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ArticleTitle	Periodic Orbits for Planar Piecewise Smooth Systems with a Line of Discontinuity	
Article Sub-Title		
Article CopyRight	Springer Science+Business Media New York (This will be the copyright line in the final PDF)	
Journal Name	Journal of Dynamics and Differential Equations	

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	Received	6 November 2013
Schedule	Revised	29 May 2014
	Accepted	

Abstract	In this work we examine the existence of periodic orbits for planar piecewise smooth dynamical systems with a line of discontinuity. Unlike existing works, we consider the case where the line does not contain the equilibrium point. Most of the analysis is for a family of piecewise linear systems, and we discover new phenomena which produce the birth of periodic orbits, as well as new bifurcation phenomena of the periodic orbits themselves. A model nonlinear piecewise smooth systems is examined as well.

Keywords (separated by '-')	Piecewise smooth systems - Periodic orbits - Bifurcation - Filippov - Hopf
Mathematics Subject Classification (1991) (separated by '-')	34C29 - 37G15
Footnote Information	

J Dyn Diff Equat
DOI 10.1007/s10884-014-9380-3

Periodic Orbits for Planar Piecewise Smooth Systems with a Line of Discontinuity

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Received: 6 November 2013 / Revised: 29 May 2014

Abstract In this work we examine the existence of periodic orbits for planar piecewise smooth dynamical systems with a line of discontinuity. Unlike existing works, we consider the case where the line does not contain the equilibrium point. Most of the analysis is for a family of piecewise linear systems, and we discover new phenomena which produce the birth of periodic orbits, as well as new bifurcation phenomena of the periodic orbits themselves. A model nonlinear piecewise smooth systems is examined as well.

Keywords Piecewise smooth systems · Periodic orbits · Bifurcation · Filippov · Hopf

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1 Introduction

An important mechanism for the appearance of periodic orbits in smooth dynamical systems is through a Hopf bifurcation, a well understood situation fundamentally characterized by having a pair of eigenvalues of the linearization at an equilibrium crossing the imaginary axis. Stability characterization and (local) bifurcation phenomena for the periodic orbits themselves are also well understood, and rely heavily on the use of Floquet multipliers and the Poincaré map. Likewise, also global bifurcation phenomena of the periodic orbits, e.g. homoclinic bifurcations, have been explored for a long time. All of these phenomena above are well understood, although they continue to be studied also for planar systems because of their importance in applications.

In recent times, piecewise smooth systems have attracted considerable attention, both because of their ability to model dynamical behaviors arising in applications, and because of

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their intrinsic mathematical interest; e.g., see [3]. Our specific interest is for planar piecewise smooth systems of the following type:

$$\dot{\mathbf{x}} = \begin{cases} f_1(\mathbf{x}), \; h(\mathbf{x}) < 0, \\ f_2(\mathbf{x}), \; h(\mathbf{x}) > 0, \end{cases} \tag{1}$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, f_1, f_2 are sufficiently smooth, $f_{1,2}(0) = 0$, and also $h : \mathbb{R}^2 \to \mathbb{R}$ is a sufficiently smooth function. Further, we let $\Sigma = \{(x, y) \in \mathbb{R}^2, \;\; h(x, y) = 0\}$, and we assume that $\nabla h(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \Sigma$, so that Σ is a simple smooth curve, separating (locally) the plane into two regions R_1 and R_2 where $h(\mathbf{x}) < 0$, respectively $h(\mathbf{x}) > 0$.

For a system like (1), Filippov convexification method is a powerful first order technique which resolves the ambiguity of what to do when $\mathbf{x} \in \Sigma$. In its simplest form, it relies on the following classification.

(i) *Crossing*. At $\mathbf{x}_0 \in \Sigma$: $(\nabla h^T f_1)(\nabla h^T f_2) > 0$. The crossing is from below if $\nabla h^T f_{1,2} > 0$, in which case the trajectory continues in R_2 with vector field f_2, and the crossing is from above if $\nabla h^T f_{1,2} < 0$, in which case the trajectory continues in R_1 with vector field f_1.

(ii) *Sliding*. At $\mathbf{x}_0 \in \Sigma$: $(\nabla h^T f_1)(\nabla h^T f_2) < 0$. One has *attractive sliding* if $\nabla h^T f_1 > 0$, and the trajectory moves on Σ with Filippov vector field given by the convexification of f_1, f_2:

$$\dot{\mathbf{x}} = (1-\mu) f_1 + \mu f_2 \quad \mu = \frac{\nabla h^T f_2}{\nabla h^T (f_1 - f_2)}. \tag{2}$$

Instead, one has *repulsive* sliding when $\nabla h^T f_1 < 0$, in which case the problem is not well posed (a trajectory could slide according to (2), or continue in R_1 or R_2).

(iii) *Exit*. At $\mathbf{x}_0 \in \Sigma$, for one –and only one– index $i = 1, 2$, $\nabla h^T f_i = 0$, $f_i(\mathbf{x}_0) \neq 0$, and the function $\nabla h^T f_i$ changes sign for $\mathbf{x} \in \Sigma$ through $\mathbf{x}_0$.

Naturally, more things can happen at a point $\mathbf{x} \in \Sigma$, and higher order corrections may be needed (e.g., see [4,6] for a host of bifurcations in planar Filippov systems), but the above scenario is sufficient for our purposes.

Relative to systems like (1), there has been considerable interest in studying and approximating periodic orbits and their stability properties and bifurcations. In the present context, however, it is not even fully transparent what an appropriate extension of the Hopf bifurcation theorem should look like. As far as we know, a fully rigorous Hopf bifurcation theorem was recently proved in the important work [12], but see also the related works [5,7,8].

A most important mathematical (and modeling) feature of the Hopf bifurcation theorem of [12] is that the curve Σ is given by one of the coordinate axes [in particular, it contains the origin, isolated equilibrium for the system (1)]. That is, in [12], the authors consider the following problem depending on a real parameter λ:

$$\dot{\mathbf{x}} = \begin{cases} A_1(\lambda)\mathbf{x} + g_1(\mathbf{x}, \lambda), \; y < 0, \\ A_2(\lambda)\mathbf{x} + g_2(\mathbf{x}, \lambda), \; y > 0, \end{cases} \tag{3}$$

with $g_{1,2}(\mathbf{x}, \lambda) = \mathcal{O}(x^2 + y^2)$ as $(x, y) \to 0$, and where the matrices $A_{1,2}(\lambda)$ have a pair of complex conjugate eigenvalues $\alpha_{1,2}(\lambda) \pm i\omega_{1,2}(\lambda)$. Ordinarily (that is, for smooth systems where $A_1 = A_2$, $g_1 = g_2$, etc.), Hopf bifurcation requires $\alpha(0) = 0$, $\frac{d}{d\lambda}\alpha|_{\lambda=0} \neq 0$, and

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$\omega(0) \neq 0$. Relative to (3), in [12] Zou, Kuepper and Beyn prove that a Hopf bifurcation now takes place if

$$B(0) = 1 \quad B'(0) \neq 0, \quad \text{where} \quad B(\lambda) = \exp\big[\pi\left(\alpha_1(\lambda)/\omega_1(\lambda) + \alpha_2(\lambda)/\omega_2(\lambda)\right)\big].$$

With respect to the same model (3), hence with the same assumption on the form of the discontinuity line, interesting bifurcation phenomena for the periodic orbits have been recently examined in [2], and also in [10,11], where the authors further considered a special planar system with an additional equilibrium (not on the line of discontinuity) and existence of homoclinic orbits to this equilibrium.

Our goal in this work is to study periodic orbits for (1) when the function $h(\mathbf{x})$ is a **general** line. Namely, we will consider the function $h(\mathbf{x})$ in the form:

$$h(x, y) = y - qx - m. \tag{4}$$

As we will see, this seemingly innocent generalization produces some totally new phenomena with respect to those observed in [12]. For example, depending on the coefficients of the linear terms, one may have or not have periodic orbits, and they may be with crossing or partially sliding behavior.

To perform our analysis, we will use a combination of explicit solution formulas, and the Poincaré map, similarly to what is done in [2,10,12]. Unlike these other works, we will also use the characterization of stability for the periodic orbits relying on the multipliers associated to the fundamental matrix solution. Our results extend the existing theoretical results, and cover important situations arising in practical applications, where one has planar systems with a discontinuity line not containing equilibria (e.g., see [8,9]).

Remark 1 As it is well understood, in dynamical systems studies it is important to consider models whose formulation is robust. For example, perturbing the horizontal line $y = 0$, and even restricting just to other horizontal lines, makes it necessary to consider the family of lines $y = m$ (for $m \lesseqgtr 0$), as we will do in this work.

A plan of the paper is as follows. In Sect. 2, we will consider the appropriate setting for the general piecewise linear problem and will set forth two models: a special (canonical) form, and a more general case. Section 3 is devoted to the complete analysis of the "canonical form", while Sect. 4 is dedicated to specific instances of the general form. Finally, in Sect. 5, we give an extension to the nonlinear case.

2 Setting of the Problem

Our main goal is to study periodic orbits for piecewise smooth planar systems with a line of discontinuity (not necessarily passing through the origin). As it turns out, the piecewise linear case is already sufficient to understand what we may observe in general and thus we will first restrict to this case.

So, presently, the basic problem we consider is the following family of piecewise smooth linear systems [see (1)]:

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x}, & y - qx - m < 0, \\ A_2\mathbf{x}, & y - qx - m > 0, \end{cases} \tag{5}$$

with $A_1, A_2 \in \mathbb{R}^{2\times 2}$, and the discontinuity line is $\Sigma = \{(x, y) \in \mathbb{R}^2,\ \ y - qx - m = 0\}$.

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The (only) interesting case is when each of the coefficient matrices has complex conjugate eigenvalues, and the two linear systems (viewed separately) have a stable (respectively, unstable) spiraling behavior toward (respectively, away from) the origin, consistently clockwise or counterclockwise. In this situation, stable and unstable periodic orbits appear also when the eigenvalues are not purely imaginary and different bifurcation phenomena, both local and global, can be observed. In light of these considerations, we thus make the following assumptions on piecewise linear systems like (5).

Assumption 1 The eigenvalues of A_1 are of the type $\{c \pm id\}$ and those of A_2 are of the type $\{-a \pm ib\}$, with $a, b, c, d > 0$.

Assumption 2 $A_i = \begin{pmatrix} a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i \end{pmatrix}$, $a_{12}^i > 0$, $i = 1, 2$. That is, each system (separately) has orbits spiraling clockwise. The case $a_{12}^i < 0$ with orbits that spiral anticlockwise is analogous.

2.1 Canonical Form

A great simplification takes place if we assume that the matrices $A_{1,2}$ have the following form:

$$A_1 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a & b \\ -b & -a \end{pmatrix}. \tag{6}$$

In this situation, without loss of generality, one can assume that $h(x, y) = y - m$. Indeed, given $h(x, y) = y - qx - p$, let $\theta = \arctan(q)$ and consider the rotation matrix $Q = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Then if $(x, y) \in \Sigma$, $Q \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \hat{x} \\ p\cos(\theta) \end{pmatrix} = \begin{pmatrix} \hat{x} \\ m \end{pmatrix}$. Since Q commutes with A_1 and A_2 in (6), the coefficient matrices are left unchanged by the coordinate change $\begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow Q \begin{pmatrix} x \\ y \end{pmatrix}$.

Remark 2 In [12], the authors further considered the case of $m = 0$. However, this is a restriction which we want to avoid since if we translate the line $y = m$ to $y = 0$, the same translation will contribute a nonhomogeneity to the piecewise linear system. [And see Remark 1 as well].

Henceforth, we thus refer to the following as the **canonical form** of the piecewise linear systems:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & y < m, \\ \begin{pmatrix} -a & b \\ -b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & y > m, \end{cases} \quad \text{with} \quad a, b, c, d > 0\,. \tag{7}$$

Remark 3 As it will become clear in Sect. 3, it is the ratios $\frac{a}{b}$ and $\frac{c}{d}$ which are the relevant quantities, rather than the value of a, b, c and d per se. We will study the changes in dynamics with respect to these ratios, and the value of m.

2.2 General Form

Unfortunately, in general, one cannot transform both matrices in (5) to canonical form with the same coordinate transformation, hence the form (7) is restrictive. However, one can

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always assume that one of the two matrices $A_{1,2}$ in (5) is in canonical form and the other in real Schur form. This statement can be justified as follows.

Let V be such that $A_1 = V \begin{pmatrix} c & d \\ -d & c \end{pmatrix} V^{-1}$ and consider the change of variables $\mathbf{x} \longleftarrow V\mathbf{x}$. Then, without loss of generality, we assume A_1 in (5) to be in the canonical form $A_1 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$. Next, let Q be a rotation matrix that takes A_2 into real Schur form: $A_2 = QTQ^T$, $T = \begin{pmatrix} -a & \frac{b}{\alpha} \\ -\alpha b & -a \end{pmatrix}$, and observe that **Assumption 2** implies that $\alpha > 0$. Now, since matrices of the form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ commute with one another, the further change of variable: $\mathbf{x} \longleftarrow Q\mathbf{x}$, will leave A_1 in canonical form and take A_2 into real Schur form. It follows that, without loss of generality, we can work with A_1 in canonical form and A_2 in real Schur form. Finally, consider two different time scalings in the two regions R_1 and R_2, $\tau_1 = \frac{t}{b}$, $\tau_2 = \frac{t}{d}$. The new system is now discontinuous with respect to time as well (if $b \neq d$), but the orbits of this system are the same as the ones of the original system.

Thus, we will consider the following family of systems as prototypes of the **general form** of piecewise linear systems with a line of discontinuity:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & y - qx - m < 0 \\ \begin{pmatrix} -a & \frac{b}{\alpha} \\ -b\alpha & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & y - qx - m > 0 \end{cases} \tag{8}$$

Notice that (8) depends on 5 values, namely $\frac{a}{b} > 0$, $\frac{c}{d} > 0$, $\alpha > 0$, m and q, and it is not easy to perform an exhaustive analysis for these problems. For this reason, in Sect. 4 we will make some simplifications to the structure (8); namely, we will restrict to the case of $q = 0$, $a = b = d = 1$ and let c, α, and m, vary.

