k=1}^{n} \beta_k^j(t+\tau)\Sigma(t+\tau)V^T(t+\tau)v_k(t+\tau)\| \\ &= M\|\sum_{k=1}^{n} \beta_k^j(t+\tau)\sigma_k(t+\tau)e_k\| \ge M|\beta_i^j(t+\tau)|\sigma_i(t+\tau). \end{aligned} \tag{40}$$

Using (40), we get

$$\begin{aligned} \chi^s(\beta_i^j) &= \limsup_{t\to+\infty} \frac{1}{t}\log|\beta_i^j(t)| \\ &\le \limsup_{t\to+\infty} \left(\frac{1}{t}\log\sigma_j(t) - \frac{1}{t}\log\sigma_i(t+\tau)\right) \le \lambda_j^s - \lambda_i^i \, . \end{aligned}$$

Observe that we do not know if the right hand side of this bound gives a negative value, which we will need to be the case for $\chi^s(\beta_i^j)$. This is the reason for the diversified bound in (38), which we now prove.

First look at the behavior of $\frac{\sigma_i(t+\tau)}{\sigma_j(t)}$. From (26) and

$\frac{\sigma_i(t+\tau)}{\sigma_i(t)} = \exp\Big(\int_t^{t+\tau}(U^TAU)_{ii}(s)ds\Big)$, we have

$$\frac{\sigma_i(t+\tau)}{\sigma_j(t)} = \frac{\sigma_i(t+\tau)}{\sigma_i(t)}\frac{\sigma_i(t)}{\sigma_j(t)} \geq e^{-\|A\|\tau}e^{a(j-i)t}\kappa^{j-i},$$

for all $t \geq 0$. It follows

$$\frac{1}{t}\log\left(\frac{\sigma_i(t+\tau)}{\sigma_j(t)}\right) \geq \frac{1}{t}((j-i)\log\kappa - \|A\|\tau) + a(j-i), \quad \forall t \geq 0\ . \qquad (41)$$

Let now $\{t_n\}$ be a sequence such that $\lim_{t_n\to+\infty}\frac{1}{t_n}\log(|\beta_i^j(t_n+\tau)|) = \chi^s(\beta_i^j)$, then from (40) and (41) we obtain

$$-a(j-i) \geq \limsup_{t_n\to+\infty}\frac{1}{t_n}\log\frac{\sigma_j(t_n)}{\sigma_i(t_n+\tau)} \geq \chi^s(\beta_i^j)\,,$$

which completes (38) for $\chi^s(\beta_i^j)$ in the case of $j > i$.

As far as the bounds for $\chi^i(\beta_i^j)$, from (40) we get

$$\liminf_{t\to+\infty}\frac{1}{t}\log\sigma_j(t) \geq \liminf_{t\to+\infty}\frac{1}{t}\log|\beta_i^j(t+\tau)| + \liminf_{t\to+\infty}\frac{1}{t}\log\sigma_i(t+\tau)$$

and therefore

$$\chi^i(\beta_i^j) \leq \lambda_j^i - \lambda_i^i\ .$$

Alternatively, we can also get the bound

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{t}\log\sigma_j(t) &\geq \limsup_{t\to+\infty}\left(\frac{1}{t}\log\sigma_i(t+\tau) + \frac{1}{t}\log|\beta_i^j(t+\tau)|\right)\\ &\geq \limsup_{t\to+\infty}\frac{1}{t}\log\sigma_i(t+\tau) + \liminf_{t\to+\infty}\frac{1}{t}\log|\beta_i^j(t+\tau)|\end{aligned}$$

and therefore

$$\chi^i(\beta_i^j) \leq \lambda_j^s - \lambda_i^s\ ,$$

so that (37) and (38) are proven for $j > i$. Notice that we are guaranteed that the bounds given for $\chi^i(\beta_i^j)$ are negative.

