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A Filippov sliding vector field on an attracting co-dimension 2 discontinuity surface, and a limited loss-of-attractivity analysis ☆

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ABSTRACT

We consider sliding motion, in the sense of Filippov, on a discontinuity surface Σ of co-dimension 2. We characterize, and restrict to, the case of Σ being *attractive through sliding*. In this situation, we show that a certain Filippov sliding vector field f_F (suggested in Alexander and Seidman, 1998 [2], di Bernardo et al., 2008 [6], Dieci and Lopez, 2011 [10]) exists and is unique. We also propose a characterization of *first order exit conditions*, clarify its relation to generic co-dimension 1 losses of attractivity for Σ, and examine what happens to the dynamics on Σ for the aforementioned vector field f_F. Examples illustrate our results.

1. Introduction

In this work, we discuss some theoretical questions related to differential equations with discontinuous right-hand side. The setting we consider is the classical one of Filippov, see [11], whereby one seeks a solution of an initial value problem of ordinary differential equations in which the right-hand

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side (the vector field) varies discontinuously as the solution trajectory reaches one or more surfaces, called *discontinuity or switching surfaces*, but it is otherwise smooth. In the literature, these are called "piecewise smooth systems," hereafter PWS systems for short (see [6]). The values where a trajectory reaches a discontinuity surface are called *events*, and we will henceforth assume that the events are isolated. In general, there are a number of possible outcomes as the solution reaches a discontinuity surface. For example, in so-called *impact* systems, the solution experiences a jump discontinuity; see [24]. However, in this work, we will only consider the case where the solution remains continuous (though not necessarily differentiable) past an event point. In this case, loosely speaking, there are two things which can occur as we reach a surface of discontinuity: we may cross it, or we may stay on it, in which case a description of the motion on the surface, *sliding motion*, will be required. This latter case is particularly interesting and important, and calls for a separate theoretical and numerical analysis.

Systems with discontinuous right-hand sides appear pervasively in applications of various nature. For a sample of references in the context of control, see e.g. [26,27,25], and in the context of biological systems, see e.g. [4,5,13,22]; for works on the class of complementarity systems, see [14], for works from the point of view of bifurcations of dynamical-systems see [7,16,15,19]; and, of course, see the classical references [3,11,26,27] for a thorough theoretical introduction to these systems. Because of their ubiquity in applications of different nature, PWS systems are receiving a lot of attention, and to witness, we mention the recent books [1,6] which deal with specific questions of bifurcations and simulations for PWS systems. Indeed, many studies on PWS systems rely on numerical simulation, and the cited text [1] has a nice collection of different case studies for which specific numerical methods have been devised.

But, in spite of the attention that PWS systems have been receiving, systems with discontinuous right-hand sides still present several outstanding theoretical and practical challenges. In particular, the widely adopted Filippov extension to define the vector field in a sliding regime is ambiguous, in general, when sliding has to take place on a surface of co-dimension 2 intersection of two co-dimension 1 surfaces (see [11]), even if the surface attracts nearby dynamics. Simple situations when this ambiguity is absent are in [21,23,26], but for general PWS systems of the type we are going to consider these approaches are not generally applicable, and the techniques which are used in practice are essentially: (i) *globally smoothing* out the vector field, see for instance [8,20,22], but see also [17,18] for pitfalls caused by altering the dynamics; (ii) *blending*, i.e., essentially interpolating, the vector fields in the neighborhood of the discontinuity surfaces (see for instance [2]); (iii) impose further constraints on the class of Filippov vector fields, in order to further regularize the problem on the co-dimension 2 surface.

Here, we will focus on (iii), and our main goal in this work is to validate a certain choice of Filippov sliding vector field in the case of a co-dimension 2 sliding surface which attracts nearby dynamics. We will characterize the type of attractive surfaces of interest and show that the proposed Filippov vector field is always well defined in these cases. We will further examine and characterize first order exit conditions.

The vector field we will consider, (8), was first suggested in [2] and there justified for what we will call "nodally attractive" Σ (see below). Later, this formulation was also suggested in [6, p. 88], but with no justification about its feasibility. Recently, it was again reconsidered in [10], where it was proven to be well defined in a few important cases (see below). Finally, besides the above references where the specific vector field (8) was proposed, we would also like to mention its relation to the limiting vector field associated to a natural (global) spatial regularization of the original discontinuous problem (under nodal attractivity conditions; see [8]).

The model we consider is the following PWS initial value problem (IVP)

$$\dot{x} = f(x), \qquad f(x) = f_i(x), \quad x \in R_i,\ i = 1, 2, 3, 4, \tag{1}$$

with initial condition $x(0) = x_0$, and where $t \in [0, T]$. In the above, for $i = 1, 2, 3, 4$, $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and we may as well think that $\mathbb{R}^n = \overline{\bigcup_i R_i}$. Moreover, each f_i is smooth on R_i and $\mathbb{R}^n \setminus \bigcup_i R_i$ has zero (Lebesgue) measure. Further, we will assume that the R_i's are separated (locally) by implicitly defined smooth surfaces of co-dimension 1, $\Sigma_1 = \{x\colon\ h_1(x) = 0,\ h_1 : \mathbb{R}^n \to \mathbb{R}\}$ and $\Sigma_2 = \{x\colon\ h_2(x) = 0,\ h_2 : \mathbb{R}^n \to \mathbb{R}\}$, as follows (this labeling is with no loss of generality):

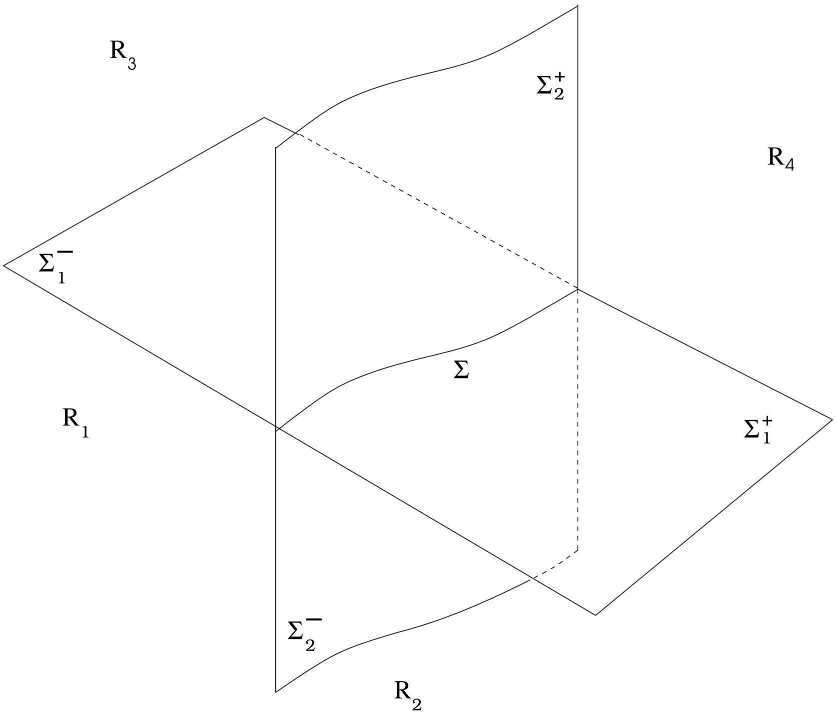
$$R_1\colon\ \text{when } h_1 < 0,\ h_2 < 0, \qquad R_2\colon\ \text{when } h_1 < 0,\ h_2 > 0,$$
$$R_3\colon\ \text{when } h_1 > 0,\ h_2 < 0, \qquad R_4\colon\ \text{when } h_1 > 0,\ h_2 > 0. \tag{2}$$

Our interest is to understand what happens on the co-dimension 2 surface Σ, intersection of the two co-dimension 1 discontinuity surfaces Σ_1 and Σ_2. So, we'll consider

$$\Sigma = \left\{x \in \mathbb{R}^n\colon\ h(x) = 0,\ h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}\right\},$$

and assume that for all $x \in \Sigma$: $\nabla h_j(x) \neq 0$, $h_j \in \mathcal{C}^k$, $k \geqslant 2$, $j = 1, 2$, and $\nabla h_1(x)$, $\nabla h_2(x)$, are linearly independent.

For later use, we will also adopt the notation $\Sigma^+_{1,2}$ and $\Sigma^-_{1,2}$ to denote the set of points $x \in \Sigma_{1,2}$ for which we also have $h_{2,1}(x) > 0$ or $h_{2,1}(x) < 0$. E.g., $\Sigma^+_1 = \{x \in \Sigma_1$ and $h_2(x) < 0\}$; see the figure below.



Regions $\Sigma^{\pm}_{1,2}$ and the co-dimension 2 discontinuity surface Σ.

Now, suppose that Σ *attracts in finite time* (see below) nearby dynamics, so that trajectories become constrained to remain on Σ (*sliding motion*): *What is the vector field associated to this sliding motion*?

Following [11], we will call Filippov sliding vector field (on Σ) any vector field of the form (for $x \in \Sigma$)

$$F(x) = \sum_{i=1}^{4} \lambda_i(x) f_i(x), \quad \text{where } \lambda_i(x) \geqslant 0, \quad \text{and} \quad \sum_{i=1}^{4} \lambda_i(x) = 1, \tag{3}$$

subject to the constraint that $F(x)$ lies in T_Σ, the tangent plane to Σ at x:

$$\left(\nabla h_j(x)\right)^T F(x) = 0, \quad \text{for } j = 1, 2. \tag{4}$$

Unfortunately, as it is immediately clear, (3)–(4) in general fail to select uniquely the coefficients λ_i, $i = 1, 2, 3, 4$, since we have three equations in four unknowns.