Remark 4 Note that a smooth planar linear dynamical system is topologically equivalent to the system with coefficient matrix in canonical form. However, this is not true for planar piecewise linear dynamical systems. Indeed, new phenomena such as folds of periodic orbits, that do not appear in the canonical form, arise in the more general setting (8); see below.

3 Canonical Form

In this section we study piecewise linear systems in the canonical form (7). We use n to indicate the normal to the line Σ:

$$\Sigma = \{(x, y) : \ y = m\} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Obviously, the only equilibrium for the system is the origin.

As we already remarked in Sect. 2, the dynamics of (7) depend on the values of $\frac{a}{b}$, $\frac{c}{d}$, and m. Our goal below is to explore the changes in dynamics in function of the two values $\frac{c}{d}$ and m while leaving $\frac{a}{b}$ fixed.

As we will see, the value $m = 0$ is a bifurcation value for any value of $\frac{c}{d}$. The phenomena we will present, however, are totally different from the Hopf bifurcation phenomenon observed in [12]. In [12], m cannot be taken as a free parameter and the line of discontinu-

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ity must contain the origin (the equilibrium). In the present paper, instead, we take m as a bifurcation parameter and observe quite different dynamical behaviors as m is varied.

3.1 Case $m = 0$

The case $m = 0$ has been studied already in [5,12]. We briefly summarize the main results for system (7).

The solutions intersect Σ transversally everywhere except at the origin: $n^T f_1(x, y) = -dx$ and $n^T f_2(x, y) = -bx$. Take an initial condition on Σ, $(x_1, 0)$, $x_1 > 0$. Then after time $t_1 = \frac{\pi}{d}$, the solution trajectory $e^{A_1 t}\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ meets Σ again at $(x_2, 0)$, with $x_2 = -e^{\frac{c}{d}\pi} x_1$, crosses Σ and enters R_2 to meet Σ again at time $t_1 + t_2 = \frac{\pi}{b} + \frac{\pi}{d}$ at $(x_3, 0)$ with $x_3 = e^{(-\frac{a}{b}+\frac{c}{d})\pi} x_1$. Thus, in order for the solution to be periodic it must be $\frac{a}{b} = \frac{c}{d}$ and since this condition is independent of x_1, the periodic orbit is not isolated and it is stable. There is a family of periodic orbits that bifurcates from the origin as $\frac{c}{d} - \frac{a}{b}$ crosses 0: the value $\frac{c}{d} = \frac{a}{b}$ is a bifurcation value, the origin changes from a stable focus to an unstable focus through the appearence of a family of periodic orbits.

3.2 Case $m > 0$

For $m > 0$, Σ has an attractive sliding region. Take $(x, m) \in \Sigma$, then $n^T f_1(x, m) = -dx + cm$ and $n^T f_2(x, m) = -bx - am$. Hence the sliding region on Σ is $\bar{S}$ (the closure of S) with

$$S = \left\{(x, y) \in \Sigma, -\frac{a}{b}m < x < \frac{c}{d}m\right\}.$$

The two points $(-\frac{a}{b}m, m)$, $(\frac{c}{d}m, m)$ are *tangential points*, i.e. points $\mathbf{x}$ in the phase space such that $(n^T f_1(\mathbf{x}))(n^T f_2(\mathbf{x})) = 0$. In [3, Chapter 19], these are called *singular points* of Class 2a. (A complete classification of singular points is in [3, Chapter 4].) The Filippov sliding vector field on $\bar{S}$ is well defined [see (2)]:

$$f_F = (1 - \mu) f_1 + \mu f_2, \qquad \mu = \frac{n^T f_1}{n^T (f_1 - f_2)}. \tag{9}$$

Hence

$$f_F(x) = \frac{(x^2 + m^2)(ad + bc)}{(cm - dx) + (bx + am)}, \tag{10}$$

which is always positive. Hence, once on S, the solution slides along Σ and exits at $\frac{c}{d}m$ to enter R_1.

There are three different types of periodic orbits that can occur in system (7): orbits that have isolated points in common with Σ, orbits that have a sliding segment in Σ and a nonempty intersection with both regions R_1 and R_2, orbits that have a sliding segment on Σ and empty intersection with one of the two regions R_1, R_2. Following [6], we call them respectively *crossing periodic orbits, crossing and sliding periodic orbits*, and *sliding periodic orbits*. We will see below that, for a given value of $\frac{c}{d}$, the existence of one type of periodic orbit rules out the possibility of existence for the other two. The system will either have one periodic orbit which attracts all initial conditions except the origin ($\frac{c}{d} < \frac{a}{b}$), or it will have no periodic orbits and all the solutions (except the origin) will be unbounded ($\frac{c}{d} \geq \frac{a}{b}$).

To detect the existence of periodic orbits, we need to study the intersections of the flow with Σ. We will use the following notation:

$$S^+ = \{(x, y) \in S,\ \ x \geq 0\}, \quad S^- = \{(x, y) \in S,\ \ x < 0\}$$
$$\Sigma^+ = \left\{(x, y) \in \Sigma,\ \ x \geq \frac{c}{d} m\right\}, \quad \Sigma^- = \left\{(x, y) \in \Sigma,\ \ x \leq -\frac{a}{b} m\right\}, \tag{11}$$

so that $\Sigma = \Sigma^- \cup S^- \cup S^+ \cup \Sigma^+$, and further let $\varphi_i(t, x_0, y_0)$ be the solution of $\dot{\mathbf{x}} = A_i \mathbf{x}$, $i = 1, 2$, such that $\mathbf{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Take an initial condition $(x, m) \in \Sigma^+$. Then $\varphi_1(t, x, m)$ will reach Σ again at unique first time $t = t_1(x)$. Define the return map

$$P_1 : \Sigma^+ \to \Sigma^- \cup S^-, \quad P_1(x) = \varphi_1(t_1(x), x, m). \tag{12}$$

In the same way, let $(x, m) \in \Sigma^-$ and let $t_2(x)$ be the first time for which $\varphi_2(t_2(x), x, m)$ meets Σ again. Define the return map

$$P_2 : \Sigma^- \to S \cup \Sigma^+,\ \ P_2(x) = \varphi_2(t_2(x), x, m). \tag{13}$$

Clearly, P_1 and P_2 are smooth maps, and when $P_1(x) \in \Sigma^-$ we can define the composite (Poincaré) map

$$P(x) = P_2(P_1(x)) : \Sigma^+ \to S \cup \Sigma^+, \tag{14}$$

which is again smooth, and it is explicitly given by:

$$P(x) = e^{-at_2(P_1(x)) + ct_1(x)}(\cos(\hat{t})x + \sin(\hat{t})m), \tag{15}$$

where $\hat{t} = bt_2(P_1(x)) + dt_1(x)$.

Crossing Periodic Orbit

We first explore the existence of crossing periodic orbits for (7), i.e., fixed points for the Poincaré map (14). To have a crossing periodic orbit, the corresponding trajectory must satisfy

$$e^{-a\bar{t}_2 + c\bar{t}_1} \begin{pmatrix} \cos(\hat{t}) & \sin(\hat{t}) \\ -\sin(\hat{t}) & \cos(\hat{t}) \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ m \end{pmatrix} = \begin{pmatrix} \bar{x}_1 \\ m \end{pmatrix}, \tag{16}$$

with $\hat{t} = b\bar{t}_2 + d\bar{t}_1$ and $\bar{t}_1 = t_1(\bar{x}_1)$, $\bar{t}_2 = t_2(P_1(\bar{x}_1))$. Formula (16) requires that a rotation multiplied by the factor $e^{-a\bar{t}_2 + c\bar{t}_1}$ takes the vector $(\bar{x}_1, m)^\top$ in itself. This is the case only if

$$-a\bar{t}_2 + c\bar{t}_1 = 0,\ \ b\bar{t}_2 + d\bar{t}_1 = 2\pi, \tag{17}$$

which gives the following values for $\bar{t}_1$ and $\bar{t}_2$:

$$\bar{t}_1 = \frac{a}{ad + bc} 2\pi\ quad \bar{t}_2 = \frac{c}{ad + bc} 2\pi \quad \text{or}$$
$$d\bar{t}_1 = 2\pi \frac{\frac{a}{b}}{\frac{a}{b} + \frac{c}{d}} \quad b\bar{t}_2 = 2\pi \frac{\frac{c}{d}}{\frac{c}{d} + \frac{a}{b}}. \tag{18}$$

Lemma 5 *The following is a necessary condition for the existence of a crossing periodic orbit:*

$$\frac{a}{b} > \frac{c}{d}. \tag{19}$$

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Proof Consider system (5) and take as initial condition a point $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$. Then the trajectory meets $y = 0$ again after time $\bar{t} = \frac{\pi}{d}$. So, the return time to Σ must be greater than $\bar{t}$. This together with the value for $\bar{t}_1$ in (18) implies $\frac{a}{b} > \frac{c}{d}$. □

For the remainder of this case $m > 0$, we will work under the assumption that (19) holds.

Proposition 6 *If a system* (5) *admits a crossing periodic orbit* γ, *then this is an isolated periodic orbit. No other crossing periodic orbit exists.*

Proof The trajectory at time $\bar{t}_1$ given by (18) must satisfy the following

$$e^{c\bar{t}_1} \begin{pmatrix} \cos(d\bar{t}_1) & \sin(d\bar{t}_1) \\ -\sin(d\bar{t}_1) & \cos(d\bar{t}_1) \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ m \end{pmatrix} = \begin{pmatrix} \bar{x}_2 \\ m \end{pmatrix}.$$

In particular, from the second component of this equality we get the following expression for a unique value of $\bar{x}_1$, and in turn of $\bar{x}_2$:

$$\bar{x}_1 = \frac{\cos(d\bar{t}_1) - e^{-c\bar{t}_1}}{\sin(d\bar{t}_1)} m, \quad \bar{x}_2 = \frac{e^{c\bar{t}_1} - \cos(d\bar{t}_1)}{\sin(d\bar{t}_1)} m. \tag{20}$$

□

It will be handy to use a more compact notation for $\bar{x}_1$ and $\bar{x}_2$ in (20):

$$\bar{x}_1 = \frac{\cos(w) - e^{-\alpha z w}}{\sin(w)}, \qquad \bar{x}_2 = \frac{e^{\alpha z w} - \cos(w)}{\sin(w)}, \quad \text{where}$$
$$z = \frac{a}{b}, \quad \frac{c}{d} = \alpha z, \quad \alpha \in (0, 1)\,; \quad w = d\bar{t}_1 = \frac{2\pi z}{z + \alpha z} = \frac{2\pi}{1 + \alpha}. \tag{21}$$

Observe that (20) is uniquely defined, and therefore, if a crossing periodic exists, it must be unique. Moreover, we have that a crossing periodic orbit exists whenever $\bar{x}_1$ and $\bar{x}_2$ are outside the sliding region S. The following proposition shows that if $\bar{x}_1 \in \Sigma^+$ then $\bar{x}_2 \in \Sigma^-$.

Proposition 7 *Let* $\bar{x}_1$ *and* $\bar{x}_2$ *be defined as in* (20). *If* $\bar{x}_1 \geq \frac{c}{d} m$ *then* $\bar{x}_2 < -\frac{a}{b} m$.

Proof Using the notation of (21), let $g_1(z, \alpha) = \bar{x}_1 - \frac{c}{d} m = \frac{\cos(w) - e^{-z\alpha w}}{\sin(w)} m - \alpha z m$ and $g_2(z, \alpha) = \bar{x}_2 + \frac{a}{b} m = \frac{e^{z\alpha w} - \cos(w)}{\sin(w)} m + z m$. We claim that $g_1(z, \alpha) \geq 0$ implies $g_2(z, \alpha) < 0$. Since $\pi < w < 2\pi$, we have $\sin(w) < 0$ and we can rewrite $g_1(z, \alpha) \geq 0$ as

$$\cos(w) \leq e^{-\alpha z w} + \alpha z \sin(w). \tag{22}$$

Now, $g_2(z, \alpha) < 0$ can be rewritten as $e^{\alpha z w} - \cos(w) + z \sin(w) > 0$ and, using (22), $g_2(z, \alpha) < 0$ if the following inequality is verified:

$$e^{\alpha z w} - e^{-\alpha z w} + \sin(w) z (1 - \alpha) > 0. \tag{23}$$

We first show that for $\alpha \in (\alpha^*, 1)$, $\alpha^* = -2\pi + \sqrt{4\pi^2 + 1}$, we have

$$e^{\alpha z w} - e^{-\alpha z w} - z(1 - \alpha) > 0, \tag{24}$$

and hence (23). Then we show that for $\alpha \in (0, \alpha^*]$ we have

$$e^{\alpha z w} - e^{-\alpha z w} + \sin(w) z > 0, \tag{25}$$

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and this will imply (23) since $-\alpha\sin(w) > 0$. We will use Taylor expansions. To show (24) we expand the exponentials with respect to their argument and we get

$$2\alpha z w + R(\alpha z w) - z(1-\alpha) > 0,$$

where $R = O((\alpha w z)^2)$ indicates the reminder of the Taylor expansion of $e^{\alpha z w} - e^{-\alpha z w}$, and is always positive. For $\alpha > \alpha^*$, $2\alpha w - (1-\alpha) > 0$ (use $w = \frac{2\pi}{1+\alpha}$), so that (24) is verified. To show that (25) is verified in $(0, \alpha^*]$ we use the Taylor expansion of $e^{\alpha z w}$ with respect to its argument and the second degree Taylor polynomial of $\sin(w)$ about $\alpha = 0$. We have

$$e^{\alpha z w} - e^{-\alpha z w} = \frac{4\pi z\alpha}{1+\alpha} + R(\alpha),$$

with $R(\alpha) = O(\alpha^3) > 0$, while

$$z\sin\left(\frac{2\pi}{1+\alpha}\right) = -2\pi z\alpha + 2\pi z\alpha^2$$
$$+\frac{\alpha^3}{6}\frac{4\pi z}{(1+\alpha_0)^4}\left(\cos(w_0)\frac{2\pi^2}{(1+\alpha_0)^2} + \sin(w_0)\frac{6\pi}{1+\alpha} - 3\cos(w_0)\right),$$

where the last quantity is z times the Lagrange reminder of the Taylor polynomial for the sine function, $\alpha_0 \in (0, \alpha^*)$ and $w_0 = \frac{2\pi}{1+\alpha_0}$. We can bound from below the coefficient of $z\frac{\alpha^3}{6}$ with the following quantity: $R_1 = 4\pi(-2\pi^2 - 6\pi - 3)$. Now, given that $R(\alpha)$ is positive, we have

$$e^{\alpha z w} - e^{-\alpha z w} + \sin(w) z > z\left(\frac{2\pi\alpha + 2\pi\alpha^3}{1+\alpha} + \frac{\alpha^3}{6}R_1\right) =: zR_2(\alpha).$$

In the interval under study the function $R_2(\alpha)$ is positive since $R_2(0) = 0$ and $\frac{d}{d\alpha}R_2(\alpha) = \frac{2\pi+6\pi\alpha^2+4\pi\alpha^3}{(1+\alpha)^2} + \frac{1}{2}\alpha^2 R_1$, and this is positive in $[0, \alpha^*]$. Hence (25) is verified as well and the proof is complete. □

Proposition 7 together with Proposition 6 insures that a unique crossing periodic orbit of (5) exists if $\bar{x}_1 > \frac{c}{d}m$.