Case $j < i$.
Again, using (39) and the smooth SVD of X we get

$$\begin{aligned}\sigma_i(t+\tau) =&\|\Phi(t+\tau)v_i(t+\tau)\| \geq L\|\Phi(t)v_i(t+\tau)\|\\ =&L\|\Sigma(t)V^T(t)v_i(t+\tau)\| \geq L|\beta_i^j(t+\tau)|\sigma_j(t),\end{aligned} \qquad (42)$$

and the bounds for $\chi^s(\beta_i^j)$ and $\chi^i(\beta_i^j)$ can be recovered by (42) with a procedure analogous to the one for the case $j > i$. □

Corollary 5.3 *With same hypotheses and notation of Lemma 5.2, if the LEs exist as limits (i.e., (4) is regular), then for all $i, j = 1, \dots, n$, $i \neq j$, we have*

$$\chi^s(\beta_i^j) \leq -|\lambda_i - \lambda_j|.$$

Proof The statement follows easily from the proof of Lemma 5.2 by noticing that $\lambda_j = \lambda_j^s = \lambda_j^i$ for all $j = 1, \dots, n$. $\square$ Lemma 5.2 is the stepping stone to understand the rate of exponential convergence of V to $\overline{V}$. The next theorem gives bounds for $\chi^s(e_j V^T(\cdot)\overline{V} e_i)$.

Theorem 5.4 *Let the same assumptions of Lemma 5.2 hold, let $\overline{v}_j$ be the j-th column of $\overline{V}$, $j = 1, \dots, n$, and $\alpha_i^j(t)$ given by (34), $i, j = 1, \dots, n$, and $t \geq 0$. Then, for all $i, j = 1, \dots, n$, $i \neq j$, we have*

$$\chi^s(\alpha_i^j) \leq A|\lambda_i^s - \lambda_j^s| \ , \tag{43}$$

where A is given by $A = \max_{k \neq l} \frac{\chi^s(\beta_k^l)}{|\lambda_k^s - \lambda_l^s|}$ and thus $A < 0$.

Moreover, for all $j = 1, \dots, n$,

$$\chi^s(1 - \alpha_j^j) \leq 2 \max_{i \neq j} \chi^s(\alpha_i^j). \tag{44}$$

Proof Rewrite $\overline{v}_j$ in the basis $(v_1(t), \dots, v_n(t))$: $\overline{v}_j = \sum_{i=1}^n \alpha_i^j(t) v_i(t)$. Call respectively $\mathcal{W}(t)$ and $\mathcal{W}$ the flags of the subspaces $W_i(t) = \text{span}\{v_i(t), \dots, v_n(t)\}$ and $W_i = \text{span}\{\overline{v}_i, \dots \overline{v}_n\}$. Then by Lemma 5.1 and the fact that $d(\mathcal{W}(t), \mathcal{W}) \to 0$ as $t \to +\infty$ (since $V(t) \to \overline{V}$), we have

$$d(\mathcal{W}(t), \mathcal{W}) \leq \sum_{m=1}^{\infty} d(\mathcal{W}(t + (m-1)\tau), \mathcal{W}(t + m\tau)) \ , \tag{45}$$

where $0 < \tau \leq 1$ is fixed. Consider the following orthogonal projections, $P_j(t) = v_j(t)v_j(t)^T$ and $P_i(t+\tau) = v_i(t+\tau)v_i(t+\tau)^T$, and notice that $\|P_j(t)P_i(t+\tau)\| = |v_i(t+\tau)^T v_j(t)| = |\beta_i^j(t+\tau)|$. Then

$$\begin{aligned} d(\mathcal{W}(t + (m-1)\tau), \mathcal{W}(t + m\tau)) &= \max_{i \neq j} \|P_j(t + (m-1)\tau)P_i(t + m\tau)\|^{\Delta/|\lambda_i^s - \lambda_j^s|} \\ &= \max_{i \neq j} |\beta_i^j(t + m\tau)|^{\Delta/|\lambda_i^s - \lambda_j^s|}, \end{aligned} \tag{46}$$

and in the same way

$$d(\mathcal{W}(t), \mathcal{W}) = \max_{i \neq j} \|P_j(t)P_i\|^{\Delta/|\lambda_i^s - \lambda_j^s|} = \max_{i \neq j} |\alpha_i^j(t)|^{\Delta/|\lambda_i^s - \lambda_j^s|}. \tag{47}$$