Example 1. The case when Σ is of co-dimension $p = 1$ is well understood, and Filippov construction for attractive Σ is not ambiguous in such case. We remark that, in its original form, Filippov theory is a first order theory, since it relies on non-vanishing of the projections of the vector field onto the tangent space of Σ. To witness, one has two regions, R_1, R_2, with vector fields f_1 and f_2, and $\Sigma = \{x \in \mathbb{R}^n\colon\ h(x) = 0,\ h : \mathbb{R}^n \to \mathbb{R}\}$. Now in R_1 we have $h(x) < 0$, and in R_2 we have $h(x) > 0$. In this case, (first order) attractivity of Σ means that for $x \in \Sigma$ (hence, near it) we have

$$\nabla h(x)^T f_1(x) > 0 \quad \text{and} \quad \nabla h(x)^T f_2(x) < 0, \tag{5}$$

and the Filippov sliding vector field on Σ is uniquely defined as (cf. with (3)–(4)):

$$x' = (1-\alpha) f_1 + \alpha f_2, \quad \alpha = \frac{\nabla h(x)^T f_1(x)}{\nabla h(x)^T f_1(x) - \nabla h(x)^T f_2(x)}. \tag{6}$$

We further observe that, when the quantities $\nabla h(x)^T f_{1,2}(x)$ are bounded away from 0, then Σ is reached in finite time by trajectories starting near Σ.

For completeness, let us also point out that in case at $x \in \Sigma$ one has

$$\nabla h(x)^T f_1(x) < 0 \quad \text{and} \quad \nabla h(x)^T f_2(x) > 0, \tag{7}$$

then Σ is called repulsive. Eq. (6) still defines a Filippov sliding vector, giving so-called *repulsive sliding* motion; however, solutions are no longer unique (at any $x \in \Sigma$, satisfying (7), we may leave with f_1, f_2, or proceed according to repulsive sliding motion).

To reiterate, in the case of a co-dimension 1 attractive discontinuity surface, Filippov sliding motion is well defined according to (6). Indeed, the situation of sliding motion in this co-dimension 1 case is much better understood and Filippov theory is a powerful (and widely adopted) first order theory clarifying not only how sliding motion on Σ will take place, but also when one should leave Σ and enter in R_1 or R_2. Namely, at first order, one will leave Σ and enter R_1, or R_2, when – at a value x – one (but not both) of these conditions is satisfied: $\nabla h(x)^T f_1(x) = 0$, or $\nabla h(x)^T f_2(x) = 0$.

To avoid the aforementioned ambiguity of Filippov sliding vector field on Σ, in [2,6,10] the authors restricted to a special convex combination. Namely, they considered using

$$f_F = (1-\alpha)(1-\beta) f_1 + (1-\alpha)\beta f_2 + \alpha(1-\beta) f_3 + \alpha\beta f_4, \tag{8}$$

where α and β need to be smooth functions of $x \in \Sigma$, to take values in $[0, 1]$, and their values need to be chosen so that $\nabla h_1^T f_F = \nabla h_2^T f_F = 0$. By letting

$$w_j^1 = \nabla h_1^T f_j, \quad \text{and} \quad w_j^2 = \nabla h_2^T f_j, \quad j = 1, 2, 3, 4, \tag{9}$$

then α and β will need to satisfy the nonlinear system

$$(1-\alpha)(1-\beta)\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix} + (1-\alpha)\beta\begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix} + \alpha(1-\beta)\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix} + \alpha\beta\begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix} = 0. \tag{10}$$

Remark 2. In (8)–(10), and elsewhere in this work, we will omit explicitly writing the dependence on x when it is clear from the context. For example, it is understood that (8) is meaningful for $x \in \Sigma$ and that (10) has to be satisfied for $x \in \Sigma$. However, the quantities w^i_j in (9), $i = 1, 2$, $j = 1, 2, 3, 4$, will need to be well defined in a neighborhood of Σ.

The main goal of this work will be precisely to understand solvability of the algebraic system (10) under realistic assumptions on the dynamics of the trajectories of (1) near Σ. As far as we know, the present work is the most complete effort to date on justifying selection of a Filippov vector field, based upon attractivity properties of a co-dimension 2 discontinuity surface Σ as we have. In fact, we do not know of any other choice that would pass as rigorous an examination as the one we have provided, under similar attractivity assumptions (giving a well-defined and smoothly varying vector field on Σ), let alone the analysis of first order exit conditions.

2. Sliding vector field on attractive Σ of co-dimension 2

In this section, we tackle this problem: "*Given $x_0 \in \Sigma$, show when the vector field* (8) *exists, unique, and is smooth*." In other words, we study when (10) has a unique solution $(\alpha, \beta) \in [0, 1]^2$, smoothly varying for $x \in \Sigma$.

Below, we give a general theorem which shows that the selection of the specific Filippov vector field on Σ given by (8) is justified whenever Σ is attractive in the following sense to be more precisely clarified below: "We will require Σ to attract nearby dynamics and to be reachable through attractive sliding motion on at least one of the sub-surfaces $\Sigma^{\pm}_{1,2}$."

To begin with, let us assume that trajectories of the PWS system (1) exist in a neighborhood U of Σ, deprived of Σ itself, in the sense of Filippov. This must be understood to imply that in case the value of x_0 is on either Σ_1 or Σ_2 (but not on $\Sigma_1 \cap \Sigma_2$), there may be sliding motion on Σ_1 or Σ_2 according to Filippov's first order theory; see Example 1. However, we also remark that motion in a neighborhood of Σ may not be uniquely defined, such as when $x_0 \in \Sigma_1$ (or Σ_2) but sliding motion on Σ_1 is repulsive. With this in mind, we will still write $\phi^t(x_0)$ to indicate a continuous Filippov trajectory of the system.

The general characterization of *attractive* Σ will require that Σ be stable (with respect to the initial conditions) and approached by trajectories of the system. For completeness, we propose the following definition of stability of Σ: "*For any $x_0 \in \Sigma$, and for any $\epsilon > 0$ sufficiently small, there is $\delta > 0$ such that if $u \in B_\delta(x_0)/(\Sigma \cap B_\delta(x_0))$, then $d(\phi^t(u) - \Sigma) \leqslant \epsilon$ for all $t \geqslant 0$.*"

However, in many situations of interest, we will want a more restrictive condition requiring not only that Σ is attractive, but it is also approached in finite time.

Definition 1. Σ attracts in finite time trajectories of (1), if:

(i) Σ is stable;
(ii) for any $x_0 \in U/(\Sigma \cap U)$, where U is a neighborhood of Σ, there exists a first (finite) time $\tau(x_0) \geqslant 0$ such that $\phi^{\tau(x_0)}(x_0) \in \Sigma$.

We stress once more that Definition 1 does not require right uniqueness of solutions in a neighborhood of Σ.

Now, given our goal to show that, when $x_0 \in \Sigma$ and Σ satisfies certain first order attractivity conditions, sliding motion according to (8) is well defined, we will consider a certain type of attractivity of Σ: *attractivity through sliding* on (some of) $\Sigma^{\pm}_{1,2}$, which we will also call *nodal or partially nodal attractivity*. In practice, with the exception of spiral like attractivity (see [10] for a possible characterization of this case, which means that a trajectory starting near Σ will reach it through repeated crossings of $\Sigma^{\pm}_1$, $\Sigma^{\pm}_2$), this includes all cases when Σ is attractive with respect to a first order theory.

2.1. Nodal and partially nodal attractivity

Nodal attractivity means that there is attractive sliding motion along each of $\Sigma^{\pm}_{1,2}$ towards Σ. (This case has been already treated in [2,10].) Partially nodal attractivity means that there is attractive sliding motion along at least one of Σ^{+}_{1}, Σ^{-}_{1}, Σ^{+}_{2} or Σ^{-}_{2}, though not along all of them. A chief novelty for this case is that the solution in a neighborhood of Σ might fail to be right unique. Indeed, along one of the $\Sigma^{\pm}_{1,2}$ we might have repulsive sliding towards Σ, while the solution on Σ will still be well defined and all the trajectories in a neighborhood of Σ are still attracted towards Σ. See Theorem 7.

To characterize the above mentioned cases, the setting is the following. First of all, we will assume that (recall (9))

$$w^i_j(x) \text{ are bounded away from } 0, \quad i=1,2,\ j=1,2,3,4,\ x\in\Sigma, \tag{11}$$

and are well defined in a neighborhood of Σ. Note that (11) is a first order condition, essentially implying that no trajectory can approach Σ tangentially from a region R_j, $j=1,2,3,4$. Moreover, the assumption of the w^i_j's being bounded away from 0 for $x\in\Sigma$ will eventually imply that (in the attractive cases) Σ is reached in finite time; see below.

We are now ready for the following assumptions on the w^i_j, for $j=1,2,3,4$, and $i=1,2$.

Assumptions 1.

(a) $(w^1_j(x), w^2_j(x))$ do not have the same signs as $(h_1(x), h_2(x))$ for $x\in R_j$, $j=1,2,3,4$.