Corollary 8 *Let $\bar{t}_1$ and $\bar{t}_2$ be defined as in* (18) *and $\bar{x}_1$ and $\bar{x}_2$ as in* (20). *If $\bar{x}_1 \geq \frac{c}{d}m$ then* (5) *admits a crossing periodic orbit γ with first return time to Σ equal to $\bar{t}_1$ and with period $\bar{t}_1 + \bar{t}_2$.* □

Remark 9 Proposition 7 tells us that, if $\bar{x}_1$ in (20) is such that $\bar{x}_1 \geq \frac{c}{d}m$, then the system has an isolated crossing periodic orbit γ. Notice that for $\frac{c}{d} \to \left(\frac{a}{b}\right)^-$, $\bar{t}_1 \to \pi^+$ and hence, $\bar{x}_1$ in (20) goes to $+\infty$. Since $\bar{x}_1$ is a continuous function of $\frac{c}{d}$, for $\frac{c}{d}$ sufficiently large (and $\frac{c}{d} < \frac{a}{b}$), the system has a crossing periodic orbit.

Below, we show that γ is asymptotically stable for $\bar{x}_1 \geq \frac{c}{d}m$. We do this by direct computation of the derivative of the Poincaré map; equivalently, we could have computed the Floquet multiplier(s) using the monodromy matrix, and we will use this approach in Sect. 4. Further, in Theorem 15 below we will show that γ attracts every initial condition except the origin.

Definition 10 A periodic orbit γ is said to be *stable and finitely reached* if it is asymptotically stable and in an open neighborhood of γ there are orbits that reach γ in finite time.

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Proposition 11 *Let $\bar{x}_1$ in (20) be such that $\bar{x}_1 \geq \frac{c}{d}m$. Then, the crossing periodic orbit γ is asymptotically stable. For $\bar{x}_1 = \frac{c}{d}m$, γ is stable and finitely reached.*

Proof We have $\bar{x}_1 \geq \frac{c}{d}m$ and $\bar{x}_2 < -\frac{a}{b}m$, and the Poincaré map P in (15) is well defined and differentiable in a neighborhood of $\bar{x}_1$. Let $x_1 \in \Sigma^+$, and let $x_2 = P_1(x_1)$ and $x_3 = P_2(x_2) = P(x_1)$. To express $\frac{dP}{dx}(x)$, the following identities will be handy

$$x_2 = e^{ct_1(x_1)}\left[\cos\left(dt_1(x_1)\right)x_1 + \sin\left(dt_1(x_1)\right)m\right], \tag{26}$$

$$m = e^{ct_1(x_1)}\left[-\sin\left(dt_1(x_1)\right)x_1 + \cos\left(dt_1(x_1)\right)m\right], \tag{27}$$

$$x_3 = e^{-at_2(x_2)}\left[\cos(bt_2(x_2))x_2 + \sin(bt_2(x_2))m\right], \tag{28}$$

$$m = e^{-at_2(x_2)}\left[-\sin(bt_2(x_2))x_2 + \cos(bt_2(x_2))m\right]. \tag{29}$$

We need to compute $\frac{dx_3}{dx_1} = \frac{dx_3}{dx_2}\frac{dx_2}{dx_1}$. Using Eq. (27) and implicit differentiation we get $\frac{dt_1}{dx_1} = \frac{sin(dt_1)e^{ct_1}}{cm-dx_2}$ and differentiating (26) with respect to x_1 we obtain the following

$$\frac{dx_2}{dx_1} = e^{ct_1}\left(\frac{cx_2+dm}{cm-dx_2}\sin(dt_1) + \cos(dt_1)\right). \tag{30}$$

In the same way we can compute $\frac{dx_3}{dx_2}$ differentiating Eq. (28) with respect to x_2, use implicit differentiation in Eq. (29) to get $\frac{dt_2}{dx_2}$, and obtain the following

$$\frac{dx_3}{dx_2} = e^{-at_2}\left[\frac{ax_3-bm}{am+bx_3}\sin(bt_2) + \cos(bt_2)\right]. \tag{31}$$

Moreover, using Eq. (17), we can derive the following handy identities

$$\cos(b\bar{t}_2) = \cos(d\bar{t}_1), \quad \sin(b\bar{t}_2) = -\sin(d\bar{t}_1), \quad e^{-a\bar{t}_2} = e^{-c\bar{t}_1}.$$

On the periodic orbit the value of x_3 will be again $\bar{x}_1$. So, we have

$$\begin{aligned}\frac{dx_3}{dx_1}(\bar{x}_1) &= \left[-\frac{a\bar{x}_1-bm}{am+b\bar{x}_1}\sin(d\bar{t}_1)+\cos(d\bar{t}_1)\right]\left[\frac{c\bar{x}_2+dm}{cm-d\bar{x}_2}\sin(d\bar{t}_1)+\cos(d\bar{t}_1)\right]\\ &= \frac{a(-\bar{x}_1\sin(d\bar{t}_1)+\cos(d\bar{t}_1)m)+b(\cos(d\bar{t}_1)\bar{x}_1+m\sin(d\bar{t}_1))}{am+b\bar{x}_1}\\ &\quad+\frac{c(-\sin(d\bar{t}_1)\bar{x}_2+\cos(d\bar{t}_1)m)-d(\sin(d\bar{t}_1)m+\cos(d\bar{t}_1)\bar{x}_2)}{cm-d\bar{x}_2}\end{aligned}$$

and using (26), (27), (29) and (20) we obtain

$$DP(\bar{x}_1) = \left(\frac{am+b\bar{x}_2}{am+b\bar{x}_1}\right)\left(\frac{cm-d\bar{x}_1}{cm-d\bar{x}_2}\right) = \left(\frac{\bar{x}_2+\frac{a}{b}m}{\bar{x}_2-\frac{c}{d}m}\right)\left(\frac{\bar{x}_1-\frac{c}{d}m}{\bar{x}_1+\frac{a}{b}m}\right). \tag{32}$$

On the crossing periodic orbit, $\bar{x}_1 > \frac{c}{d}m$ and $\bar{x}_2 < -\frac{a}{b}m$, so that $DP(\bar{x}_1) < 1$ and γ is asymptotically stable.

If $\bar{x}_1 = \frac{c}{d}m$, then $DP(\bar{x}_1) = 0$ and γ is stable and finitely reached. □

Sliding Periodic Orbit

Assume now that system (7) admits a sliding periodic orbit γ_1, and notice that γ_1 will always exist for $\frac{c}{d}m$ sufficiently small. Indeed, let x_{-1} be the counterimage of $-\frac{a}{b}m$ on Σ under $\varphi_1(\cdot,\cdot,\cdot)$. For $\frac{c}{d}m < x_{-1}$, system (7) admits a sliding periodic orbit.

In the next theorem we show that γ_1 is stable and finitely reached and it attracts all initial conditions except the origin.

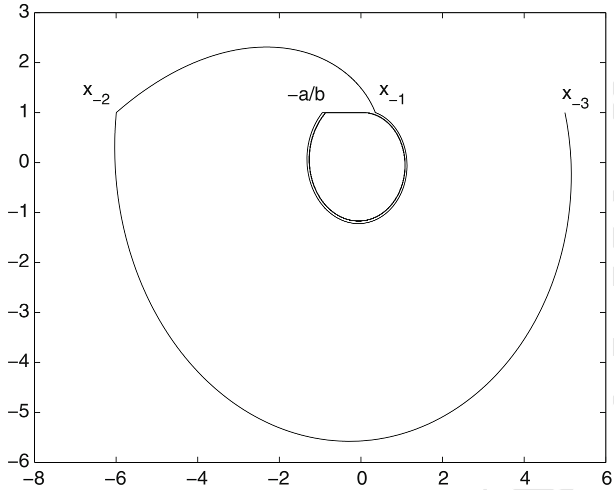

Fig. 1 Sliding periodic orbit and first elements of the sequence in the proof of Theorem 12

Theorem 12 *Assume that system (5) has a sliding periodic orbit γ_1. Then γ_1 is stable and finitely reached and it attracts all initial conditions except the origin.*

Proof We first prove global stability. For $(x, y) \in \Sigma$, let $t_1(x)$ be the first return time of $\varphi_1(t, x, y)$ to Σ. The existence of γ_1 implies that $\varphi_1(t_1(\frac{c}{d}m), \frac{c}{d}m, m) = (\bar{x}, m)$, with $\frac{c}{d}m > \bar{x} \geq -\frac{a}{b}m$. Then, let x_{-1} be the counterimage of $-\frac{a}{b}m$ on Σ under $\varphi_1(\cdot, \cdot, \cdot)$; i.e., let x_{-1} be the point on Σ such that $\varphi_1(\tau_1(x_{-1}), x_{-1}, m) = (-\frac{a}{b}m, m)$. Clearly $x_{-1} \geq \frac{c}{d}m$ and it must exist since $\varphi_1(\cdot, \cdot, \cdot)$ is a diffeomorphism of $\mathbb{R}^2$. Let x_{-2} be the point on Σ such that $\varphi_2(\tau_2(x_{-2}), x_{-2}, m) = (x_{-1}, m)$. Again the linearity of the vector field ensures existence of x_{-2}. Moreover $x_{-2} < -\frac{a}{b}m$. This same reasoning can be applied to x_{-2} and we can generate two sequences $\{x_{-2k}\}$ and $\{x_{-2k-1}\}$ such that: $\varphi_2(\tau_2(x_{-2k}), x_{-2k}, m) = x_{-2k+1}$, $\varphi_1(\tau_1(x_{-2k-1}), x_{-2k-1}, m) = x_{-2k}$ and $x_{-2k-1} > x_{-2k+1}$, $x_{-2k-2} < x_{-2k}$. See Fig. 1. Notice that the sequences can not possibly accumulate on a crossing periodic orbit γ, since γ, if it exists, is isolated and asymptotically stable. This implies global stability (except for the initial condition at the origin) of γ_1. To prove that the orbit is stable and finitely reached, notice that any x_0 in the region inside γ is such that $\varphi(t_1(x_0), x_0, m) \in \bar{S}$. Moreover any x_0 in $[\frac{c}{d}m, x_{-1}]$ is such that $\varphi_1(t_1(x_0), x_0, m) \in \bar{S}$. If $x_{-1} = \frac{c}{d}m$ then any $x_0 \in [x_{-1}, x_{-3}]$ is such that $\varphi_2 \circ \varphi_1(t_1(x_0), x_0, m) \in \bar{S}$. See Fig. 1. □

Crossing and Sliding Periodic Orbit

We noticed above that for $\frac{c}{d}$ small enough, system (7) has a sliding periodic orbit. Let $\frac{c_0}{d_0}$ be such that $P_1(\frac{c_0}{d_0}) = -\frac{a}{b}m$, then for $\frac{c}{d}$ in a right neighborhood of $\frac{c_0}{d_0}$, system (7) has a crossing-and-sliding periodic orbit γ_2 (see Fig. 2).

Theorem 13 *Assume that system (7) has a crossing-and-sliding periodic orbit γ_2. Then γ_2 is stable and finitely reached and it attracts all initial conditions except the origin.*

Proof The proof is similar to the one in Theorem 12 for a sliding periodic orbit. Let x_{-2} be the point on Σ^- such that $\varphi_2(\tau_2(x_{-2}), x_{-2}, m) = (\frac{c}{d}m, m)$. The linearity of the vector field ensures existence of x_{-2}. The rest of the proof is the same as in Theorem 12. Notice that

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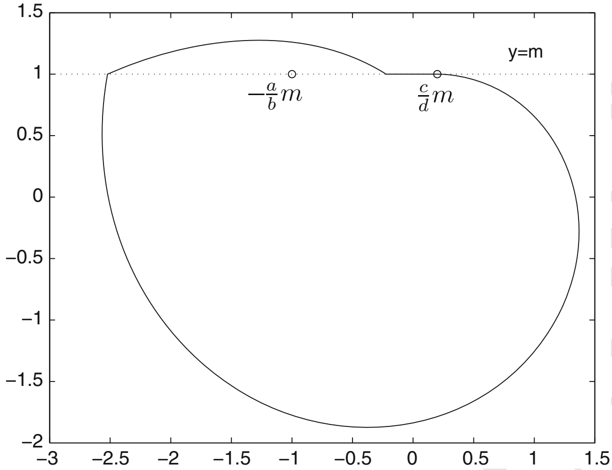

Fig. 2 Sliding and crossing periodic orbit

the sequences x_{-2k} and x_{-2k-1} cannot accumulate on a crossing periodic orbit γ since this would contradict the asymptotic stability of γ. □

The proofs of Theorem 12 and 13 rule out the possibility that two different kind of periodic orbits coexist.

Proposition 14 *Assume that system (5) has a crossing, or a crossing-and-sliding, or a sliding, periodic orbit γ. Then γ is the unique periodic orbit of system (7).* □

Theorems 12 and 13, and Proposition 14, allow to infer the following.