Using (46) and (47) we can rewrite (45) as

$$\max_{i \neq j} |\alpha_i^j(t)|^{\Delta/|\lambda_i^s - \lambda_j^s|} \leq \sum_{m=1}^{\infty} \max_{i \neq j} |\beta_i^j(t + m\tau)|^{\Delta/|\lambda_i^s - \lambda_j^s|}. \tag{48}$$

Now, $\chi^s(\beta_i^j) < 0$ for all $i \neq j$ and Property 2.2-(c) applies to (48), so that for any $j \neq i$,

$$\frac{\Delta}{|\lambda_i^s - \lambda_j^s|}\chi^s(\alpha_i^j) \leq \max_{k \neq l} \frac{\Delta}{|\lambda_k^s - \lambda_l^s|}\chi^s(\beta_k^l),$$

and (43) follows.
When $i = j$, we know from (29) that $\lim_{t\to+\infty} \alpha_j^j(t) = \lim_{t\to+\infty} v_j(t)^T \overline{v}_j = 1$. Moreover, $1 = \|\overline{v}_j\|^2 = \sum_{i=1}^n (\alpha_i^j(t))^2$ so that $\chi^s(1 - (\alpha_j^j)^2) \leq 2\max_{i\neq j}\chi^s(\alpha_i^j)$. Then (44) follows by $\chi^s(1-(\alpha_j^j)^2) = \chi^s\Big((1-\alpha_j^j)(1+\alpha_j^j)\Big) = \limsup_{t\to+\infty}\frac{1}{t}\log(1-\alpha_j^j) + \lim_{t\to+\infty}\frac{1}{t}\log(1+\alpha_j^j(t)) = \chi^s(1-\alpha_j^j)$. □

Corollary 5.5 *With same assumptions and notations of Theorem 5.4, if system* (4) *is regular then for* $i \neq j$ *we have*

$$\chi^s(\alpha_i^j) \leq -|\lambda_i - \lambda_j|.$$

Remark 5.6 Comparing the rate expressed in the above Corollary with the rate (43) in the non regular case, we notice that in the non regular case there may be a slower rate of convergence, in the sense that the value of A in (43) may be close to 0.

An important component of Lemma 5.2 and Theorem 5.4 is the key role played by the integral separation constant $a > 0$ in the rate of exponential convergence of V to $\overline{V}$; see (43) and the bounds (38) for $\chi^s(\beta_i^j)$. It is therefore of interest being able to estimate this constant a. The following Proposition gives a way to estimate a. As it turns out, the technique of this Proposition 5.7 emerges as a natural byproduct of a way of computing the SVD of X in the first place (see [8]).

Proposition 5.7 *Let* (4) *have stable and distinct LEs, let* Φ *be its principal matrix solution,* $\Phi = U\Sigma V^T$ *its smooth SVD, and let* $C = U^TAU$*. For* $j = 1, \ldots, (n-1)$*, denote with* $[a_j, b_j]$ *the exponential dichotomy interval of the scalar differential equation* $\dot{x} = (c_{jj}(t) - c_{j+1,j+1}(t))x$*. Then* $a_j > 0$ *and the constant* $a > 0$ *in formula* (26) *can be taken to be*

$$a = \min_{j=1,\ldots,n-1} a_j.$$

Proof By Theorem 4.2, the singular values of the principal matrix solution Φ are integrally separated and (26) is satisfied. Using (23), rewrite (26) as

$$\int_s^t (c_{jj}(\tau) - c_{j+1,j+1}(\tau) - a)d\tau \geq \log\kappa, \qquad \text{for all} \quad j = 1, \ldots, n-1, \quad \text{and for all} \quad t \geq s \geq 0\ . \tag{49}$$