(b) At least one pair of the relations [(1^+) and (1^+_a)], or [(1^-) and (1^-_a)], or [(2^+) and (2^+_a)], or [(2^-) and (2^-_a)], is satisfied on Σ and in a neighborhood of Σ, where

$$\begin{aligned}
&(1^+)\quad w^1_2>0,\ w^1_4<0, &&(1^+_a)\quad \frac{w^2_2}{w^1_2}-\frac{w^2_4}{w^1_4}<0,\\
&(1^-)\quad w^1_1>0,\ w^1_3<0, &&(1^-_a)\quad \frac{w^2_3}{w^1_3}-\frac{w^2_1}{w^1_1}<0,\\
&(2^+)\quad w^2_3>0,\ w^2_4<0, &&(2^+_a)\quad \frac{w^1_3}{w^2_3}-\frac{w^1_4}{w^2_4}<0,\\
&(2^-)\quad w^2_1>0,\ w^2_2<0, &&(2^-_a)\quad \frac{w^1_2}{w^2_2}-\frac{w^1_1}{w^2_1}<0.
\end{aligned}$$

(c) If any of $(1^{\pm})$ or $(2^{\pm})$ is satisfied, then $(1^{\pm}_a)$ or $(2^{\pm}_a)$ must be satisfied as well.

Let us clarify the meaning of Assumptions 1 insofar as the dynamics of the system. Assumption 1(a) implies that the vector fields f_j, $j=1,\dots,4$, must point towards at least one of $\Sigma_{1,2}$. Assumption 1(b) guarantees that there is attractive sliding towards Σ along at least one of the $\Sigma^{\pm}_{1,2}$. Assumption 1(c) states that if attractive sliding occurs along $\Sigma^{\pm}_{1,2}$ it must be towards Σ.

Remark 3. Observe that Assumptions 1 do not exclude the case of repulsive sliding along $\Sigma^{\pm}_{1,2}$. However, Assumption 1(a) guarantees that, if repulsive sliding occurs, then it must be towards Σ. This means that even if the solution in a neighborhood of Σ might be not right unique, every trajectory in a neighborhood of Σ is still attracted towards Σ.

The following lemma clarifies that Assumptions 1 guarantee that Σ is attractive in finite time.

Lemma 4. *Let Assumptions 1 be satisfied and further let* (1_a^+), (1_a^-), (2_a^+), *and* (2_a^-) *hold uniformly*; *that is,* (1_a^+) *be replaced by* $\frac{w_2^2}{w_2^1} - \frac{w_4^2}{w_4^1} \leqslant -\lambda_1^+ < 0$, *and similarly for the others. Then,* Σ *is attractive in finite time.*

Proof. Because of 1(a), then – for at least one $j = 1, 2, 3, 4$ – the vector field f_j point towards at least one of Σ_1 or Σ_2, and a solution with initial condition in R_j will reach Σ_1 or Σ_2 (or Σ itself). If it reaches Σ_1 or Σ_2, then it will cross it and enter into a new region, or – because of Assumption 1(c) – it will slide towards Σ. To complete the proof, we observe that because of Assumption 1(b), we must slide towards Σ on at least one of the $\Sigma_{1,2}^{\pm}$. For example, suppose that this is happening on Σ_1^+, so that Assumptions 1(b)(1^+) and (1_a^+) are satisfied. Consider the function $V(x(t)) = h_2(x(t))$ (which is positive for $x \in \Sigma_1^+$ and not on Σ). Taking the total derivative, we obtain

$$\frac{dV}{dt} = \nabla h_2\big(x(t)\big)^T \big(\big(1-\alpha^+\big) f_2 + \alpha^+ f_4\big)$$

where α^+ is found as in (6): $\alpha^+ = \frac{w_2^1}{w_2^1 - w_4^1}$. Therefore, using Assumptions 1(b)(1^+) and (1_a^+), we get $\frac{dV}{dt} \leqslant \frac{-\lambda_1^+}{w_2^1 - w_4^1} < 0$ and the claim follows. Similarly in the other cases. □

We are now ready to proceed justifying the choice (8). Rewrite the nonlinear system (10) for α and β as

$$\begin{gathered} \beta\big(L_2^i(\alpha) - L_1^i(\alpha)\big) + L_1^i(\alpha) = 0, \\ L_1^i(\alpha) = (1-\alpha)w_1^i + \alpha w_3^i, \qquad L_2^i(\alpha) = (1-\alpha)w_2^i + \alpha w_4^i, \end{gathered} \tag{12}$$

$$\begin{gathered} \alpha\big(C_2^j(\beta) - C_1^j(\beta)\big) + C_1^j(\beta) = 0, \\ C_1^j(\beta) = (1-\beta)w_1^j + \beta w_2^j, \qquad C_2^j(\beta) = (1-\beta)w_3^j + \beta w_4^j, \end{gathered} \tag{13}$$

with $i, j = 1, 2$, and $j \neq i$. Using Eqs. (12)–(13) we can write β and α as

$$\beta = \frac{L_1^i(\alpha)}{L_1^i(\alpha) - L_2^i(\alpha)}, \quad \text{for } L_1^i(\alpha) - L_2^i(\alpha) \neq 0,\ i = 1 \text{ or } 2, \tag{14}$$

$$\alpha = \frac{C_1^j(\beta)}{C_1^j(\beta) - C_2^j(\beta)}, \quad \text{for } C_1^j(\beta) - C_2^j(\beta) \neq 0,\ j = 1 \text{ or } 2. \tag{15}$$

Substituting (15) into (10), we obtain the following function of β:

$$f_1^j(\beta) = \frac{C_1^j(\beta)[(1-\beta)w_3^i + \beta w_4^i] - C_2^j(\beta)[(1-\beta)w_1^i + \beta w_2^i]}{C_1^j(\beta) - C_2^j(\beta)}.$$

This is a rational function and its zeros for $C_1^j(\beta) - C_2^j(\beta) \neq 0$ are the zeros of the second degree polynomial given by its numerator. We will use the notation $P(\beta)$ for this polynomial when $i = 1$ and $j = 2$ and the notation $\hat{P}(\beta)$ when $i = 2$ and $j = 1$. For completeness, let us explicitly write these formulas; as long as the values of α are well defined (see (15)), we are using:

$$\begin{cases} \alpha = g(\beta) = \dfrac{(1-\beta)w_1^2 + \beta w_2^2}{[(1-\beta)w_1^2 + \beta w_2^2] - [(1-\beta)w_3^2 + \beta w_4^2]}, \\ P(\beta) = \big[(1-\beta)w_1^2 + \beta w_2^2\big]\big[(1-\beta)w_3^1 + \beta w_4^1\big] - \big[(1-\beta)w_3^2 + \beta w_4^2\big]\big[(1-\beta)w_1^1 + \beta w_2^1\big], \end{cases} \tag{16}$$

and

$$\begin{cases} \alpha = \hat{g}(\beta) = \dfrac{(1-\beta)w_1^1 + \beta w_2^1}{[(1-\beta)w_1^1 + \beta w_2^1] - [(1-\beta)w_3^1 + \beta w_4^1]}, \\ \hat{P}(\beta) = \big[(1-\beta)w_1^1 + \beta w_2^1\big]\big[(1-\beta)w_3^2 + \beta w_4^2\big] - \big[(1-\beta)w_3^1 + \beta w_4^1\big]\big[(1-\beta)w_1^2 + \beta w_2^2\big]. \end{cases} \tag{17}$$

Alternatively, we can use (14) into (10) to obtain the following function of α

$$f_2^i(\alpha) = \frac{L_1^i(\alpha)[(1-\alpha)w_2^j + \alpha w_4^j] - L_2^i(\alpha)[(1-\alpha)w_1^j - \alpha w_3^j]}{L_1^i(\alpha) - L_2^i(\alpha)},$$

and the zeros of $f_2^i(\alpha)$ for $L_1^i(\alpha) - L_2^i(\alpha) \neq 0$ are the zeros of the second degree polynomial given by the numerator. We will use the notation $Q(\alpha)$ for this polynomial when $i = 1$ and $j = 2$ and the notation $\hat{Q}(\alpha)$ when $i = 2$ and $j = 1$. That is, as long as the values of β are well defined (see (14)), we use:

$$\begin{cases} \beta = h(\alpha) = \dfrac{(1-\alpha)w_1^1 + \alpha w_3^1}{[(1-\alpha)w_1^1 + \alpha w_3^1] - [(1-\alpha)w_2^1 + \alpha w_4^1]}, \\ Q(\alpha) = \big[(1-\alpha)w_1^1 + \alpha w_3^1\big]\big[(1-\alpha)w_2^2 + \alpha w_4^2\big] - \big[(1-\alpha)w_2^1 + \alpha w_4^1\big]\big[(1-\alpha)w_1^2 + \alpha w_3^2\big], \end{cases} \tag{18}$$

and

$$\begin{cases} \beta = \hat{h}(\alpha) = \dfrac{(1-\alpha)w_1^2 + \alpha w_3^2}{[(1-\alpha)w_1^2 + \alpha w_3^2] - [(1-\alpha)w_2^2 + \alpha w_4^2]}, \\ \hat{Q}(\alpha) = \big[(1-\alpha)w_1^2 + \alpha w_3^2\big]\big[(1-\alpha)w_2^1 + \alpha w_4^1\big] - \big[(1-\alpha)w_2^2 + \alpha w_4^2\big]\big[(1-\alpha)w_1^1 + \alpha w_3^1\big]. \end{cases} \tag{19}$$