Theorem 15 *Assume that a system (7) admits a crossing periodic orbit γ. Consider any initial condition different from the origin. Then the corresponding solution trajectory approaches γ. If $\bar{x}_1$ in (20) is such that $\bar{x}_1 = \frac{c}{d}m$ then γ is also stable and finitely reached (from inside).*

Proof To prove this result we can select an initial condition on Σ. Indeed, any other initial condition will lead to a solution that meets Σ in finite time.

(i) $\bar{x}_1 > \frac{c}{d}m$.
Without loss of generality, we can take the initial condition (x_1, m), with $\frac{c}{d}m \le x_1 < \bar{x}_1$. Indeed if $x_1 \in S$, then the corresponding solution will slide on Σ to exit Σ at $\frac{c}{d}m$. Then, since a sliding orbit cannot exist, $x_2 = P_1(x_1)$ is such that $-\frac{a}{b}m > x_2 > \bar{x}_2$ and $x_3 = P_3(x_2)$ is such that $x_3 < \bar{x}_1$. We claim that $x_3 > x_1$. To prove this, we write the solution explicitly as

$$e^{-a(t_2-t_1)+ct_1} Q_1 Q_2 \begin{pmatrix} x_1 \\ m \end{pmatrix} = \begin{pmatrix} x_3 \\ m \end{pmatrix}, \tag{33}$$

with $Q_1 = \begin{pmatrix} \cos(dt_1) & \sin(dt_1) \\ -\sin(dt_1) & \cos(dt_1) \end{pmatrix}$ and $Q_2 = \begin{pmatrix} \cos(bt_2) & \sin(bt_2) \\ -\sin(bt_2) & \cos(bt_2) \end{pmatrix}$. Since $x_1 < \bar{x}_1$ and $x_2 > \bar{x}_2$, the rotation that takes (x_1, m) to (x_2, m) must satisfy: $d\bar{t}_1 < dt_1$. Indeed, if we denote with $\angle x_1 x_2$ the angle between the two vectors $(x_1, m)^\top$ and $(x_2, m)^\top$, then $\angle x_1 x_2 = dt_1$ and this must be greater than $\angle \bar{x}_1 \bar{x}_2 = d\bar{t}_1$. Similarly, $x_2 > \bar{x}_2$ and

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$x_3 < \bar{x}_1$ implies $bt_2 < b\bar{t}_2$. Then $-at_2 + ct_1 > -a\bar{t}_2 + c\bar{t}_1 = 0$ so that $e^{-at_2+ct_1} > 1$ and $\|(x_3, m)\| > \|(x_1, m)\|$, i.e. $x_3 > x_1$. Since γ is the unique crossing periodic orbit, for any $x_1 < \bar{x}_1$, the corresponding solution approaches γ. Similarly, we can show that for any $x_1 > \bar{x}_1$, $x_1 > x_3 = P(x_1) > \bar{x}_1$, and, again, uniqueness of γ implies that for any $x_1 > \bar{x}_1$, the corresponding solution approaches γ.

(ii) $\bar{x}_1 = \frac{c}{d}m$.
If $x_1 < \bar{x}_1$ then the solution will slide on Σ until it reaches $\bar{x}_1$ in finite time. Hence γ is stable and finitely reached from the inside. For $x_1 > \bar{x}_1$, the proof is the same as for case i).

This completes the proof of the theorem. □

All previous results have used condition (19): $\frac{c}{d} < \frac{a}{b}$. If this condition is not satisfied, all solutions (except the origin) are unbounded.

Proposition 16 *For $\frac{c}{d} \geq \frac{a}{b}$, no periodic orbit exists, and all solutions of system (5) (except the origin) are unbounded.*

Proof Because of Lemma 5, there cannot be crossing periodic orbits. Next, if $x_1 \in S^+$ then the solution slides along Σ until it reaches $\frac{c}{d}m$. Hence take $x_1 \geq \frac{c}{d}m$. Let $(x_2, m) = \varphi_1(t_1(x_1), x_1, m)$. It must be $x_2 < -\frac{c}{d}m$, since $\|\varphi_1(t, x_1, m)\| = e^{ct}$ is a monotone increasing function of t. Let $(x_3, m) = \varphi_2(t_2(x_2), x_2, m)$. We claim that $x_3 > x_1$. Indeed

$$\begin{pmatrix} x_3 \\ m \end{pmatrix} = e^{ct_1(x_1)-at_2(x_2)} Q_2(x_2) Q_1(x_1) \begin{pmatrix} x_1 \\ m \end{pmatrix},$$

with $Q_1(x) = \begin{pmatrix} \cos(dt_1(x)) & \sin(dt_1(x)) \\ -\sin(dt_1(x)) & \cos(dt_1(x)) \end{pmatrix}$ and $Q_2(x) = \begin{pmatrix} \cos(bt_2(x)) & \sin(bt_2(x)) \\ -\sin(bt_2(x)) & \cos(bt_2(x)) \end{pmatrix}$. Since for $m > 0$ there exists $\epsilon_0 > 0$ such that $dt_1(x_1) > \pi + \epsilon_0$ and $bt_2(x_2) < \pi - \epsilon_0$, then $ct_1(x_1) - at_2(x_2) > (\frac{c}{d} - \frac{a}{b})\pi + \epsilon_0(\frac{1}{b} + \frac{1}{d})$, and hence $x_3 > e^{\epsilon_0(\frac{1}{b}+\frac{1}{d})}x_1$. Applying the same reasoning to x_3 we generate a sequence $\{x_{2k+1}\}$ such that $x_{2k+1} > e^{k\epsilon_0(\frac{1}{b}+\frac{1}{d})}x_1$. This proves the theorem. □

Varying $\frac{c}{d}$

Next, we study the behavior of the system as we vary the value of $\frac{c}{d}$, still holding $\frac{a}{b}$ fixed. Looking at (21), this requires studying $\bar{x}_1$ as a function of $\frac{c}{d}$, hence of α.

Proposition 17 *Let $\bar{x}_1$ be defined by (21). If (7) has a crossing periodic orbit, then $\bar{x}_1$ is increasing as a function of α.*

Proof If there is a crossing periodic orbit γ, then $\bar{x}_2(\alpha) < -\frac{a}{b}m = -zm$. Hence

$$e^{\alpha zw} - \cos(w) + z\sin(w) > 0. \tag{34}$$

For the first derivative of $\bar{x}_1$ with respect to α we have

$$\frac{d}{d\alpha}\bar{x}_1(\alpha) = \frac{\frac{2\pi}{(1+\alpha^2)}\left(1 + e^{-\alpha zw}(z\sin(w) - \cos(w))\right)}{\sin(w)^2}m, \tag{35}$$

which is positive because of (34). □

Notice that for $\frac{c}{d} \to (\frac{a}{b})^-$, $\bar{x}_1 \to +\infty$ [see (20) and (18)]. Hence Proposition 17, together with Corollary 8, ensure that (7) admits a crossing periodic orbit whose length goes to ∞

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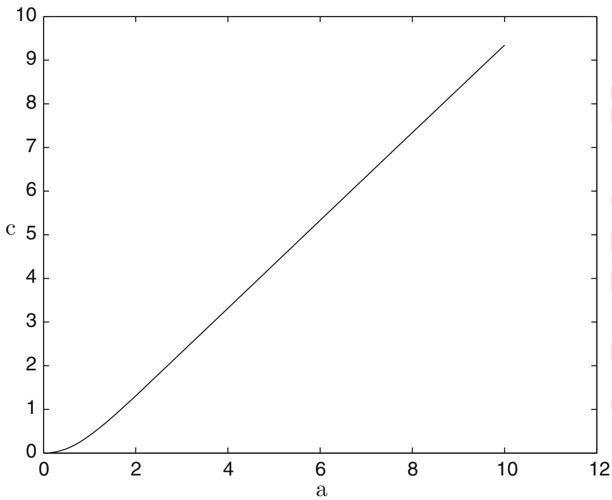

Fig. 3 Parameter values $b = d = m = 1$. Portion of the curve of crossing bifurcation values in a and c

when $\frac{c}{d} \to (\frac{a}{b})^-$. Because of Proposition 17, as $\frac{c}{d}$ decreases, $\bar{x}_1$ decreases, and hence the length of the crossing periodic orbit decreases as well. Notice moreover that, if for a certain parameter value $\frac{\bar{c}}{\bar{d}}$, system (7) admits a crossing-and-sliding periodic orbit γ_2, then for $\frac{c}{d} < \frac{\bar{c}}{\bar{d}}$, there is either a sliding-and-crossing periodic orbit or a sliding periodic orbit contained in the interior region of γ_2, this rules out the existence of a crossing periodic orbit. It follows that there exists a $\frac{c_1}{d_1}$ such that: (i) $\bar{x}_1 = \frac{c_1}{d_1}m$; (ii) for $\frac{c}{d} > \frac{c_1}{d_1}$ system (7) has a crossing periodic orbit; (iii) for $\frac{c}{d} < \frac{c_1}{d_1}$ ($\frac{c}{d} > \frac{c_0}{d_0}$, see below) system (7) has a crossing-and-sliding periodic orbit. The parameter value $\frac{c_1}{d_1}$ is a *crossing bifurcation* value.

Example 18 (Curve of Crossing Bifurcations). In Fig. 3, we show (a piece of) the curve of the crossing bifurcation value in the two parameters $\frac{c}{d}$ and $\frac{a}{b}$. This is obtained looking at the solution curve of $\bar{x}_1 - \frac{c}{d}m = 0$, with $\bar{x}_1$ in (20).

As $\frac{c}{d}$ decreases, there is a parameter value $\frac{c}{d} = \frac{c_0}{d_0}$ such that $P_1(\frac{c_0}{d_0}m) = -\frac{a}{b}m$. This is a *buckling bifurcation*: γ persists, but it becomes a sliding periodic orbit. See Fig. 4 obtained for $a = b = d = 1$ and for different values of c. The bold periodic orbit at the buckling bifurcation $c \simeq 0.0634$ is a sliding periodic orbit. The one at $c = 0.1$ is a sliding and crossing periodic orbit and the one at $c = 0.03$ is a sliding periodic orbit.

Example 19 (Curve of Buckling Bifurcations) For the buckling bifurcation we do not have a close formula for the first return time $t(\frac{c}{d}m)$ to Σ. Hence we need to solve for t as well. The curve is obtained through continuation techniques applied to the following nonlinear system

$$e^{ct}(\cos(dt)\frac{c}{d} + \sin(dt)) = \frac{a}{b}$$

$$e^{ct}(-\sin(dt)\frac{c}{d} + \cos(dt)) = m.$$

In Fig. 5, we plot (a piece of) the curve of the buckling bifurcation holding $b = d = m = 1$.

Below we summarize the behavior of the system in the case of $m > 0$, as $\frac{c}{d}$ varies. In Fig. 6 we plot the bifurcation diagram for this case. The top diagram represents the behavior

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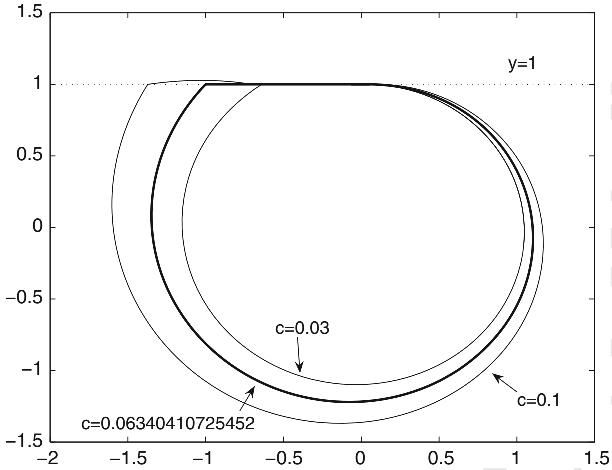

Fig. 4 Parameter values $a = b = d = m = 1$. As c decreases the sliding and crossing periodic orbit becomes a sliding periodic orbit. The value $c \simeq 0.0634$ is the buckling bifurcation value

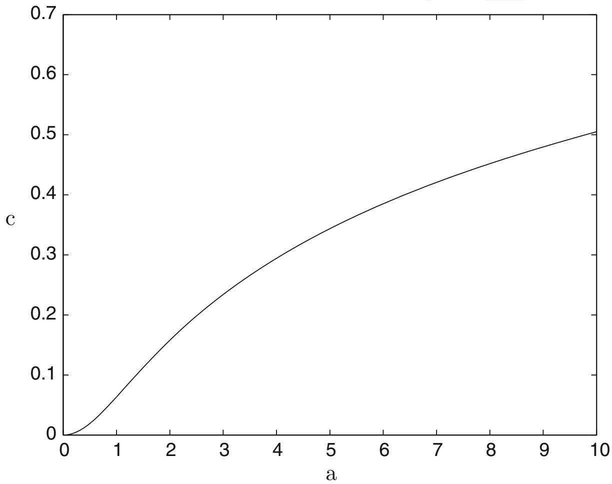

Fig. 5 Parameter values $b = d = m = 1$. A piece of the curve of buckling bifurcation

of the origin, the only equilibrium of the system. The other plots, depict the stable periodic orbits that occur for different parameter values. Recall that the values c_1/d_1 and c_0/d_0 are those we have just defined above: $\bar{x}_1 = \frac{c_1}{d_1}m$, $P_1(\frac{c_0}{d_0}m) = -\frac{a}{b}m$.

$\frac{c}{d} < 0$ The origin is a globally asymptotically stable focus for (7).

$\frac{c}{d} = 0$ This is a bifurcation value for the origin: the origin is stable but not asymptotically stable. There is a family of stable periodic orbits of radius $\rho \le m$: $\rho(\cos dt, \sin dt)$. For $\rho = m$, the orbit is tangent to Σ at $(0, m)$ and it is stable and finitely reached from outside and stable from inside. This is a bifurcation value: a grazing bifurcation of periodic orbits. See Fig. 7.