Now, fix j. Because of integral separation, the scalar problem $\dot{x} = \Big(c_{jj}(t) - c_{j+1,j+1}(t)\Big)x$, must have dichotomy interval strictly contained in the positive real axis. Moreover (see [12, Lemma 8.2]), the resolvent is the same as the set of all μ's such that one of the following two conditions is satisfied for all $t \geq s \geq 0$

$$\int_s^t \Big[\Big(c_{jj}(\tau) - c_{j+1,j+1}(\tau)\Big) - \mu\Big] d\tau \geq \alpha(t-s) - b,$$
$$\int_s^t \Big[\mu - \Big(c_{jj}(\tau) - c_{j+1,j+1}(\tau)\Big)\Big] d\tau \geq \alpha(t-s) - b,$$

where $\alpha > 0$ and $b \geq 0$. In particular, for all μ's satisfying the first inequality, we must have that $a_j > \mu$. From this and (49) it follows that $a \leq a_j$, for all $j = 1, \dots, n-1$. □

We are now ready to prove the main set of results of this Section. Recall the definition of the subspaces W_j's in (13), and that they form a filtration (14).

Theorem 5.8 *Assume system (4) has stable and distinct LEs, and let $\Phi = U\Sigma V^T$ be the smooth SVD of its principal matrix solution. Let $\overline{V} = \{\overline{v}_1, \dots, \overline{v}_n\}$ be the limit of the factor V. Then we have*

$$\chi^s(\Phi(\cdot)\overline{v}_j) = \lambda_j^s \ , \quad \textit{and} \quad \chi^i(\Phi(\cdot)\overline{v}_j) = \lambda_j^i \, , \quad j = 1, \dots, n \, .$$

That is, $\overline{V}$ is a normal basis for the upper and lower Lyapunov exponents.

Proof First of all, for all $j = 1, \dots, n$, we have

$$\|\Phi(t)\overline{v}_j\| = \| \begin{bmatrix} \sigma_1(t)\alpha_1^j(t) \\ \vdots \\ \sigma_n(t)\alpha_n^j(t) \end{bmatrix} \| \geq \sigma_j(t)|\alpha_j^j(t)| \ ,$$

so that

$$\begin{aligned} &\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)\overline{v}_j\| \geq \limsup_{t\to+\infty} \frac{1}{t} \log\Big(\sigma_j(t)|\alpha_j^j(t)|\Big) = \lambda_j^s \ , \quad \text{and} \\ &\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(t)\overline{v}_j\| \geq \liminf_{t\to+\infty} \frac{1}{t} \log\Big(\sigma_j(t)|\alpha_j^j(t)|\Big) = \lambda_j^i \, . \end{aligned} \tag{50}$$

From (50), the result follows for $j = 1$. Define $\overline{L}_1 = \text{span}\{\overline{v}_1\}$ and observe that, because of (50), we must have $\overline{L}_1 \subset W_1 \backslash W_2$. Next, let us continue the proof just for the lim sup's.