The following formal equalities are immediately verified and tell us that effectively $\hat{P}$ and $\hat{Q}$ are not needed and we can just consider the polynomials P and/or Q:

$$\hat{P}(\beta) = -P(\beta), \qquad \hat{Q}(\alpha) = -Q(\alpha). \tag{20}$$

The following relations are immediately obtained from (16), (18) and will be useful

$$P(0) = w_1^2 w_3^1 - w_3^2 w_1^1, \qquad P(1) = w_2^2 w_4^1 - w_4^2 w_2^1,$$
$$Q(0) = w_1^1 w_2^2 - w_2^1 w_1^2, \qquad Q(1) = w_3^1 w_4^2 - w_4^1 w_3^2.$$

Also, we let $(\alpha^+, 1)$, $(\alpha^-, 0)$ $(1, \beta^+)$, $(0, \beta^-)$ be the values of (α, β) that would correspond to sliding (attractive or repulsive) along $\Sigma_1^{\pm}$ and $\Sigma_2^{\pm}$ respectively; that is, as long as the denominators are nonzero, we let:

$$\alpha^+ = \frac{w_2^1}{w_2^1 - w_4^1}, \qquad \alpha^- = \frac{w_1^1}{w_1^1 - w_3^1}, \qquad \beta^+ = \frac{w_3^2}{w_3^2 - w_4^2}, \qquad \beta^- = \frac{w_1^2}{w_1^2 - w_2^2}, \tag{21}$$

and further observe that we have

$$P(\beta^-) = C_2^2(\beta^-)\frac{Q(0)}{w_1^2 - w_2^2}, \qquad P(\beta^+) = -C_1^2(\beta^+)\frac{Q(1)}{w_3^2 - w_4^2},$$
$$Q(\alpha^-) = L_2^1(\alpha^-)\frac{P(0)}{w_1^1 - w_3^1}, \qquad Q(\alpha^+) = -L_1^1(\alpha^+)\frac{P(1)}{w_2^1 - w_4^1}. \tag{22}$$

The next result establishes the relation between the signs of P and Q at 0 and 1, and sliding on $\Sigma_{1,2}^{\pm}$.

Lemma 5. *Under Assumptions 1 the following are true*

(1) $P(1) < 0$ (*respectively,* $Q(1) < 0$) *if repulsive sliding occurs on* Σ_1^+ (*respectively,* Σ_2^+);
(2) $P(0) > 0$ (*respectively,* $Q(0) > 0$) *if repulsive sliding occurs on* Σ_1^- (*respectively,* Σ_2^-);
(3) $\mathrm{sign}(P(1))$ *is undetermined when* $w_2^1 w_4^1 > 0$, $w_2^2 w_4^2 > 0$;
(4) $\mathrm{sign}(Q(1))$ *is undetermined when* $w_3^1 w_4^1 > 0$, $w_3^2 w_4^2 > 0$;
(5) $\mathrm{sign}(P(0))$ *is undetermined when* $w_1^1 w_3^1 > 0$, $w_1^2 w_3^2 > 0$;
(6) $\mathrm{sign}(Q(0))$ *is undetermined when* $w_1^1 w_2^1 > 0$, $w_1^2 w_2^2 > 0$;
(7) $P(1) > 0$ (*respectively,* $Q(1) > 0$), *if there is attractive sliding on* Σ_1^+ (*respectively,* Σ_2^+) *towards* Σ;
(8) $P(0) < 0$ (*respectively,* $Q(0) < 0$), *if there is attractive sliding on* Σ_1^- (*respectively,* Σ_2^-) *towards* Σ.

Proof. We prove (1) and (3), the other cases follow in a similar way.

For (1), if repulsive sliding occurs on Σ_1^+, then we must have: $w_2^1 < 0$, $w_4^1 > 0$. Due to Assumption 1(a), at the same time we must have: $w_2^2 < 0$ and $w_4^2 < 0$ so that (1) is proven. The statements (2), (7) and (8) are proven in a similar way.

For (3), notice that $\mathrm{sign}(w_2^2 w_4^1) = \mathrm{sign}(w_4^2 w_2^1)$ so that $\mathrm{sign}(P(1))$ depends on whether $w_2^2 w_4^1 > w_4^2 w_2^1$ or not. We have two feasible cases.

(i) Case $w_2^1 > 0, w_2^2 < 0$. Here, $P(1) > 0$ if the vector $w_2 = \left[\begin{smallmatrix} w_2^1 \\ w_2^2 \end{smallmatrix}\right]$ is steeper than the vector $w_4 = \left[\begin{smallmatrix} w_4^1 \\ w_4^2 \end{smallmatrix}\right]$ and $P(1) < 0$ otherwise. Either way, (3) follows.
(ii) Case $w_2^1 < 0, w_2^2 < 0$. Here, $P(1) > 0$ if w_4 is steeper than w_2 and $P(1) < 0$ otherwise. Again, (3) follows.

The proof for the statements (4), (5) and (6) is similar to the one above for (3). □

The cases $P(1) = 0$, $Q(1) = 0$, $P(0) = 0$, $Q(0) = 0$ are dealt with in Section 3, since they are first order exit conditions (co-dimension 1 losses of attractivity).

The following proposition is useful to establish the sign of $P(\beta^{\pm})$ and $Q(\alpha^{\pm})$, and it is easy to verify.

Proposition 6. *Using the notation of* (12) *and* (13), *we have the following.*

(1) $L_1^1(\alpha) > 0$ (*respectively,* $L_1^1(\alpha) < 0$) *if there is crossing on* Σ_1^- *in the direction of* R_3 (*respectively,* R_1).
(2) $L_2^1(\alpha) > 0$ (*respectively,* $L_1^1(\alpha) < 0$) *if there is crossing on* Σ_1^+ *in the direction of* R_4 (*respectively,* R_2).
(3) $C_1^2(\beta) > 0$ (*respectively,* $C_1^2(\beta) < 0$) *if there is crossing on* Σ_2^- *in the direction of* R_2 (*respectively,* R_1).
(4) $C_2^2(\beta) > 0$ (*respectively,* $C_2^2(\beta) < 0$) *if there is crossing on* Σ_2^+ *in the direction of* R_4 (*respectively,* R_3).

Finally, the following theorem establishes the existence of a unique solution of system (12)–(13) under Assumptions 1, i.e. it ensures the existence of a unique solution to (10) on the sliding surface Σ.

Theorem 7. *Let Assumptions 1 hold. Then, there exists a unique solution $(\bar{\alpha}, \bar{\beta})$ of system (12)–(13) in $(0, 1) \times (0, 1)$.*

Proof. Different cases need to be addressed in this proof, depending along which part(s) of Σ_1 and Σ_2 there is attractive sliding towards Σ. In essence, there are four scenarios: (i) attractive sliding occurs along only one of Σ_1^+, Σ_1^-, Σ_2^+ or Σ_2^-; (ii) attractive sliding occurs exactly along any two of these; (iii) attractive sliding occurs exactly along three of these; and (iv) attractive sliding occurs along each of $\Sigma_{1,2}^{\pm}$. In the proof below, we only examine different cases up to obvious equivalences between them; namely, in case (i) it is sufficient to see what happens when attractive sliding occurs along Σ_1^+, in case (ii) when attractive sliding occurs along Σ_1^+ and either of Σ_1^- or Σ_2^+, and in case (iii) when it occurs along Σ_1^+, Σ_1^- and Σ_2^+. With this in mind, there are 13 possible cases to examine. To examine these cases, we will use one of the reformulations (16) or (18), making sure that it is well defined for the case under scrutiny.

The following quantities will be repeatedly used. For the function $g = g(\beta)$ in (16), we have

$$g(0) = \frac{w_1^2}{w_1^2 - w_3^2}, \qquad g(1) = \frac{w_2^2}{w_2^2 - w_4^2}, \qquad \beta_S = \frac{w_1^2 - w_3^2}{(w_1^2 - w_3^2) - (w_2^2 - w_4^2)}, \tag{23}$$

where β_S is the singularity of g. For the function $h = h(\alpha)$ in (18), we have

$$h(0) = \frac{w_1^1}{w_1^1 - w_2^1}, \qquad h(1) = \frac{w_3^1}{w_3^1 - w_4^1}, \qquad \alpha_S = \frac{w_1^1 - w_2^1}{(w_1^1 - w_2^1) - (w_3^1 - w_4^1)}, \tag{24}$$

where α_S is the singularity of h.

Case $S_{\Sigma_1^+}$ **Attractive sliding towards Σ along Σ_1^+ only:**

This is "spiral case 2" of [10]. Two possible configurations are feasible.

Case $(S_{\Sigma_1^+}: 1)$ The signs of the entries of w^1 and w^2 are as in Table 1 and the following condition is satisfied (cf. with condition (1_a^+) of Assumptions 1)

$$\frac{w_2^2}{w_2^1} < \frac{w_4^2}{w_4^1}, \tag{25}$$

guaranteeing attractive sliding on Σ_1^+ towards Σ, see Fig. 1. For this case, we can use (16).

By Lemma 5, $P(1) > 0$, and, because of Table 1, $P(0) < 0$, so that there exists a unique $\bar{\beta} \in (0, 1)$ such that $P(\bar{\beta}) = 0$. Moreover using $\alpha = g(\beta)$, where g is defined in (16), we have $g(\beta) \in (0, 1)$ for $\beta \in [0, 1]$ since $C_1^2(\beta) < 0$ and $C_2^2(\beta) > 0$ in Eq. (13). Thus, $\bar{\alpha} = g(\bar{\beta}) \in (0, 1)$, and there exists a unique sliding vector field (8) on Σ.