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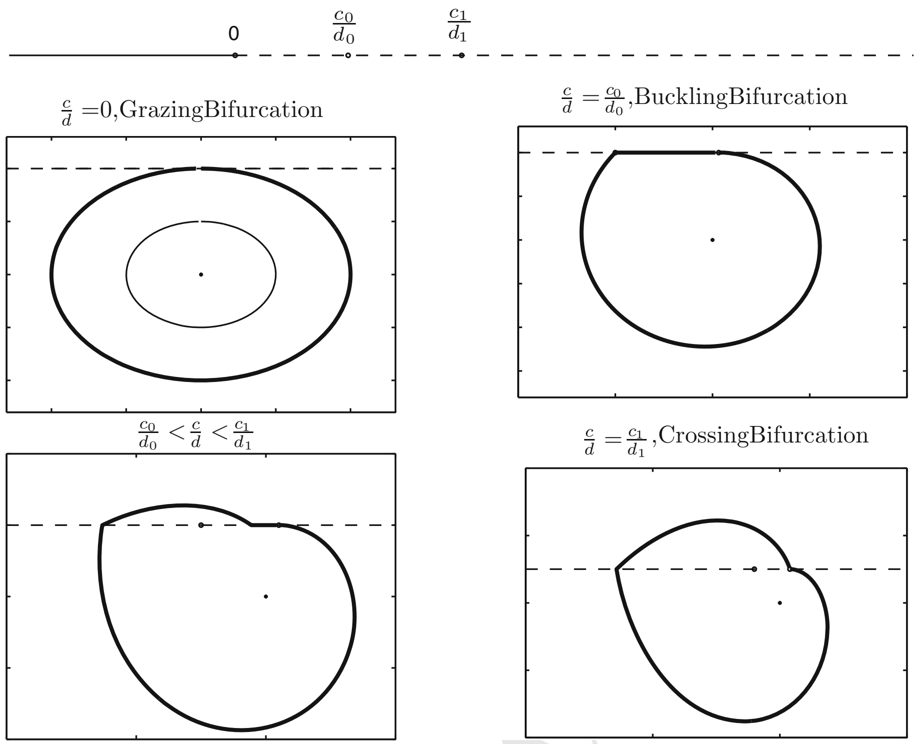

Fig. 6 Bifurcation diagram for $m > 0$

$\frac{c_0}{d_0} > \frac{c}{d} > 0$ The origin is an unstable focus. The periodic orbit tangent to Σ in $\frac{c}{d} = 0$ survives and it becomes a sliding periodic orbit. The orbit is stable and finitely reached from inside.

$\frac{c}{d} = \frac{c_0}{d_0}$ This a buckling bifurcation value. See Fig. 4. The value $c \simeq 0.0634$ is a buckling bifurcation value.

$\frac{c_1}{d_1} > \frac{c}{d} > \frac{c_0}{d_0}$ The periodic orbit retains its stability but changes type and it becomes a crossing-and-sliding periodic orbit. See Fig. 4 with $c = 0.1$.

$\frac{c}{d} = \frac{c_1}{d_1}$ This is a crossing bifurcation value.

$\frac{a}{b} > \frac{c}{d} > \frac{c_1}{d_1}$ The system has a globally stable (except for the origin) crossing periodic orbit of constant period $\bar{t}_1 + \bar{t}_2$ as in (18), and length that approaches ∞ for $\frac{c}{d} \to (\frac{a}{b})^-$.

$\frac{c}{d} \geq \frac{a}{b}$ There are no periodic orbits, all orbits are unbounded (see Proposition 16).

3.3 Case $m < 0$

To begin with, we observe that system (7) with $m < 0$ is equivalent to the case $m > 0$ in backward time with counterclockwise rotation and with $\frac{c}{d}$ replaced by $\frac{a}{b}$. So, we again consider the case of $\frac{a}{b}$ fixed and let $\frac{c}{d}$ vary. We use same notations as for the case $m > 0$.

The system exhibits repulsive sliding on Σ for $\frac{c}{d}m \leq x \leq -\frac{a}{b}m$. Let $S = \{x \in \Sigma,\ \frac{c}{d}m < x < -\frac{a}{b}m\}$, $\Sigma^+ = \{x \in \Sigma,\ x \geq -\frac{a}{b}m\}$, $\Sigma^- = \{x \in \Sigma,\ x \leq \frac{c}{d}m\}$. The origin is a

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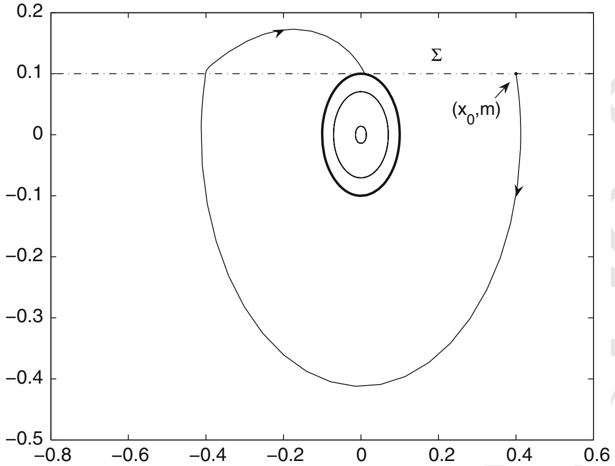

Fig. 7 Here, $\frac{c}{d} = 0$. Grazing bifurcation of periodic orbits. The orbit in *bold* is tangent to Σ at $(0, m)$

stable focus and it is the only equilibrium point. The two points $(-\frac{a}{b}m, m)$ and $(\frac{c}{d}m, m)$ are tangential points.

Since the system exhibits repulsive sliding on $\bar{S}$, the only orbits that may cross, or slide on, $\bar{S}$ are those that start in $\bar{S}$, although no orbit that slides on (a portion of) $\bar{S}$ is uniquely defined in forward time. Nonetheless, the Filippov's vector field (10) is still well defined and $f_F(x) < 0$ for all $x \in S$.

In order to study the crossing periodic orbits of the system, we reason as in the case $m > 0$, and we get the same quantities as in (18), (20), and the analog of Proposition 6 holds true as well. Notice that now $d\bar{t}_1 < \pi$ so that we get the following Lemma (cfr. with Lemma 5).

Lemma 20 *The following is a necessary condition for the existence of a crossing periodic orbit:*

$$\frac{c}{d} > \frac{a}{b}. \tag{36}$$

□

Proposition 21 *Let* (36) *hold and suppose that system* (7) *has a crossing periodic orbit* γ. *Then* γ *is unstable.*

Proof Let $\bar{x}_1$ and $\bar{x}_2$ be defined as in (20). Take $x_1 > \bar{x}_1$ and let $x_2 = P_1(x_1)$ and $x_3 = P_2(x_2)$. Then $x_2 < \bar{x}_2$ and we claim that $x_3 > x_1$. Indeed

$$\begin{pmatrix} x_3 \\ m \end{pmatrix} = e^{ct_1 - at_2} Q_2 Q_1 \begin{pmatrix} x_1 \\ m \end{pmatrix},$$

with $Q_1 = \begin{pmatrix} \cos(dt_1) & \sin(dt_1) \\ -\sin(dt_1) & \cos(dt_1) \end{pmatrix}$, $Q_2 = \begin{pmatrix} \cos(bt_2) & \sin(bt_2) \\ -\sin(bt_2) & \cos(bt_2) \end{pmatrix}$ and t_1 and t_2 are first and second return time to Σ. Moreover, $x_1 > \bar{x}_1$ and $x_2 < \bar{x}_2$ imply that $t_1 > \bar{t}_1$, while $x_2 < \bar{x}_2$ and $x_3 > \bar{x}_1$ imply that $t_2 < \bar{t}_2$. Hence $e^{ct_1 - at_2} > 1$ and $x_3 > x_1$. In a similar way we can show that γ is unstable from inside as well. □

As we will see below, system (7) exhibits crossing-and-sliding or sliding periodic orbits for $\frac{c}{d}$ sufficiently large. In Fig. 8 the curve in bold is a sliding-and-crossing periodic orbit. It

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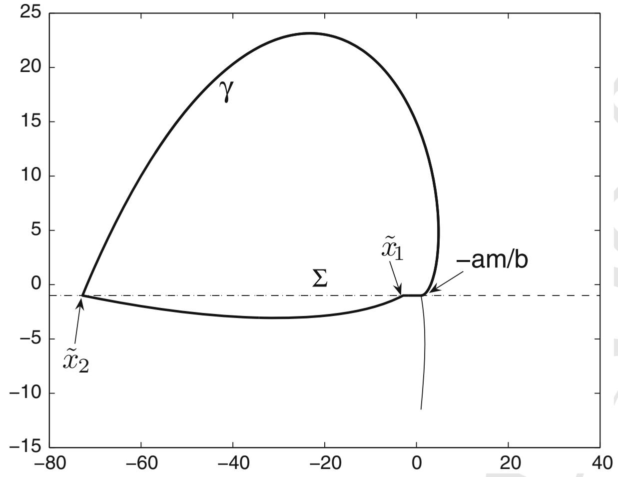

Fig. 8 γ in *bold* is a sliding-and-crossing periodic orbit

starts at $(-\frac{a}{b}m, m)$, slides on Σ towards $\tilde{x}_1$, it leaves Σ and enters R_1. It crosses Σ again at $\tilde{x}_2$ and enters R_2 until it reaches Σ at $(-\frac{a}{b}m, m)$. The following identities are satisfied

$$\tilde{x}_2 = P_2^{-1}\left(-\frac{a}{b}m\right), \quad \tilde{x}_1 = P_1^{-1}(\tilde{x}_2). \tag{37}$$

Remark 22 Notice that any solution of system (7) with initial condition on Σ is not uniquely defined in forward time. This means that, if γ is a crossing-and-sliding or sliding periodic orbit, a solution of (7) with initial condition on γ can leave γ at any point that belongs to the intersection $\gamma \cap \bar{S}$. However, if we consider the time change $\tau \to -t$ and we consider the system obtained taking the derivative with respect to τ, then γ is an invariant object for the new system. Notice that at $(-\frac{a}{b}m, m)$ the sliding vector field f_F is f_2, and μ in (9) is equal to 1. However, $\dot{\mu}$ is different from zero at $(-\frac{a}{b}m, m)$. Hence a solution with initial condition $(-\frac{a}{b}m, m)$ might either stay in Σ, or leave Σ tangentially to enter R_2, or leave Σ transversally to enter R_1.

Proposition 23 *Let* (36) *hold and suppose that a system* (7) *has a crossing-and-sliding or a sliding periodic orbit. Then they are unstable.*

Proof We will prove the theorem for γ crossing-and-sliding. The proof for γ sliding is analogous. Let γ be a crossing-and-sliding periodic orbit of (7). In Fig. 8 γ is in bold. It starts at $(-\frac{a}{b}m, m)$, slides on Σ towards $\tilde{x}_1$, it leaves Σ and enters R_1. It crosses Σ again at $\tilde{x}_2$ and enters R_2 until it reaches Σ at $(-\frac{a}{b}m, m)$. Consider now the solution that starts at $(-\frac{a}{b}m, m)$ and, instead of sliding on Σ, enters R_1 (a small piece of curve in Fig. 8). To show instability of γ, we need to show that this solution moves away from γ. We have $x_2 = P_1(-\frac{a}{b}m) < P_1(\tilde{x}_1) = \tilde{x}_2$ so that $x_3 = P_2((P_1(-\frac{a}{b}m)) > -\frac{a}{b}m$. We can apply the same reasoning to x_3 so to generate two sequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ on Σ with $x_{2k+1} > x_{2k-1}$ and $x_{2k} < x_{2k-2}$. Hence instability from outside is proven.

In order to prove instability from the inside, we just need to notice that any initial condition in the region inside γ approaches the origin. Denote this region with Γ. Any solution with initial condition in $R_1 \cap \Gamma$ cannot cross γ so that it must enter $R_2 \cap \Gamma$. Once in $R_2 \cap \Gamma$, the solution cannot meet Σ again (since $n^T f_2(\mathbf{x}) > 0$ on $\bar{S}$) and hence it approaches the origin.

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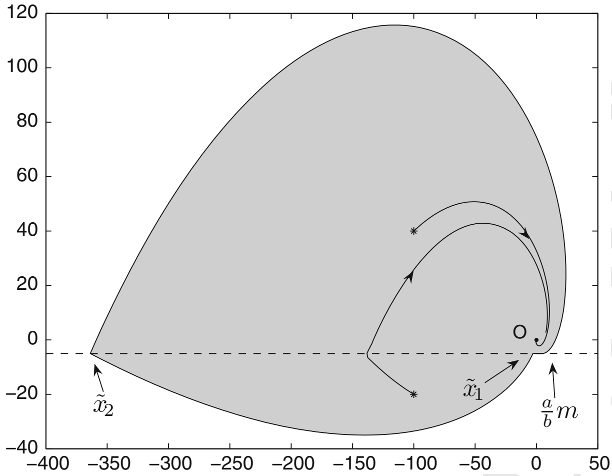

Fig. 9 Sliding and crossing periodic orbit. All the solutions *inside the shaded region* approach the origin

In Fig. 9, the region inside γ is filled in gray. The asterisks denote initial conditions in Γ and the corresponding solutions move towards the origin. □

The sequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ in the proof of Proposition 23 cannot accumulate on a crossing periodic orbit due to Proposition 21. Hence the following holds true.

Proposition 24 *System* (7) *with line of discontinuity* $y = m$ *and* $m < 0$, *admits at most one periodic orbit.* □

The sliding-and-crossing (or sliding) periodic orbit acts as a separatrix of the phase space. All the initial conditions inside it lead to the origin, while the solutions outside γ are unbounded.

The following proposition establishes sufficient conditions for the existence of a crossing periodic orbit γ. Its proof follows from the observation that for any $x \in \Sigma^+$, $P_1(x) \in \Sigma^-$, since $n^T A_1 x < 0$ for $x \in S \cup \Sigma^+$.

Proposition 25 *Let* (36) *hold, and let* $\bar{t}_1$, $\bar{t}_2$, $\bar{x}_1$ *and* $\bar{x}_2$ *be defined as in* (18), (20). *If* $\bar{x}_1 > -\frac{a}{b}m$, *then* (7) *has a crossing periodic orbit* γ. □

In order to study what happens at (or past) the value $\frac{c}{d} = \frac{a}{b}$, we set $\frac{a}{b} = z$, $\frac{c}{d} = \alpha\frac{a}{b}$ (with $\alpha > 1$), and study $\bar{x}_1$ as a function of α, similarly to what we did in (21).