Define $\overline{L}_2 = \text{span}\{\overline{v}_1, \overline{v}_2\}$ and take $w \in \overline{L}_2$, $w \neq 0$: $w = c_1\overline{v}_1 + c_2\overline{v}_2$. We have

$$\|\Phi(t)w\| = \| \begin{bmatrix} \sigma_1(t)(c_1\alpha_1^1(t) + c_2\alpha_1^2(t)) \\ \vdots \\ \sigma_n(t)(c_1\alpha_n^1(t) + c_2\alpha_n^2(t)) \end{bmatrix} \| \geq \sigma_1(t)|c_1\alpha_1^1(t) + c_2\alpha_1^2(t)| \ .$$

Now, if $c_1 \neq 0$, this gives

$$\limsup_{t\to+\infty} \frac{1}{t}\log\|\Phi(t)w\| \geq \limsup_{t\to\infty}\left(\frac{1}{t}\log\sigma_1(t) + \frac{1}{t}\log|c_1\alpha_1^1(t)+c_2\alpha_1^2(t)|\right) = \lambda_1^s\,,$$

where the last equality follows from the fact that $(c_1\alpha_1^1(t)+c_2\alpha_1^2(t))$ approaches c_1. In particular, if $c_1 \neq 0$, $w \notin W_2$. If, instead, $c_1 = 0$, then we still have from (50) $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)w\| \geq \lambda_2^s$. But we cannot have $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)\overline{v}_2\| = \lambda_1^s$ because

$$\dim(W_2) = n-1\,, \quad \text{and} \quad \dim(\overline{L}_2) = 2\,,$$

and thus $\overline{L}_2 \cap W_2 \neq \emptyset$, and the vector in this intersection must be $\overline{v}_2$. Thus, $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)\overline{v}_2\| = \lambda_2^s$ and $\overline{v}_2 \in W_2\backslash W_3$.

We consider one more step in this process, and the general case will follow inductively. So, define $\overline{L}_3 = \text{span}\{\overline{v}_1,\overline{v}_2,\overline{v}_3\}$ and take $w \in \overline{L}_3$, $w \neq 0$: $w = c_1\overline{v}_1 + c_2\overline{v}_2 + c_3\overline{v}_3$. We have

$$\begin{aligned}\|\Phi(t)w\| &= \left\|\begin{bmatrix}\sigma_1(t)(c_1\alpha_1^1(t)+c_2\alpha_1^2(t)+c_3\alpha_1^3(t))\\ \vdots \\ \sigma_n(t)(c_1\alpha_n^1(t)+c_2\alpha_n^2(t)+c_3\alpha_n^3(t))\end{bmatrix}\right\| \\ &\geq \sigma_1(t)|c_1\alpha_1^1(t)+c_2\alpha_1^2(t)+c_3\alpha_1^3(t)|\,.\end{aligned}$$

Now, if $c_1 \neq 0$, this gives (like before) $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)w\| \geq \lambda_1^s$, and so if $c_1 \neq 0$, $w \notin W_2$. If $c_1 = 0$, but $c_2 \neq 0$, then $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)w\| \geq \lambda_2^s$, and so if $c_1 = 0$, but $c_2 \neq 0$, $w \notin W_3$. Finally, if $c_1 = c_2 = 0$, then from (50) $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)w\| \geq \lambda_3^s$. But we cannot have $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)\overline{v}_3\| > \lambda_3^s$ because

$$\dim(W_3) = n-2\,, \quad \text{and} \quad \dim(\overline{L}_3) = 3\,,$$

and thus $\overline{L}_3 \cap W_3 \neq \emptyset$, and all vectors w with nonzero component in $\overline{v}_1$ or $\overline{v}_2$ cannot be in this intersection, and so the only vector in the intersection must be $\overline{v}_3$. Thus, $\limsup_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)\overline{v}_3\| = \lambda_3^s$ and $\overline{v}_3 \in W_3\backslash W_4$.

Continuing in this way, we obtain the result for the lim sup's. The result for the lim inf's is identical, upon recalling Proposition 2.10. □

Remark 5.9 Upon noticing that $X(t) = U(t)\Sigma(t)V^T(t)$ implies $X^{-T}(t) = U(t)\Sigma^{-1}(t)V^T(t)$, for all $t \geq 0$, and in the same situation of Theorem 5.8, the matrix $\overline{V}[e_n,\dots,e_1]$ is a normal basis for the adjoint problem.