Further, it is possible to show that $\bar{\alpha} \in (\alpha^+, 1)$. Using Eq. (17) we see that for $\bar{\alpha}$ to be in $(0, 1)$, since $C_1^1(\bar{\beta}) > 0$, we need $C_1^1(\bar{\beta}) - C_2^1(\bar{\beta}) > 0$. Using this and the signs in Table 1 it is easy to verify that $\bar{\alpha} > \alpha^+ = \frac{w_2^1}{w_2^1 - w_4^1}$. Indeed, $\bar{\alpha} > \alpha^+$ rewrites as $-C_1^1(\bar{\beta}) w_4^1 > -C_2^1(\bar{\beta}) w_2^1$ and, using the expressions for $C_1^1(\bar{\beta})$ and $C_2^1(\bar{\beta})$ in (13), this is equivalent to $w_1^1 w_4^1 < w_2^1 w_3^1$. The signs in Table 1 guarantee that this is always true. Hence $(\bar{\alpha}, \bar{\beta}) \in (\alpha^+, 1) \times (0, 1)$.

Case $(S_{\Sigma_1^+}: 2)$ The signs of the entries of w^1 and w^2 are as in Table 2, and condition (25) is satisfied (again, cf. with condition (1_a^+) of Assumptions 1); see Fig. 2. Again we can use (16).

Table 1
Case $S_{\Sigma_1^+}:1$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	>0	<0
w_i^2	<0	<0	>0	>0

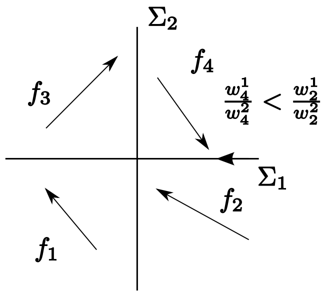


Fig. 1. Case $S_{\Sigma_1^+}:1$.

Table 2
Case $S_{\Sigma_1^+}:2$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	<0	>0	<0	<0
w_i^2	>0	>0	<0	<0

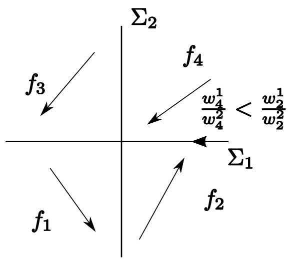


Fig. 2. Case $S_{\Sigma_1^+}:2$.

As in Case $(S_{\Sigma_1^+:1})$ we have $P(1)>0$, $P(0)<0$, so that there is a unique $\bar{\beta}\in(0,1)$ such that $P(\bar{\beta})=0$. Moreover $C_1^2(\beta)>0$ while $C_2^2(\beta)<0$ so that $\bar{\alpha}=g(\bar{\beta})\in(0,1)$ and there exists a unique sliding vector field (8) on Σ.
We can better localize $\bar{\alpha}$ using Eq. (17) for $\bar{\alpha}$. Since $C_2^1(\beta)<0$ for $\beta\in(0,1)$, for $\bar{\alpha}$ to be in $(0,1)$ we need $C_1^1(\beta)>0$. In particular we have $C_1^1(\beta)-C_2^1(\beta)>0$ and this, together with the signs in Table 2, implies that $\bar{\alpha}\in(0,\alpha^+)$.

Case $S_{\Sigma_1^+,\Sigma_2^+}$ **Attractive sliding towards Σ along both Σ_1^+ and Σ_2^+ only:** There are five different possible cases.

Case $(S_{\Sigma_1^+,\Sigma_2^+}:1)$ The signs of the entries of w^1 and w^2 are as in Table 3 and the following condition is satisfied (cf. with condition (2_a^+) of Assumptions 1)

$$\frac{w_3^1}{w_3^2} < \frac{w_4^1}{w_4^2}, \tag{26}$$

see Fig. 3. For this configuration, we are going to use the reformulation (16).

Table 3
Case $S_{\Sigma_1^+,\Sigma_2^+}:1$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	>0	<0
w_i^2	<0	<0	>0	<0

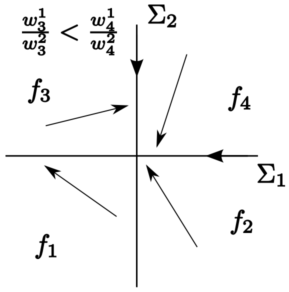
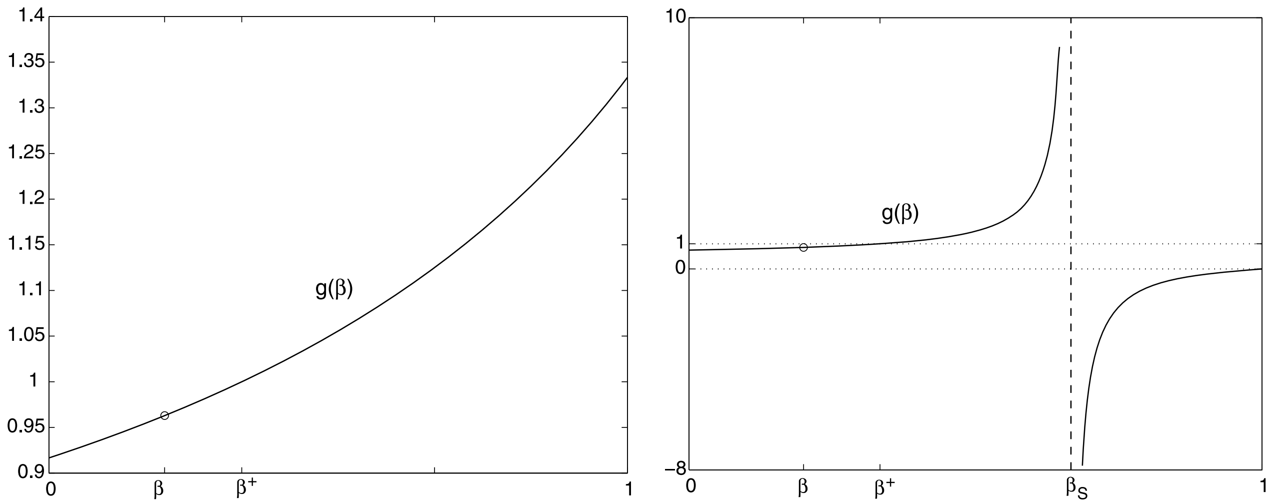


Fig. 3. Case $S_{\Sigma_1^+,\Sigma_2^+}:1$.

Using Table 3, Lemma 5 and Proposition 6, we have $P(1)>0$, $P(0)<0$, $P(\beta^+)>0$. Thus, there exists a unique $\bar\beta$ in $[0,1]$, indeed $\bar\beta \in (0,\beta^+)$, such that $P(\bar\beta)=0$. Next, we need to show that $\bar\alpha = g(\bar\beta)$ is in $[0,1]$. Notice that, given the signs in Table 1, we have $g(0)\in(0,1)$ and $g(\beta^+)=1$, where $g(0)$ is given explicitly in (23). We do not know whether $\beta_S \in [0,1]$ or not.
If $\beta_S \notin [0,1]$ then $\bar\alpha \in [0,1]$ since g is an increasing function in $[0,1]$ and the proof is complete.
If $\beta_S \in [0,1]$, instead, since $w_1^2 - w_3^2 < 0$ it must be $w_2^2 - w_4^2 > 0$ (see explicit expression for β_S in (23)). Given these inequalities, it is easy to verify that $\beta_S > \beta^+$, so that g has no singularities in $[0,\beta^+]$ and it is monotone increasing. Now $g(0)\in(0,1)$ and $g(\beta^+)=1$ implies $0<\bar\alpha = g(\bar\beta)<1$. See the figure below for a graphical interpretation of the proof. Reasoning as in Case $(S_{\Sigma_1^+}:1)$, we further have that $(\bar\alpha,\bar\beta)\in(\alpha^+,1)\times(0,\beta^+)$.



Case $(S_{\Sigma_1^+,\Sigma_2^+}:1)$: plots of $\alpha = g(\beta)$. On the left for $\beta_S \notin [0,1]$ and on the right for $\beta_S \in [0,1]$.

Case $(S_{\Sigma_1^+,\Sigma_2^+}:2)$ The signs of the entries of w^1 and w^2 are as in Table 4 and (25) is satisfied (cf. with condition (1_a^+) of Assumptions 1); see Fig. 4. For this configuration, we are going to use the reformulation (18).

Table 4
Case $S_{\Sigma_1^+,\Sigma_2^+}:2$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	<0	>0	<0	<0
w_i^2	>0	>0	>0	<0

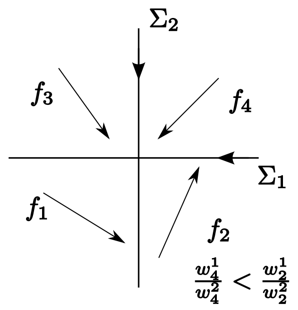


Fig. 4. Case $S_{\Sigma_1^+,\Sigma_2^+}:2$.