Proposition 26 *Let* (36) *hold. If the system admits a crossing periodic orbit, then the function* $\bar{x}_1 + \frac{a}{b}m$ *is a decreasing function of* $\frac{c}{d}$.

Proof If γ exists, we have $\bar{x}_2 < \frac{c}{d}m$. Using same notations as in (21) and noticing that $\sin(w) > 0$ we have

$$e^{\alpha z w} - \cos(w) > \alpha z \sin(w). \tag{38}$$

Let $g_1(z, \alpha) = \bar{x}_1 + zm$, so that $\partial g_1/\partial\alpha = d\bar{x}_1/d\alpha$, and this is given by (35). Using (38) then gives that (35) is negative. □

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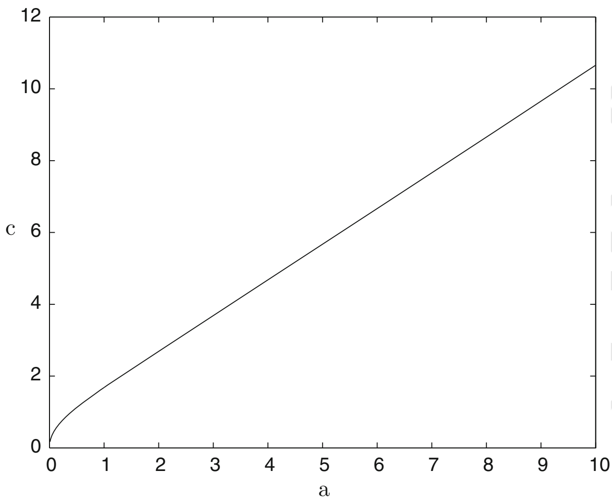

Fig. 10 Crossing bifurcation curve. Parameter values $b = d = 1, m = -1$

Observe that when $\frac{c}{d} \to (\frac{a}{b})^+$, $\bar{x}_1 \to +\infty$ and the length of the periodic orbit γ tends to ∞, while for $\frac{c}{d} \to \infty$, $\bar{x}_1 \to -\infty$. Hence, there must be a value of $\frac{c}{d}$, say $\frac{c}{d} = \frac{c_2}{d_2}$, such that $\bar{x}_1 = -\frac{a}{b}m$. This is a *crossing bifurcation* value. Due to Proposition 26 this crossing bifurcation value must be unique.

Example 27 (Curve of Crossing Bifurcations) In Fig. 10, we show a portion of the curve of the crossing bifurcation values in the two parameters c and a. The other parameters are $b = d = 1, m = -1$.

Next, let $\frac{c}{d} = \frac{c_3}{d_3}$ be such that $P_2(\frac{c_3}{d_3}m) = -\frac{a}{b}m$. Then, the solution that starts at $(-\frac{a}{b}m, m)$ slides on Σ, exits Σ for $x = \frac{c_3}{d_3}$ to enter R_1 and reaches Σ again for $x = -\frac{a}{b}m$: it is a sliding periodic orbit. Denote it with γ. As $\frac{c}{d}$ increases beyond $\frac{c_3}{d_3}$ the repulsive sliding region $\bar{S}$ becomes larger, but γ is not affected by the parameter change: it starts at $(-\frac{a}{b}m, m)$, slides on Σ, exits Σ at $x = \frac{c_3}{d_3}$ to enter R_1 and reaches Σ again at $x = -\frac{a}{b}m$. Hence for $\frac{c}{d} \geq \frac{c_3}{d_3}$, the system admits a sliding periodic orbit γ that is independent on the value of $\frac{c}{d}$.

The value $\frac{c}{d} = \frac{c_3}{d_3}$ for which $P_2(\frac{c_3}{d_3}m) = -\frac{a}{b}m$ is a *buckling bifurcation*.

Example 28 (Curve of Buckling Bifurcations) In Fig. 11, we plot (part of) the curve of buckling bifurcation values in the two parameters c and a. The other parameter values are $b = d = 1, m = -1$.

When (36) is violated, and $\frac{a}{b} > \frac{c}{d}$, the following proposition shows that the origin is globally asymptotically stable.

Proposition 29 *Assume* $\frac{c}{d} < \frac{a}{b}$. *Then the origin is globally stable for* (7).

Proof Let $x_1 \in \Sigma^+$, $x_2 = P_1(x_1)$ and $x_3 = P_2(x_2)$. Then, as in the proof of Proposition 21, we have

$$(x_3^2 + m^2) = e^{2(ct_1 - at_2)}(x_1^2 + m^2),$$

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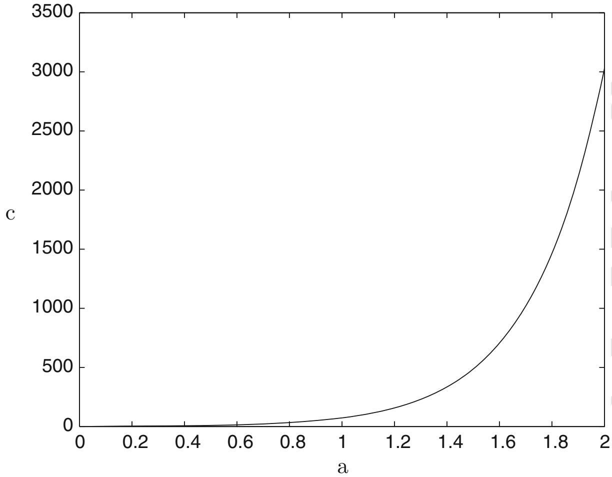

Fig. 11 Buckling bifurcation curve. Parameter values $b = d = 1$, $m = -1$

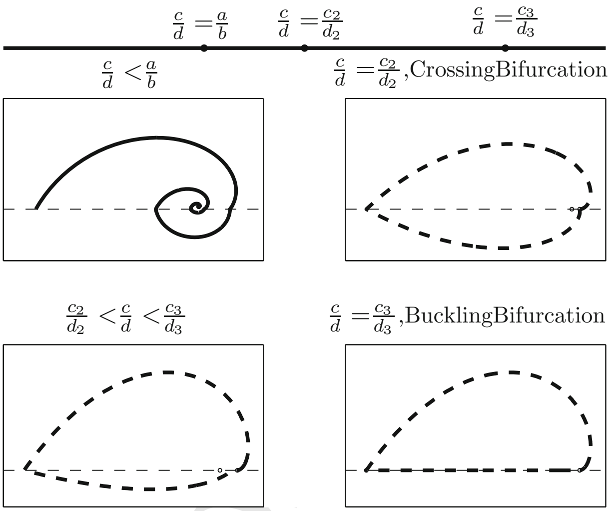

Fig. 12 Bifurcation diagram for the case $m < 0$

and $ct_1 - at_2 < \left(\frac{c}{d} - \frac{a}{b}\right)\pi < 0$. Hence $x_3 < x_1$. We repeat for x_3 the reasoning we used for x_1, and so on. Hence, we generate two sequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ with $x_{2k} > x_{2k-2}$ and $x_{2k+1} < x_{2k-1}$, with their differences bounded away from 0. Let $\bar{x}$ be such that $P_2(\bar{x}) = -\frac{a}{b}m$. Then there exists a finite k such that $x_{2k} \geq \bar{x}$ and hence $\varphi_2(t, x_{2k}, m)$ approaches the origin for $t \to \infty$. □

Below we summarize the behavior of the system as $\frac{c}{d}$ varies, in this case of $m < 0$. Recall that the values c_2/d_2 and c_3/d_3 are such that, respectively: $\bar{x}_1 = -\frac{a}{b}m$, $P_2(\frac{c_3}{d_3}m) = -\frac{a}{b}m$. In Fig. 12 we plot the bifurcation diagram for this case. The top diagram represents the behavior of the origin, the only equilibrium of the system. The boxed plot that corresponds to the value

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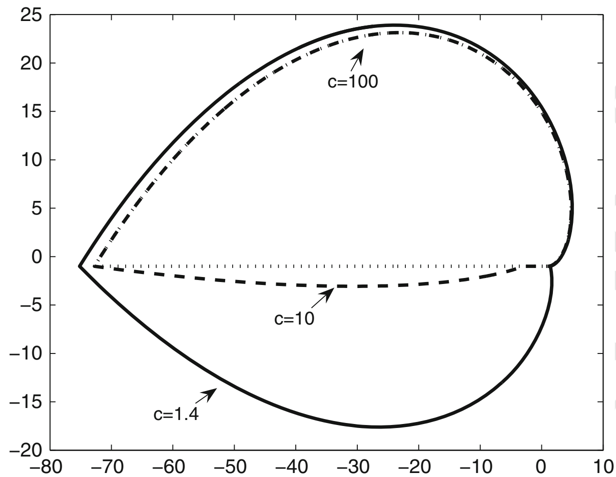

Fig. 13 Crossing, crossing and sliding and sliding periodic orbit for different values of c and $a = b = d = 1$ and $m = -1$

$\frac{c}{d} < \frac{a}{b}$, depicts the behavior of the origin as a global stable focus. The other plots, depict the unstable periodic orbits that (might) occur for different parameter values.

$\frac{c}{d} \leq \frac{a}{b}$ The origin is a global stable focus for (7). The value $\frac{c}{d} = \frac{a}{b}$ is a bifurcation value.

$\frac{c_2}{d_2} > \frac{c}{d} > \frac{a}{b}$ The origin is a locally stable focus and there is an unstable crossing periodic orbit γ acting as a separatrix: initial conditions in the region inside the periodic orbit have corresponding solutions that approach the origin, while initial conditions in the region outside γ lead to unbounded solutions. The period of γ is always finite while the orbit's length decreases as $\frac{c}{d}$ increases, and its length approaches ∞ when $\frac{c}{d} \to (\frac{a}{b})^+$. In Fig. 13 we plot the crossing periodic orbit γ with a continuous line corresponding to the value $c = 1.4$. All the other parameters are taken equal to 1.

$\frac{c}{d} = \frac{c_2}{d_2}$ This is a crossing bifurcation value.

$\frac{c_2}{d_2} < \frac{c}{d} \leq \frac{c_3}{d_3}$ The origin is a locally stable focus. There is a crossing-and-sliding periodic orbit γ which acts as separatrix: the solutions inside γ approach the origin, the solutions outside γ are unbounded. In Fig. 13 we plot the crossing and sliding periodic orbit γ with a dashed line. The corresponding value of c is 20.

$\frac{c}{d} = \frac{c_3}{d_3}$ This is a buckling bifurcation.

$\frac{c}{d} \geq \frac{c_3}{d_3}$ The origin is a locally stable focus and the system has a sliding periodic orbit γ (that is the same for all the values of $\frac{c}{d}$), which acts as separatrix for initial conditions that lead to trajectories approaching the origin and those leading to unbounded solutions. In Fig. 13 we plot the sliding periodic orbit γ with a dotted line. The corresponding value of c is 100.

3.4 m as a Bifurcation Parameter

The analysis in Sects. 3.3 and 3.2 allows us to study (7) using m as bifurcation parameter. In what follows we will distinguish between three cases $\frac{c}{d} < \frac{a}{b}$, $\frac{c}{d} = \frac{a}{b}$ and $\frac{c}{d} > \frac{a}{b}$.

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We stress that the value $m = 0$ will always be a bifurcation value, regardless of whether $\frac{a}{b} \gtreqless \frac{c}{d}$; cfr. with [12] where the authors assumed $\frac{a}{b} = \frac{c}{d}$ when $m = 0$.

$$0 < \frac{c}{d} < \frac{a}{b}$$

- $m < 0$ The origin is globally asymptotically stable.
- $m = 0$ There are no periodic orbits and the origin is still globally asymptotically stable. This is a Hopf bifurcation value.
- $m > 0$ The origin is unstable, and there is a unique, globally stable (except for the origin), periodic orbit. The periodic orbit might be a crossing, crossing-and-sliding, or sliding periodic orbit, respectively for $\frac{c}{d} \geq \frac{c_1}{d_1}$, $\frac{c_1}{d_1} > \frac{c}{d} > \frac{c_0}{d_0}$, or $\frac{c_0}{d_0} > \frac{c}{d}$.

$$\frac{c}{d} = \frac{a}{b}$$

- $m < 0$ The origin is globally asymptotically stable.
- $m = 0$ This is a bifurcation value. The origin is stable but not asymptotically stable, and there is a family of stable periodic orbits.
- $m > 0$ The origin is unstable, there are no periodic orbits, and all orbits (except the origin) are unbounded.

$$\frac{c}{d} > \frac{a}{b}$$

- $m < 0$ The origin is (locally) asymptotically stable. An unstable periodic orbit acts as separatrix of trajectories approaching the origin and those that become unbounded. The periodic orbit is either a crossing, or a crossing-and-sliding, or a sliding, periodic orbit, respectively for $\frac{c}{d} \leq \frac{c_2}{d_2}$, $\frac{c_2}{d_2} < \frac{c}{d} < \frac{c_3}{d_3}$, or $\frac{c_3}{d_3} \leq \frac{c}{d}$.
- $m = 0$ This is a Hopf bifurcation value. The origin is unstable, all other orbits are unbounded.
- $m > 0$ The origin is unstable, and all other orbits are unbounded.

4 General Form

In this section we consider the general family of systems (8). Although our study of this case is far from complete, we believe that it is still of interest since it highlights completely new phenomena which cannot occur when the family of linear systems is in canonical form.

In general, we cannot bring $\Sigma := \{(x, y) : \ y - qx - m = 0\}$ into a horizontal line without breaking the structure of the system. Hence, we need to work with $h(x, y) = y - qx - m$. Note that the sliding region for (8) is $\bar{S}$ with $S = \{(x, m) \in \Sigma, \ -\frac{a}{\alpha b}m < x < \frac{c}{d}m\}$ and it is an attractive sliding region. On it, the Filippov sliding vector field (10) is well defined and is given by

$$f_F(x) = \frac{(x^2 + m^2)ad + (\alpha x^2 + \frac{m^2}{\alpha})bc + bdmx(\alpha - \frac{1}{\alpha})}{(c + a)m + x(\alpha b - d)}. \tag{39}$$

Unfortunately, these problems depend on five parameters: $\frac{a}{b}$, $\frac{c}{d}$, α, with $a, b, c, d > 0$, and on m and q, and are too difficult to analyze in such generality. For this reason, we make the following simplifications:

$$\begin{aligned} &m > 0 \ \text{ and } \ h(x, y) = y - m, \quad \text{i.e.} \quad q = 0, \\ &\text{and } \ a = 1, \ \ b = 1, \ \ d = 1\,. \end{aligned} \tag{40}$$

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Note that, in the case (40), f_F is easily seen to be always positive.