Corollary 5.10 *With same assumptions of Theorem 5.8, assume system* (4) *is also regular. Then, for each* $j = 1,\dots,n$,

$$\lim_{t\to+\infty}\frac{1}{t}\log\|\Phi(t)\overline{v}_j\| = \lambda_j.$$

Remark 5.11 Suppose that instead of the principal matrix solution, we had considered another fundamental matrix solution X with initial conditions $X(0) = X_0$, and let $X = U\Sigma V^T$ be the SVD of this X. Then, again, as $t \to +\infty$, $V(t) \to \widetilde{V}$, and the columns of $\widetilde{V}$ now provide the appropriate directional information relative to this new fundamental solution.

Theorem 5.12 *With same assumptions of Theorem 5.8, let* $\Sigma_{\text{ED}} = \bigcup_{i=1}^{m}[a_j, b_j]$. *Then, for* $j = 1, \ldots, m$, *we have*

$$\mathcal{N}_j = \text{span}\{\overline{v}_k, \ldots, \overline{v}_l\} \ ,$$

where k *and* l, $k < l$, *are such that*

$$\lambda_{l+1}^s < a_j < \lambda_l^i, \qquad \lambda_k^s < b_j < \lambda_{k-1}^i.$$

Proof The statement follows from Theorem 5.8, Corollary 3.12, and Remark 5.9. So, $W_k = \text{span}(\overline{v}_k, \ldots, \overline{v}_n)$ and $L_{n-l+1} = \text{span}(\overline{v}_1, \ldots, \overline{v}_l)$. □

Corollary 5.13 *Under the same assumptions of Theorem 5.12, and assuming that* Σ_{ED} *is made up by at least two intervals, we have that the subspaces* $\mathcal{N}_j$ *and* $\mathcal{N}_k$ *are perpendicular to each other, for any* j, k, $j \neq k$.

Proof The proof follows from the representation of the subspaces $\mathcal{N}_j$ given in Theorem 5.12, since the vectors $\{\overline{v}_1, \ldots, \overline{v}_n\}$ are all mutually orthogonal. □

A final interesting consequence of having (4) integrally separated is that not only the system, but also the subspaces $\mathcal{N}_j$'s are pairwise integrally separated.

Theorem 5.14 *Under the same assumptions of Theorem 5.12, assume further that* Σ_{ED} *does not reduce to a unique interval, and let* $y_0 \in \mathcal{N}_j$ *and* $z_0 \in \mathcal{N}_p$, $j > p$. *Then, the functions* $y(t) = \Phi(t)y_0$ *and* $z(t) = \Phi(t)z_0$, *for all* $t \geq 0$, *are integrally separated functions.*

Proof Consider the quotient $\frac{\|y(t)\|}{\|y(s)\|}\frac{\|z(s)\|}{\|z(t)\|}$ for all $t \geq s \geq 0$. Because of Remark 3.10, we thus have for all $t \geq s \geq 0$:

$$\frac{\|y(t)\|}{\|y(s)\|}\,\frac{\|z(s)\|}{\|z(t)\|} \geq \frac{1}{K_{j-1}}e^{a_j(t-s)}\,\frac{1}{K_p}e^{-b_p(t-s)} = \frac{1}{K_{j-1}K_p}e^{(a_j-b_p)(t-s)}$$

and integral separation follows, since $a_j > b_p$ and $K_{j-1}, K_p \geq 1$. □

To complete this section, we stress that Theorem 5.8 tells us that, as long as (4) has stable and distinct LEs, the initial conditions given by $\overline{V}$ form a normal basis. But, in general, we cannot say that $\overline{V}$ leads to an integrally separated fundamental matrix solution. The next result gives us a natural condition of when this is true.