For this configuration we have $P(1)>0$, $P(\beta^+)<0$, $Q(0)<0$, $Q(1)>0$, $Q(\alpha^+)>0$ while the sign of $P(0)$ is undetermined. Then there is a unique root $\bar{\alpha}$ of Q and $\bar{\alpha}\in(0,\alpha^+)$. We need to show that $\bar{\beta}=h(\bar{\alpha})$ is in $[0,1]$, where h is the function given in (18). The proof here follows same steps as the proof of Case $(S_{\Sigma_1^+,\Sigma_2^+}:1)$, only the reasoning is applied to the function $h(\alpha)$ instead of $g(\beta)$. Note that we have: $h(0)\in(0,1)$ and $h(\alpha^+)=1$.
If $\alpha_S\notin[0,1]$ then the proof is complete since h is a monotone function.
If $\alpha_S\in[0,1]$, then since $w_1^1-w_2^1<0$ it must be $w_3^1-w_4^1>0$ and it is easy to verify that this implies $\alpha_S>\alpha^+$. So the only singularity of h is not in $[0,\alpha^+]$ and again the proof follows from the monotonicity of h.
Notice that $\bar{\beta}\in(0,1)$ implies that in Eq. (12) it must be $L_1^2(\bar{\alpha})-L_2^2(\bar{\alpha})>0$. This can be used to prove that $\bar{\beta}\in(\beta^+,1)$. Indeed the inequality rewrites as $-w_4^2L_1^2(\bar{\alpha})>-w_3^2L_2^2(\bar{\alpha})$ and this in turn, using the explicit form for $L_1^2(\alpha)$ and $L_2^2(\alpha)$, is equivalent to $-w_4^2w_1^2>-w_3^2w_2^2$. The signs in Table 4 guarantee that this is true. Hence $(\bar{\alpha},\bar{\beta})\in(\alpha^+,1)\times(\beta^+,1)$.

Case $(S_{\Sigma_1^+,\Sigma_2^+}:3)$ The signs of the entries of w^1 and w^2 are as in Table 5 and both (25) and (26) are satisfied; see Fig. 5. We are going to use the reformulation (18).
For this configuration the system exhibits repulsive sliding along Σ_2^- towards Σ. We have $Q(1)>0$, $Q(0)>0$ and $Q(\alpha^+)<0$ so that Q has two roots in $(0,1)$, in particular one of them, call it $\bar{\alpha}$, is in $(\alpha^+,1)$. [We also note that $P(1)>0$, $P(0)<0$ so that there is a unique $\beta\in(0,1)$ such that $P(\beta)=0$.] We claim that $\bar{\beta}=h(\bar{\alpha})$ is in $(0,1)$.
To verify the claim we will show that for any $\alpha\in(\alpha^+,1)$, $\beta=h(\alpha)\in(0,1)$. For this configuration we have $h(1)\in(0,1)$ and $h(\alpha^+)=1$. We do not know whether α_S is in $[0,1]$ or not.
If $\alpha_S\notin[0,1]$ then the monotonicity of h completes the proof.
If $\alpha_S\in[0,1]$, looking at the signs in Table 5, we have $w_3^1-w_4^1>0$, hence it must be $w_1^1-w_2^1<0$. This last condition implies $\alpha_S<\alpha^+$ so that, again, the proof follows from the monotonicity of h and the fact that $\bar{\alpha}\in(\alpha^+,1)$. Hence we have a unique solution $(\bar{\alpha},\bar{\beta})\in(\alpha^+,1)\times(0,1)$.

Case $(S_{\Sigma_1^+,\Sigma_2^+}:4)$ The signs of the entries of w^1 and w^2 are as in Table 6 and both (25) and (26) are satisfied; see Fig. 6.
For this configuration we have $P(1)>0$, $P(\beta^+)<0$, $Q(1)>0$, $Q(\alpha^+)<0$ and the signs of $P(0)$ and $Q(0)$ are undetermined. So, there is surely one root of P,

Table 5
Case $S_{\Sigma_1^+,\Sigma_2^+}:3$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	>0	<0
w_i^2	<0	>0	>0	<0

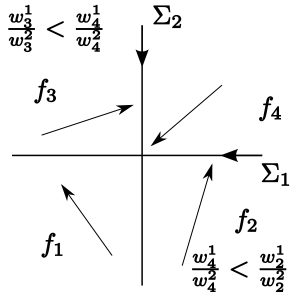


Fig. 5. Case $S_{\Sigma_1^+,\Sigma_2^+}:3$.

Table 6
Case $S_{\Sigma_1^+,\Sigma_2^+}:4$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	>0	<0
w_i^2	>0	>0	>0	<0

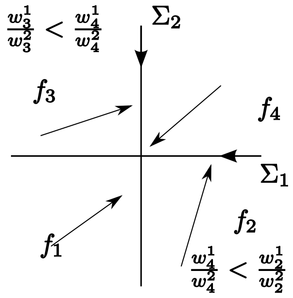
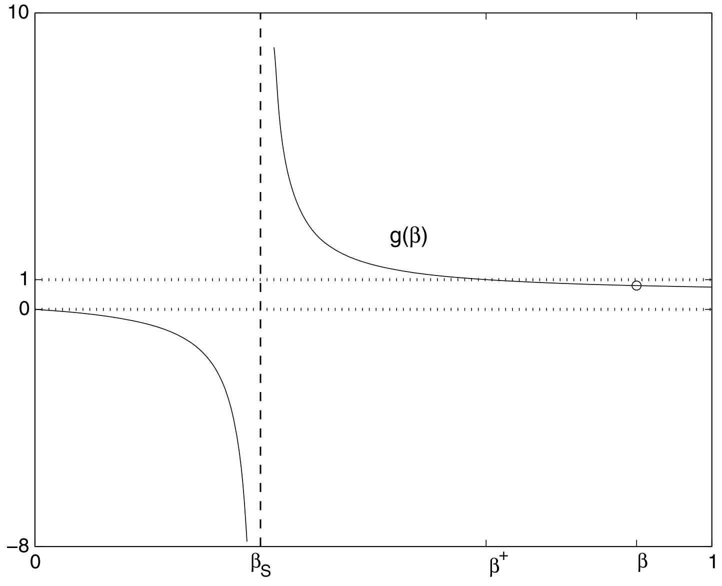


Fig. 6. Case $S_{\Sigma_1^+,\Sigma_2^+}:4$.

$\bar\beta$: $\bar\beta \in (\beta^+, 1)$, and one root of Q, $\bar\alpha$: $\bar\alpha \in (\alpha^+, 1)$. We claim that this is the only solution in $(0,1)\times(0,1)$.

Let $\alpha = g(\beta)$ for $\beta \in (0,1)$. For this case we have $g(\beta^+) = 1$ and $g(1) \in (0,1)$. We do not know whether β_S is in $[0,1]$ or not. If $\beta_S \notin [0,1]$ then g is monotone decreasing in $[0,1]$ and system (12)–(13) has one and only one solution in $[0,1]^2$. In particular $(\bar\alpha,\bar\beta) \in (0,1)\times(\beta^+,1)$. Assume instead that $\beta_S \in [0,1]$. Given the signs in Table 6, we have $w_2^2 - w_4^2 > 0$, so that, for β_S to be in $[0,1]$, it must be $w_1^2 - w_3^2 < 0$. This implies that $g(0) < 0$ and that $\beta_S < \beta^+$. Then, since g is monotone decreasing, it is negative in $[0,\beta_S)$, positive in $(\beta_S, 1)$ and it is in $[0,1]$ for $\beta \in [\beta^+, 1]$. It follows that there is only one solution $(\bar\alpha,\bar\beta)$ of (12)–(13) in $[0,1]^2$, see the figure below. Hence, it must be $(\bar\alpha,\bar\beta) \in (\alpha^+,1)\times(\beta^+,1)$.



Case $(S_{\Sigma_1^+,\Sigma_2^+}:4)$: plot of $g(\beta)$ for $\beta_S \in [0,1]$.

Case $(S_{\Sigma_1^+,\Sigma_2^+}:5)$ The signs of the entries of w^1 and w^2 are as in Table 7 and both (25) and (26) are satisfied; see Fig. 7. We use (16).

Under this configuration, repulsive sliding along Σ_1^- occurs so that the solution is not right unique in a neighborhood of Σ_1^-, however Σ is attractive. For this configuration we have $P(1) > 0$, $P(0) > 0$ and $P(\beta^+) < 0$ so that there are two roots of P in $(0,1)$, one in $(0,\beta^+)$ and the other – which we label $\bar{\beta}$ – in $(\beta^+,1)$. With $\alpha = g(\beta)$ from (16), we have $g(1) \in (0,1)$ and $g(\beta^+) = 1$. We need to locate β_S. The argument is similar to the one of Case $(S_{\Sigma_1^+,\Sigma_2^+}:4)$.

If $\beta_S \notin [0,1]$ then g is monotone decreasing in $(0,1)$ and there is only one solution of system (12)–(13) in $[0,1]^2$. Moreover $(\bar{\alpha},\bar{\beta}) \in (0,1) \times (\beta^+,1)$.

If $\beta_S \in [0,1]$ then $w_2^2 - w_4^2 < 0$ implies $w_1^2 - w_3^2 < 0$ and this in turn implies $\beta_S < \beta^+$ and $g(0) < 0$. Then there is only one solution of system (12)–(13) in $[0,1]^2$ and moreover $(\bar{\alpha},\bar{\beta}) \in (0,1) \times (\beta^+,1)$.