To reiterate, we explore (8) by allowing just c and α to vary. Still, as we will see, even in this simplified case (40), the dynamical behavior of (8) is richer than that reflected by the system in canonical form (7). However, even in this seemingly simpler case, exact analytical expressions for the solution of the problem are out of reach, and we will use a combination of analysis and computer aided simulation to highlight what can happen.

For $c < a$, we know that system (7) has a globally asymptotically stable periodic orbit. This might be a crossing, or crossing-and-sliding, or sliding, periodic orbit. The question is whether (8) retains the same dynamical behavior of the system in canonical form.

The reasoning for the existence of crossing periodic orbits is similar to the one in (16). A simple computation shows that with $T = \begin{pmatrix} -a & \frac{b}{\alpha} \\ -\alpha b & -a \end{pmatrix}$ we have

$$e^{Tt} = e^{-at} \begin{pmatrix} cos(bt) & \frac{1}{\alpha}\sin(bt) \\ -\alpha\sin(bt) & \cos(bt) \end{pmatrix}.$$

Using this expression, we see that there is a crossing periodic orbit if there exists $\bar{x}_1$, $\bar{t}_1$ and $\bar{t}_2$ such that

$$e^{c\bar{t}_1} e^{-a\bar{t}_2} \begin{pmatrix} \cos(b\bar{t}_2) & \frac{\sin(b\bar{t}_2)}{\alpha} \\ -\alpha\sin(b\bar{t}_2) & \cos(b\bar{t}_2) \end{pmatrix} \begin{pmatrix} \cos(d\bar{t}_1) & \sin(d\bar{t}_1) \\ -\sin(d\bar{t}_1) & \cos(d\bar{t}_1) \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ m \end{pmatrix} = \begin{pmatrix} \bar{x}_1 \\ m \end{pmatrix}. \tag{41}$$

Let $\bar{x}_2 = P_1(\bar{x}_1)$. Then we can write $\bar{x}_1$ and $\bar{x}_2$ in function of $\bar{t}_1$, as in (20), and we can use (41) to rewrite $\bar{t}_2$ in function of $\bar{t}_1$ and α as:

$$\begin{aligned} \bar{t}_2(\bar{t}_1, \alpha) &= \frac{1}{b} \arctan \frac{\alpha(\bar{x}1 - \bar{x}_2)}{1 + \alpha^2 x_1 x_2} \\ &= \frac{1}{b} \arctan \frac{\alpha \sin(d\bar{t}_1)\left(2\cos(d\bar{t}_1) - (e^{-c\bar{t}_1} + e^{c\bar{t}_1})\right)}{\sin(d\bar{t}_1)^2 + \alpha^2\left(\cos(d\bar{t}_1)(e^{c\bar{t}_1} + e^{-c\bar{t}_1}) - 1 - \cos(d\bar{t}_1)^2\right)}. \end{aligned} \tag{42}$$

We are left with the problem of looking for the zeros of the following function of α and $\bar{t}_1$,

$$f(\bar{t}_1, \alpha) = \alpha\sin(b\bar{t}_2)(e^{c\bar{t}_1} - \cos(d\bar{t}_1)) - \sin(d\bar{t}_1)(\cos(b\bar{t}_2) - e^{a\bar{t}_2}), \tag{43}$$

with $\bar{t}_2$ as in (42) and $b\bar{t}_2, d\bar{t}_1 \neq \pi$. In Fig. 14 we plot the curve of zeros of $f(\bar{t}_1, \alpha)$ as a function of α for different values of $c < a = 1$. Figure 15 is the same of Fig. 14 but it is obtained for c in [0.5 0.8306122]

As it is clear from the plot, it is not true that there is a unique value of $\bar{t}_1$ for all values of α and c. To clarify, in Fig. 16 we plot $\bar{t}_1$ in function of α for $c \simeq 0.83061$. As it can be seen, not all values of $\bar{t}_1$ correspond to a crossing periodic orbit. The dashed line in the plot is the value of $\bar{t}_1$ such that the corresponding $\bar{x}_1$ is equal to $\frac{c}{d}m$ and let us denote it as $t_1^{\frac{c}{d}m}$. We claim that for $\bar{t}_1 > t_1^{\frac{c}{d}m}$, there are no corresponding crossing periodic orbits and instead a crossing-and-sliding or sliding periodic orbit appears. Indeed, (41) is obtained regardless of the fact that the vector field $f_i(\mathbf{x}) = A_i\mathbf{x}$ is defined only in R_i, for $i = 1, 2$. Let us denote with $\varphi_i(t, x, y)$, the solution of $\dot{\mathbf{x}} = A_i\mathbf{x}$ with initial condition $\mathbf{x} = (x, y)^\top$. If we consider $\bar{x}_1$ in (20) as a function of $\bar{t}_1$, it is easy to verify that $\bar{x}_1$ is decreasing for $\bar{t}_1 \in (\frac{\pi}{d}, \frac{2\pi}{d})$. Hence $\bar{x}_1(\bar{t}_1) < \frac{c}{d}m$ whenever $\bar{t}_1 > t_1^{\frac{c}{d}m}$. It follows that $\varphi_1(t, \bar{x}_1, m)$ (computed regardless of the fact that f_1 is only defined in R_1) first enters R_2 and it meets Σ at a point $\bar{x}_{1/2} > \frac{c}{d}m$, then it enters R_1 and reaches Σ again at time $\bar{t}_1$ and at the point $\bar{x}_2$ given in (20). At $\bar{x}_2$, (41) considers the solution of $\dot{\mathbf{x}} = A_2\mathbf{x}$, $\varphi_2(t, \bar{x}_2, m)$, that at time $\bar{t}_2$ meets Σ again at $\bar{x}_1$. The orbit

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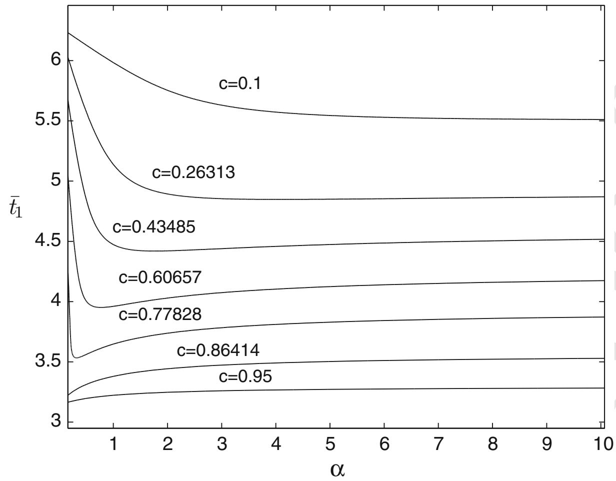

Fig. 14 Curve of $\bar{t}_1$ as a function of α for different values of c. Here $a = b = d = m = 1$

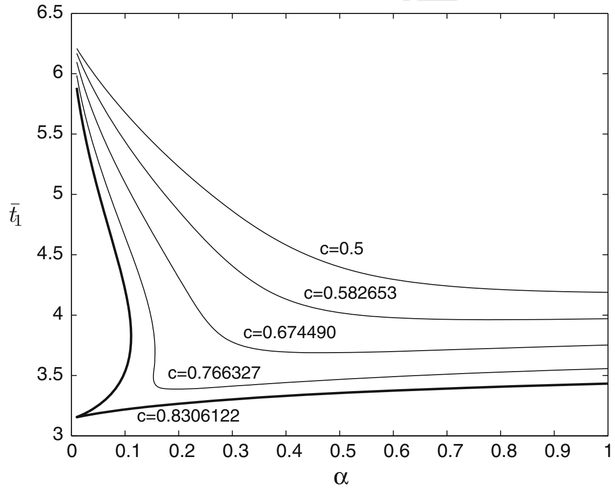

Fig. 15 Curve of $\bar{t}_1$ as a function of α for different values of c. Here $a = b = d = m = 1$

obtained through the composition of the two flows is a closed curve and we denote it with ψ. If we consider instead the initial condition $(\frac{c}{d}m, m)$, since $\frac{c}{d}m > \bar{x}_1$, $\varphi_1(t, \frac{c}{d}m, m)$, meets Σ at a point $\tilde{x}_2$, with $\tilde{x}_2 > \bar{x}_2$. Two cases may occur: (i) $\tilde{x}_2 \geq -\frac{a}{b}m$, then (8) has a sliding orbit γ contained in the interior region of ψ; (ii) $\tilde{x}_2 < -\frac{a}{b}m$, then $\varphi_2(t, \tilde{x}_2, m)$ meets Σ at a point $\tilde{x}_1 < \bar{x}_1 < \frac{c}{d}m$ and hence there is a crossing-and-sliding periodic orbit in the interior region of ψ.

Remark 30 The plot in Fig. 16 allows to determine the number of periodic orbits that system (8) has for each value of α. If, for a given α, there are three corresponding values of $\bar{t}_1$ ($\alpha = 0.1$ for example), the system has three periodic orbits. If one of the values of $\bar{t}_1$ is greater than the crossing bifurcation value, then this means that there is a sliding or crossing-and-sliding periodic orbit.

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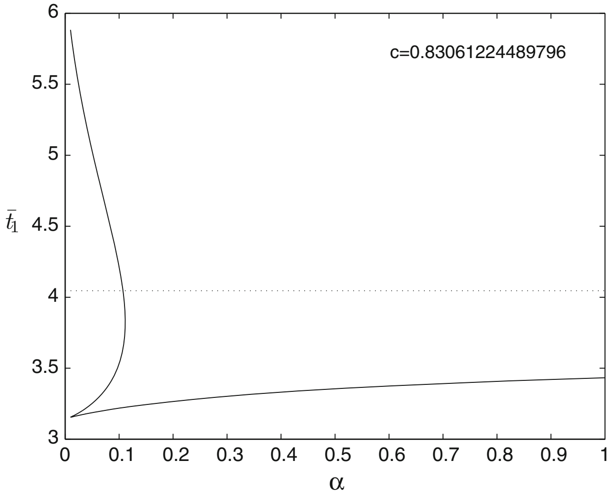

Fig. 16 Plot of $\bar{t}_1$ in function of α for $c \simeq 0.83061$. Here $a = b = d = m = 1$. The *dashed line* in the plot is the value of t_1 at the crossing bifurcation

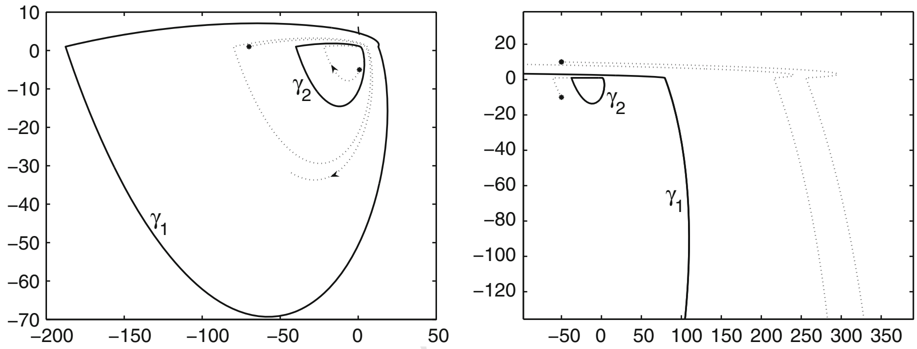

Fig. 17 Behavior of system (8) close to the periodic orbits at the fold

From the picture in Fig. 16, it is clear that there are two fold bifurcations of periodic orbits.

Example 31 (Fold of Periodic Orbits and Stability) Through standard fold location techniques, we have computed the fold points: there is one fold for $\alpha \simeq 0.1130986266212$ and one for $\alpha \simeq 0.01036814335189$. The periodic orbits at these two different parameter values are shown in Fig. 17. Consider first the case of $\alpha = 0.1130986266212$ on the left of Fig. 17. The system has two crossing periodic orbits, in bold in the figure: in the figure, γ_1 corresponds to the smaller value of $\bar{t}_1$ while γ_2 corresponds to the value at the fold. Observe that γ_2 is stable from inside and unstable from outside. The value of $\bar{t}_1$ at the fold is $\bar{t}_1 \simeq 3.82459032689332$ and, using this, we can compute $\bar{x}_1$, $\bar{x}_2$ and $\bar{t}_2$ explicitly.