Corollary 5.15 *Suppose that* (4) *is integrally separated and that* Σ_{ED} *is given by* n *disjoint subintervals:* $\Sigma_{\text{ED}} = \cup_{i=1}^{n}[a_i, b_i]$, *with* $a_1 \leq b_1 < \cdots < a_n \leq b_n$. *Then, the initial condition given by* $\overline{V}$ *leads to an integrally separated fundamental matrix solution.*

Proof This is an obvious consequence of Theorems 5.12 and 5.14. □

6 Conclusions

In this paper we have considered the use of the SVD of a fundamental matrix solution, in order to extract the Lyapunov and Dichotomy spectra of a given system. Albeit SVD based techinques have been used before for approximating Lyapunov exponents of (1) via the setup provided by the MET, a thorough justification for their use to approximate spectra had not been previously undertaken. Although the MET of Oseledec may be indirectly used in support of SVD methods, we have favored use of the assumption of stable and distinct Lyapunov exponents, hence justified our analysis for integrally separated linear systems. This is much more in tune with the practical success of a numerical technique, and it has allowed us to take the theory for SVD methods a step closer to that of QR methods, for which a thorough justification for integrally separated fundamental matrix solutions had been already done, see [12]. Still, comparison of the relative merits of SVD and QR methods was beyond our intention in this paper, and in fact our point of view is that these techniques should complement and not replace each other. Thus, it is important to appreciate what the SVD methods can offer that the QR methods (at least, with our present understanding) do not. For example, we have shown that SVD methods allow to obtain the set of directions associated to the spectral intervals, via the matrix $\overline{V}$. Possibly, also QR methods can be used to obtain this directional information, and one may want to adapt our results to QR methods, working with the matrix $\overline{Z}$ of [12, Lemma 7.4] rather than $\overline{V}$. This is yet to be done, and (even if theoretically feasible) a different implementation of QR methods would be needed since such matrix $\overline{Z}$ is not obtained with the usual implementation. On the other side, QR methods are theoretically justified ([13]) as techniques to approximate spectral intervals as long as the intervals are stable, a condition less stringent than that of having stable and distinct Lyapunov exponents. Some further comments on the relative merits of SVD versus QR techniques are in [8], to which we refer for algorithmic aspects of SVD methods as well.

Our analysis in this paper has been for linear problems. This is unavoidable, since the spectra we considered are defined for linear problems. So, what information do these spectra give when one considers the nonlinear problem (1)? What do we make of the MET, and of the measurable spectrum? Technically speaking, our results apply to the single linear system obtained by linearization along the specific trajectory of initial condition x_0. If this trajectory is on an attractor with a uniquely defined invariant and ergodic measure, then the spectral information we are obtaining is representative of the whole system. Regretfully, there are not very many theoretical results on attractors with a unique ergodic measure. Alternatively, one may need to take the point of view of Johnson-Palmer-Sell, see [18], consider all invariant measures on the attractor, and compute spectra with respect to each of these. In general, this also seems to be a daunting task. The above notwithstanding, we ought to appreciate the importance of our emphasis on the stability of the Lyapunov exponents, rather than, say, of regularity for the linear system. In any given situation, we will at best be able to accurately approximate the solution of the nonlinear system, and thus we will at best obtain a linear system close to the one we wanted to consider. This is why we have insisted on conditions guaranteeing that these systems have close spectra.

There are several directions which need to be pursued to complete and extend the present work. In particular, analysis of SVD methods assuming stable (and not stable and distinct) Lyapunov exponents remains to be done. How to estimate both constants in the integral separation relation (26), and not just a, is also a problem of interest, since it ultimately helps to give upper bounds on the value of $\bar{t}$ after which the singular values are guaranteed to be distinct. Extension of the analysis to the case of only a few (dominant) spectral intervals also needs to be carried out, as well as specialized analysis for parameter dependent systems, to see how to setup continuation of SVD factors in that context. Finally, there are a host of practical issues to be dealt with, some of them tackled/reviewed in [8]. We are presently thinking about some of these problems.

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