Table 7
Case $S_{\Sigma_1^+,\Sigma_2^+}:5$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	<0	>0	>0	<0
w_i^2	>0	>0	>0	<0

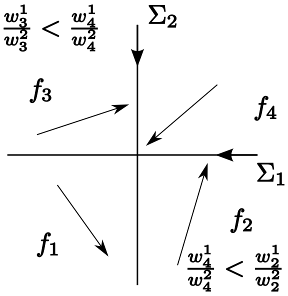


Fig. 7. Case $S_{\Sigma_1^+,\Sigma_2^+}:5$.

Table 8
Case $S_{\Sigma_1^\pm}: 1$.

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
w_i^1	> 0	> 0	< 0	< 0
w_i^2	> 0	> 0	< 0	< 0

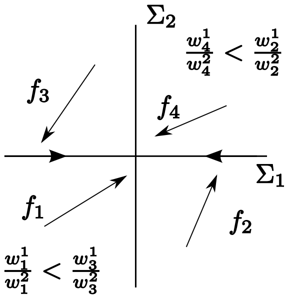


Fig. 8. Case $S_{\Sigma_1^\pm}: 1$.

Table 9
Case $S_{\Sigma_1^\pm}: 2$.

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
w_i^1	> 0	> 0	< 0	< 0
w_i^2	< 0	< 0	> 0	> 0

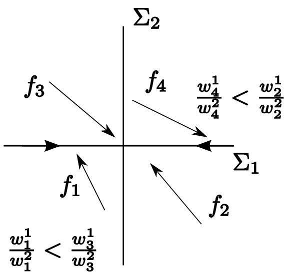


Fig. 9. Case $S_{\Sigma_1^\pm}: 2$.

Case $S_{\Sigma_1^\pm}$ **Attractive sliding towards Σ along both Σ_1^+ and Σ_1^- only:**
Here there are only two possible cases. In both of them, we must satisfy (cf. with conditions (1_a^+) and (1_a^-) of Assumptions 1)

$$\text{(a)} \quad \frac{w_3^2}{w_3^1} < \frac{w_1^2}{w_1^1} \quad \text{and} \quad \text{(b)} \quad \frac{w_2^2}{w_2^1} < \frac{w_4^2}{w_4^1}. \tag{27}$$

Case $(S_{\Sigma_1^\pm}: 1)$ The signs of the entries of w^1 and w^2 are as in Table 8, and see Fig. 8.
Case $(S_{\Sigma_1^\pm}: 2)$ The signs of the entries of w^1 and w^2 are as in Table 9, and see Fig. 9.

The following argument holds in both cases $(S_{\Sigma_1^\pm}: 1, 2)$ above. For both configurations we have $P(1) > 0$ and $P(0) < 0$ so that there exists a unique $\bar{\beta}$ in $(0, 1)$ such that $P(\bar{\beta}) = 0$. Moreover $g(0), g(1) \in (0, 1)$ and $\beta_S \notin [0, 1]$ so that $\bar{\alpha} = g(\bar{\beta}) \in (0, 1)$.

Table 10
Case $S_{\Sigma_1^{\pm},\Sigma_2^{+}}:1$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	<0	<0
w_i^2	<0	<0	>0	<0

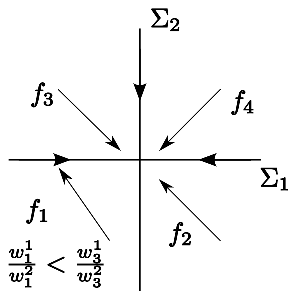


Fig. 10. Case $S_{\Sigma_1^{\pm},\Sigma_2^{+}}:1$.

Case $S_{\Sigma_1^{\pm},\Sigma_2^{+}}$ **Attractive sliding towards Σ along each of Σ_1^+, Σ_1^- and Σ_2^+ only:**
Now there are three possible (nonequivalent) configurations, and we are going to use (16) for all of them.

Case $(S_{\Sigma_1^{\pm},\Sigma_2^{+}}:1)$ The signs of the entries of w^1 and w^2 are given in Table 10 and (27)(a) is satisfied, see Fig. 10.
Under this configuration we have $P(1)>0$, $P(0)<0$, $P(\beta^+)>0$ so that there exists a unique root $\bar\beta$ of P, in $(0,1)$, and $\bar\beta\in(0,\beta^+)$. Further, $g(0)\in[0,1]$, $g(\beta^+)=1$, but we do not know whether $\beta_S\in[0,1]$ or not.
If $\beta_S\notin[0,1]$ the proof follows from the monotonicity of g.
If $\beta_S\in[0,1]$ then since $w_1^2-w_3^2<0$ it must be $w_2^2-w_4^2>0$ and this implies $\beta_S>\beta^+$. The proof now follows from the monotonicity of h in $[0,\beta^+]$ together with $\bar\beta\in(0,\beta^+)$.
Using Eq. (13) with $j=1$ and reasoning in a way similar to Case $(S_{\Sigma_1^+}:1)$ it is easy to verify that $(\bar\alpha,\bar\beta)\in[\min(\alpha^+,\alpha^-),\max(\alpha^+,\alpha^-)]\times(0,\beta^+)$.

Case $(S_{\Sigma_1^{\pm},\Sigma_2^{+}}:2)$ The signs of the entries of w^1 and w^2 are given in Table 11 and (27)(b) is satisfied; see Fig. 11.
We have $P(1)>0$, $P(\beta^+)<0$ and $P(0)<0$ so that there is only one root of P, $\bar\beta\in(\beta^+,1)$. Moreover, $g(1)\in[0,1]$ and $g(\beta^+)=1$. Again we need to locate β_S.
If $\beta_S\notin[0,1]$ we have uniqueness of solution in $[0,1]^2$.
If instead $\beta_S\in[0,1]$ reasoning in a way similar to the previous case we have $\beta_S<\beta^+$ and the proof follows from the monotonicity of g in $[\beta^+,1]$, together with $\bar\beta\in(\beta^+,1)$.
Again, it is easy to verify that $(\bar\alpha,\bar\beta)\in[\min(\alpha^+,\alpha^-),\max(\alpha^+,\alpha^-)]\times(\beta^+,1)$.

Case $(S_{\Sigma_1^{\pm},\Sigma_2^{+}}:3)$ The signs of the entries of w^1 and w^2 are given in Table 12 and (27) (both (a) and (b)) is satisfied; see Fig. 12.
Under these conditions there is repulsive sliding on Σ_2^- towards Σ so that the solution in a neighborhood of Σ is not right unique. Still, we have $P(1)>0$ and $P(0)<0$ so that there exists a unique $\bar\beta$ in $(0,1)$ such that $P(\bar\beta)=0$. Moreover we have $g(0), g(1)\in[0,1]$, while $g(\beta^+)=1$ and clearly $\beta_S\in[0,1]$. We also note that $Q(1)>0$. The proof will follow from the following claim:

"If $\beta_S<\beta^+$ (respectively, $\beta_S>\beta^+$), then $\bar\beta>\beta^+$ (respectively, $\bar\beta<\beta^+$)."

Table 11
Case $S_{\Sigma_1^\pm,\Sigma_2^+}$: 2.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	<0	<0
w_i^2	>0	>0	>0	<0

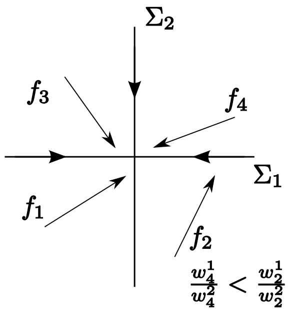


Fig. 11. Case $S_{\Sigma_1^\pm,\Sigma_2^+}$: 2.

Table 12
Case $S_{\Sigma_1^\pm,\Sigma_2^+}$: 3.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	<0	<0
w_i^2	<0	>0	>0	<0

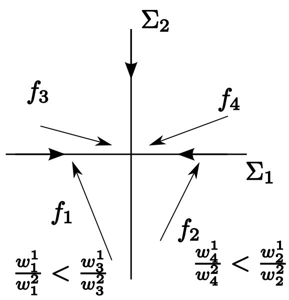


Fig. 12. Case $S_{\Sigma_1^\pm,\Sigma_2^+}$: 3.

To verify the claim, notice that $\beta_S < \beta^+$ implies $-w_1^2 w_4^2 + w_3^2 w_2^2 > 0$ and since $P(\beta^+) = \frac{w_1^2 w_4^2 - w_3^2 w_2^2}{w_3^2 - w_4^2} \frac{Q(1)}{w_3^2 - w_4^2}$, and $Q(1) > 0$, then $P(\beta^+) < 0$ and $\bar\beta \in (\beta^+, 1)$. Similarly, the case $\beta_S > \beta^+$ implies $P(\beta^+) > 0$ and $\bar\beta \in (0, \beta^+)$.

Again, it is easy to verify that $(\bar\alpha, \bar\beta) \in [\min(\alpha^+, \alpha^-), \max(\alpha^+, \alpha^-)] \times (0, 1)$.

Case $S_{\Sigma_1^\pm,\Sigma_2^\pm}$ **Attractive sliding towards Σ along each of $\Sigma_1^\pm$ and $\Sigma_2^\pm$:**
This is nodal attractivity and existence and uniqueness of the solution has been proven already in [2,10]. For completeness, we remark that the signs of w^1 and w^2 are as in Table 13; see Fig. 13. $\square$

To complete this section, we now show that – under the conditions of Theorem 7 – the unique solution $(\bar\alpha, \bar\beta)$ of (10) varies smoothly with respect to $x \in \Sigma$.