The stability properties of both orbits can be studied using the Poincaré map or, equivalently, via the monodromy matrix. We will use here the approach based on the monodromy matrix. We denote with $\bar{x}_1$ and $\bar{x}_2$ the two intersections of the periodic orbit with Σ^+ and Σ^- respectively and with $\bar{t}_1$ and $\bar{t}_2$ the first return time to Σ^- and Σ^+ respectively. To form the monodromy matrix we must take into account the saltation or jump matrices, i.e., fun-

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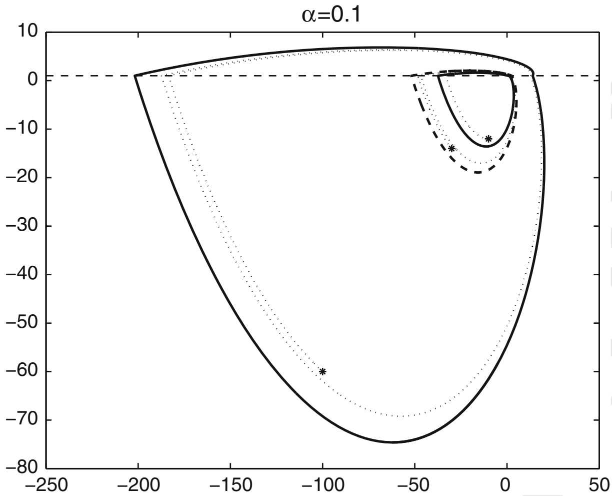

Fig. 18 Periodic orbits of system (8) for $a = b = d = m = 1$, $c \simeq 0.83061$ and $\alpha = 0.1$. The *dotted curve* is the unstable periodic orbit, the other two are the stable ones

damental matrices at the discontinuity (see [1,3,7–9]). The fundamental matrix solution has the following expression

$$X(T) = S_{21}\Phi_2(\bar{t}_2, \bar{t}_1)S_{12}\Phi_1(\bar{t}_1, 0), \tag{44}$$

where $\Phi_j(t, t_0)$ is the principal matrix solution of $\dot{X} = A_j X,\ \ X(t_0) = I$, at time t, and S_{12} and S_{21} are the two saltation matrices, defined as

$$S_{12} = I + [f_2(\bar{x}_2\ \ m) - f_1(\bar{x}_2\ \ m)]\frac{n^T(\bar{x}_2\ \ m)^T}{n^T f_1(\bar{x}_2\ \ m)},$$

$$S_{21} = I + [f_2(\bar{x}_1\ \ m) - f_1(\bar{x}_1\ \ m)]\frac{n^T(\bar{x}_1\ \ m)^T}{n^T f_1(\bar{x}_1\ \ m)}.$$

Since both periodic orbits must give a multiplier equal to 1, the other multiplier is $\det(X(T)) = e^{2(-a\bar{t}_2+c\bar{t}_1)}\det(S_{12})\det(S_{21})$ for which we have the following explicit expression

$$\det(X(T)) = e^{2(-a\bar{t}_2+c\bar{t}_1)}\frac{\bar{x}_2 + \frac{a}{\alpha b}m}{\bar{x}_2 - \frac{c}{d}m}\,\frac{\bar{x}_1 - \frac{c}{d}m}{\bar{x}_1 + \frac{a}{b\alpha}m}. \tag{45}$$

For γ_2, we obtain: $\lambda_1 \simeq 1$, and $\lambda_2 \simeq 0.998052$. On the right of Figure 17 is depicted the case for the second fold at $\alpha \simeq 0.01036814335189$. We plot only one arc of the periodic orbit γ_1 at the fold together with the sliding orbit γ_2. The asterisks are two initial conditions that do not belong to the periodic orbits and the dotted lines are the corresponding solutions. From the plot we observe that γ_1 is stable from the outside and unstable from the inside. At γ_1, the corresponding value of $\bar{t}_1$ is $\bar{t}_1 \simeq 3.15511532823135$ and $\bar{x}_1 \simeq 79.32564021634117$. Computing the eigenvalues of the monodromy matrix we obtain $\lambda_1 \simeq 1$ and $\lambda_2 \simeq 1.00009$.

Finally in Fig. 18 we plot the orbits of the system for $\alpha = 0.1$. This value is between the two folds and, looking at Fig. 16 we expect the system to have three periodic orbits. Indeed we see the three orbits plotted in bold in Fig. 18. The dotted periodic orbit is unstable while

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the two solid orbits are stable. The other orbits in the plot correspond to the initial conditions marked with the stars. The inner orbit is a crossing-and-sliding periodic orbit.

5 A Model Nonlinear Problem

In this section we consider a weak non linear perturbation of system (5) with $A_{1,2}$ as in (6), namely

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} + \epsilon g_1(\mathbf{x}),\ y < m, \\ A_2\mathbf{x} + \epsilon g_2(\mathbf{x}),\ y > m, \end{cases} \tag{46}$$

with g_1 and g_2 continuously differentiable functions and such that $g_1(0) = g_2(0) = 0$, ϵ sufficiently small and $m \geq m_0 > 0$ (and m_0 uniformly bounded away from 0). As usual, let $n = (0, 1)^T$ denote the normal to $\Sigma = \{(x, y)\ |y = m\}$.

As long as we stay away from the bifurcation values of the underlying linear problem, for ϵ small these types of systems exhibit a behavior similar to the linear case.

First of all, note that as long as ϵ is sufficiently small and $\epsilon n^T \frac{d}{dx} g_1(x, y)\Big|_{(-\frac{a}{b}m,m)} \neq b$ and $\epsilon n^T \frac{d}{dx} g_2(x, y)\Big|_{(\frac{c}{d}m,m)} \neq d$, the Implicit Function Theorem guarantees that there is an attractive sliding region $\bar{\mathcal{S}}$ for system (46). We denote with x_R and x_L respectively the right and left endpoints of $\mathcal{S}$, $\mathcal{S} = \{(x, m)|x_L < x < x_R\}$ and with $\tilde{\Sigma}_+ = \{(x, m)|x \geq x_R\}$ and $\tilde{\Sigma}_- = \{(x, m)|x \leq x_L\}$. Clearly $x_{R,L} = x_{R,L}(\epsilon)$, and $x_R(0) = \frac{c}{d}m$ and $x_L(0) = -\frac{a}{b}m$.

In what follows we will study the behavior of system (46) as $\frac{c}{d}$ varies. Below, the values $\frac{c_j}{d_j}$ $(j = 0, 1)$ are the critical values for the linear problem (5) in Sect. 3.

(a) Let $\frac{c}{d}$ be such that: $\frac{a}{b} > \frac{\bar{a}}{\bar{b}} \geq \frac{c}{d} \geq \frac{\bar{c}_1}{\bar{d}_1} > \frac{c_1}{d_1}$, where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, depend on ϵ, but are bounded away from 0 uniformly in ϵ.

The linear system (5) has an hyperbolic crossing periodic orbit γ for $\frac{c}{d}$ in this range. In Sect. 3 we already defined a Poincaré map for (5) and we denoted with $\bar{x}$ its fixed point. We want to define a Poincaré map $\tilde{P} = \tilde{P}(x, \epsilon)$ for the nonlinear problem (46) and show that there exists an $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, $\tilde{P}$ has a fixed point. We will use the Implicit Function Theorem and the fact that $\bar{x}$ is an hyperbolic fixed point of P. In what follows we will use some of the results and the notations of Sect. 3, in particular insofar as $\bar{t}_1, \bar{x}$, etc..

We denote with $\tilde{\varphi}_{1,2}(t, x, y, \epsilon)$ the flow of system (46) respectively for $y < m$ and $y > m$. For $\epsilon = 0$, $\tilde{\varphi}_{1,2}(t, x, y, 0) = \varphi_{1,2}(t, x, y)$. We denote with $\tilde{t}_{1,2}(x, \epsilon)$ the return time of $\tilde{\varphi}_{1,2}(t, x, m, \epsilon)$ to Σ. Then $\tilde{t}_{1,2}(x, 0) = t_{1,2}(x)$ and smoothness of $\tilde{t}_{1,2}$ and $\tilde{\varphi}_{1,2}$ with respect to x and ϵ implies that there exist an $\epsilon_1 > 0$ and a $\delta_1 > 0$ such that for all $\epsilon \in [0, \epsilon_1)$ and all $x \in (\bar{x} - \delta_1, \bar{x} + \delta_1)$, $\tilde{t}_{1,2}(x, \epsilon)$ is smooth and well defined and $\tilde{\varphi}_1(\tilde{t}_1(x), x, \epsilon) \in \tilde{\Sigma}_-$ and $\tilde{\varphi}_2(\tilde{t}_2(x), x, \epsilon) \in \tilde{\Sigma}_+$.

We can define the following maps for $\epsilon \in [0, \epsilon_1)$ and $x \in (\bar{x} - \delta_1, \bar{x} + \delta_1)$

$$\tilde{P}_1(x, \epsilon) : \tilde{\Sigma}_+ \to \tilde{\Sigma}_-,\ \ \tilde{P}_1(x, \epsilon) = \varphi_1(\tilde{t}_1(x, \epsilon), x, \epsilon),$$

and

$$\tilde{P}_2(x, \epsilon) : \tilde{\Sigma}_- \to \tilde{\Sigma}_+,\ \ \tilde{P}_2(x, \epsilon) = \varphi_2(\tilde{t}_2(x, \epsilon), x, \epsilon).$$

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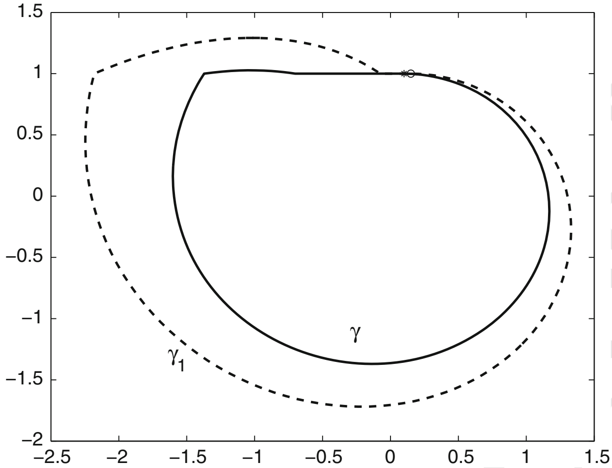

Fig. 19 Crossing and sliding periodic orbits for the linear system (*solid line*) and for its nonlinear perturbation (*dotted line*). The *asterisk* is the tangential exit point for the linear system, while the *circle* is the one for the nonlinear system

We define the Poincaré map for system (46) as $\tilde{P}(x, \epsilon) : \tilde{\Sigma}_+ \rightarrow \tilde{\Sigma}_+$, $\tilde{P}(x, \epsilon) = \tilde{P}_2(\tilde{P}_1(x, \epsilon), \epsilon)$. Clearly $\tilde{P}(x, 0) = P(x)$. Moreover $\tilde{P}_x(x, \epsilon)\Big|_{(\bar{x},0)} = P_x(\bar{x}) \neq 1$ since γ is hyperbolic. Hence there exists an $\epsilon_0 \leq \epsilon_1$ such that for all $\epsilon \in (0, \epsilon_0)$ there is an $x = x(\epsilon)$ such that $\tilde{P}(x(\epsilon), \epsilon) = x(\epsilon)$. We proved the following theorem.

Theorem 32 *In this case (a), there exists $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0)$ the system* (46) *has a unique continuous asymptotically stable crossing periodic orbit γ reducing to the crossing periodic orbit of the linear problem for $\epsilon = 0$.*

(b) Now take $\frac{c}{d}$ such that $\frac{c_1}{d_1} > \frac{\tilde{c}_1}{\tilde{d}_1} \geq \frac{c}{d} \geq \frac{\tilde{c}_0}{\tilde{d}_0} > \frac{c_0}{d_0}$, where $\tilde{c}_0, \tilde{d}_0, \tilde{c}_1, \tilde{d}_1$, are uniformly bounded away from 0 in ϵ.

System (5) has a crossing-and-sliding periodic orbit γ that is stable and finitely reached. This γ starts at $(\frac{c}{d}m, m)$, enters R_1, crosses Σ^- at $\varphi_1(t_1(\frac{c}{d}m), \frac{c}{d}m, m) = (x_1, m)$, enters R_2, meets S at $\varphi_2(t_2(x_1), x_1, m) = (x_2, m)$ and starts sliding on S until it reaches the tangential exit point $(\frac{c}{d}, m)$. Then, since $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are smooth in ϵ and x, there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ $\tilde{t}_1(x_R(\epsilon), \epsilon)$ is well defined and, for $\tilde{x}_2 = \tilde{\varphi}_1(\tilde{t}_1(x_R(\epsilon)), x_R(\epsilon), \epsilon)$, then also $\tilde{t}_2(\tilde{x}_2, \epsilon)$ is well defined and $\tilde{\varphi}_2(\tilde{t}_2(\tilde{x}_2), \tilde{x}_2, m) \in \mathcal{S}$. This implies the existence of a sliding and crossing periodic orbit also for the perturbed nonlinear system.

Theorem 33 *In this case (b), there exists $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0)$ the system* (46) *has a unique continuous crossing and sliding periodic orbit γ reducing to the crossing and sliding periodic orbit of the linear problem for $\epsilon = 0$.*

In Fig. 19 we plot the periodic orbit γ of the linear system and γ_1 of the nonlinear system for $a = b = d = m = 1$, $c = 0.1$, $\epsilon = 0.1$, $g_1(\mathbf{x}) = \left(x \;\; \frac{y}{1+y^2}\right)$, $g_2(\mathbf{x}) = \left(x^2 \;\; x^2 + y^2\right)$.

(c) Finally, take $\frac{c}{d}$ such that $\frac{c_0}{d_0} > \frac{\bar{c}_0}{\bar{d}_0} \geq \frac{c}{d} \geq \eta > 0$, where $\bar{c}_0$, $\bar{d}_0$, and η are bounded away from 0 uniformly in ϵ.

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Now system (5) with $A_{1,2}$ has a sliding periodic orbit γ that is stable and finitely reached. This γ starts at $(\frac{c}{d}m, m)$, enters R_1, meets S at $\varphi_1(t_1(\frac{c}{d}m, m), \frac{c}{d}m, m) = (x_1, m)$ and starts sliding on S until it reaches $(\frac{c}{d}m, m)$ again. Now, $\tilde{\varphi}_1$ and $\tilde{t}_1$ are smooth in x and ϵ and $\tilde{\varphi}_1(t, x, y, 0) = \varphi_1(t, x, y)$ and $\tilde{t}_1(x, 0) = t_1(x)$. Hence, there exist an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $\tilde{t}_1(x_R(\epsilon))$ is well defined and $\tilde{\varphi}_1(\tilde{t}_1(x_R(\epsilon)), x_R(\epsilon), m, \epsilon) \in \mathcal{S}$. From this, we get

Theorem 34 *In case (c), there exists* $\epsilon_0 > 0$ *such that for all* $\epsilon \in [0, \epsilon_0)$ *system* (46) *has a unique continuous sliding periodic orbit* γ*, reducing to the sliding periodic orbit of the linear problem for* $\epsilon = 0$.

Acknowledgments This work was perfomed while the second author was on leave from the University of Bari, Bari, Italy, and in visit to the School of Mathematics of the Georgia Institute of Technology, whose support is gratefully acknowledged. The first author is grateful to the University of Jilin, Changchun, China, where he spent part of the Summer 2013 as Tang Aoqing Professor.

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