Before doing so, we must stress that the proof of Theorem 7 consisted of the following steps:

(a) adopt one of the possible rewritings (16)–(18) (equivalently, (17)–(19)) of (10), in such a way that the denominator in the expression for α or β is nonzero;

Table 13
Case $S_{\Sigma_1^\pm,\Sigma_2^\pm}$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	<0	<0
w_i^2	>0	<0	>0	<0

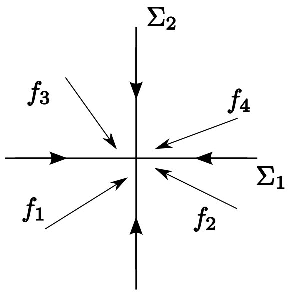


Fig. 13. Case $S_{\Sigma_1^\pm,\Sigma_2^\pm}$.

(b) show that the second degree polynomial we ended up considering (i.e., one of P or Q) has a unique root in $[0,1]$ and changes sign through this root.

As a consequence of (b) above, in the proof of Theorem 7 we have actually shown that – at the unique solution $(\bar\alpha,\bar\beta)$ of (10) – one of these relations is surely satisfied:

$$P'(\bar\beta)\neq 0,\quad \text{and/or}\quad Q'(\bar\alpha)\neq 0. \tag{28}$$

We are now ready to show the following result.

Theorem 8. *Under Assumptions 1, the unique solution $(\bar\alpha,\bar\beta)\in(0,1)^2$ of system (10) varies smoothly with respect to $x\in\Sigma$.*

Proof. Since $(\bar\alpha,\bar\beta)$ is the only solution in the square $[0,1]^2$, it will be sufficient to show that the Jacobian J of (10) is nonsingular at $(\bar\alpha,\bar\beta)$ and the result will follow.

For ease of notation, let us rename the left-hand side of (10) as follows:

$$F_1(\alpha,\beta) := (1-\alpha)(1-\beta)w_1^1 + (1-\alpha)\beta w_2^1 + \alpha(1-\beta)w_3^1 + \alpha\beta w_4^1,$$
$$F_2(\alpha,\beta) := (1-\alpha)(1-\beta)w_1^2 + (1-\alpha)\beta w_2^2 + \alpha(1-\beta)w_3^2 + \alpha\beta w_4^2.$$

We have

$$\det J = (\partial_\alpha F_1)\,(\partial_\beta F_2) - (\partial_\alpha F_2)\,(\partial_\beta F_1).$$

Now we set up to show that – at $(\bar\alpha,\bar\beta)$ – the following four equalities hold:

$$\det J = P'(\bar\beta) = -\hat P'(\bar\beta) = -Q'(\bar\alpha) = \hat Q'(\bar\alpha), \tag{29}$$

where P, $\hat P$, Q, $\hat Q$, are defined in (16)–(19); these must be understood as being formal equalities, in the sense that the denominators in the expressions for α and/or β in (16)–(19) evaluated at $(\bar\alpha,\bar\beta)$ are nonzero (we know that at least one of these is surely not zero). Thus, as a consequence of (28), the proof of the theorem will be complete. Because of (20), of course we will just show the result for P and Q.

Write (16) as

$$P(\beta) = \big(C_1^2(\beta) - C_2^2(\beta)\big)F_1\big(C_1^2/\big(C_1^2 - C_2^2\big), \beta\big), \qquad \alpha = C_1^2/\big(C_1^2 - C_2^2\big),$$

so that

$$P'(\bar\beta) = \big(C_1^2 - C_2^2\big)\left[(\partial_\alpha F_1)\left(\partial_\beta \frac{C_1^2}{C_1^2 - C_2^2}\right) + \partial_\beta F_1\right]_{\bar\beta} + \big[F_1 \partial_\beta\big(C_1^2 - C_2^2\big)\big]_{\bar\beta},$$

and the second term vanishes since $\bar\beta$ is a solution of (10). So, to show that $(\det J)_{(\bar\alpha,\bar\beta)} = P'(\bar\beta)$, we need to show that $\partial_\alpha F_1 = C_2^2 - C_1^2$, which is a trivial verification, and that $-[\partial_\beta F_2]_{(\bar\alpha,\bar\beta)} = [(C_2^2 - C_1^2)\partial_\beta \frac{C_1^2}{C_1^2 - C_2^2}]_{(\bar\alpha,\bar\beta)}$, which is an equally simple verification.

For Q, write (18) as

$$Q(\alpha) = \big(L_1^1(\alpha) - L_1^2(\alpha)\big)F_2\big(\alpha, L_1^1/\big(L_1^1 - L_2^1\big)\big), \qquad \beta = L_1^1/\big(L_1^1 - L_2^1\big),$$

so that

$$Q'(\bar\alpha) = \big(L_1^1 - L_2^1\big)\left[(\partial_\alpha F_2) + \left(\partial_\beta F_2 \partial_\alpha \frac{L_1^1}{L_1^1 - L_1^2}\right)\right]_{\bar\alpha} + \big[F_2 \partial_\alpha\big(L_1^1 - L_2^1\big)\big]_{\bar\alpha},$$

and the second term is 0. So, to show that $(\det J)_{(\bar\alpha,\bar\beta)} = -Q'(\bar\alpha)$, we need to check that $\partial_\beta F_1 = L_2^1 - L_1^1$, and that $-[\partial_\alpha F_1]_{(\bar\alpha,\bar\beta)} = [(L_1^1 - L_2^1)\partial_\alpha \frac{L_1^1}{L_1^1 - L_2^1}]_{(\bar\alpha,\bar\beta)}$, both of which are simple verifications. □

2.2. Spiral attractivity

As previously mentioned, Σ may attract nearby dynamics in different ways than those contemplated by Assumptions 1. An interesting situation is when Σ is reached by *spiraling* around it, and there is no attractive sliding motion on any of $\Sigma_{1,2}^{\pm}$. In this work, we have not justified use of (8) in this case; however, we remark that sufficient conditions for well-posedness of the choice of (8), when Σ is attractive in a spiraling way, were given in [10].

3. Co-dimension 1 losses of attractivity and first order exit conditions

Here we consider the following phenomenon. Suppose we are following a solution trajectory which is sliding on Σ according to the vector field (8), and the conditions of Theorem 7 are satisfied for $t \in [0, T)$, but at $t = T$ one of the compatibility conditions $(1_a^{\pm})$, $(2_a^{\pm})$ of Assumptions 1 becomes violated, that is, one of them becomes equality. Clearly, Σ loses attractivity, but what happens to the vector field (8)? The situation, with respect to the proof of Theorem 7, is one where the signs of the w_j^i, $j = 1, 2, 3, 4$, $i = 1, 2$, are as in the tables there, but the compatibility conditions (25) and/or (26) and/or (27) no longer hold, in the cases examined in Theorem 7. The question we want address is: what happens to the trajectory we are following on Σ? Do we lose uniqueness of solutions to the system (10)?

This problem is intimately connected with possible smooth exits from Σ. But, first, we must realize that if a trajectory is going to (smoothly, or at least continuously) leave Σ, it can do so in one of two different ways: (a) exit Σ and remain on one of Σ_1^+, Σ_1^-, Σ_2^+, or Σ_2^-, or (b) exit Σ and enter into one of the regions R_i, $i = 1, 2, 3, 4$. The second type of exit would require one of the vector fields f_i to be tangent to Σ (i.e., $\nabla h_1^T f_i = \nabla h_2^T f_i = 0$), and as such is a co-dimension 2 phenomenon. The first type of exit, instead, is a co-dimension 1 phenomenon, as we are going to clarify next. Next, let us give a formal definition of potential exit points from Σ.

Definition 2 *(First order exit conditions).* Let f_Σ be the convex hull of Filippov vector fields on Σ:

$$f_\Sigma = \left\{ F = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4,\ \lambda_i \geqslant 0,\ i = 1, 2, 3, 4, \right.$$

$$\left. \sum_{i=1}^{4} \lambda_i = 1,\ (\nabla h_1)^T F = (\nabla h_2)^T F = 0 \right\}.$$

Further, as long as the denominators in the expressions for $\alpha^\pm$ and $\beta^\pm$ in (21) are nonzero, let $f_{\Sigma_1^+}$, $f_{\Sigma_1^-}$, $f_{\Sigma_2^+}$, and $f_{\Sigma_2^-}$, be the vectors defined by $f_{\Sigma_1^+} = (1-\alpha^+) f_2 + \alpha^+ f_4$, $f_{\Sigma_1^-} = (1-\alpha^-) f_1 + \alpha^- f_3$, $f_{\Sigma_2^+} = (1-\beta^+) f_3 + \beta^+ f_4$, $f_{\Sigma_2^-} = (1-\beta^-) f_1 + \beta^- f_2$ (these would be sliding Filippov vector fields on $\Sigma_{1,2}^\pm$, if there is a well-defined sliding motion on $\Sigma_{1,2}^\pm$).

Then, we say that a first order exit condition is satisfied at $x \in \Sigma$, if one of the four vectors $f_{\Sigma_{1,2}^\pm}$ – which lies on the tangent plane to Σ_1 or Σ_2 at x – is a Filippov sliding vector field on one of $\Sigma_{1,2}^\pm$ and it is also tangent to Σ itself at x. A Filippov sliding vector field on Σ which satisfies a first order exit condition will be called an exit vector field.

A first order exit condition is called generic if, at $x \in \Sigma$, only one of the four vectors $f_{\Sigma_{1,2}^\pm}$ is an admissible Filippov sliding vector field on the corresponding $\Sigma_{1,2}^\pm$ and on Σ.

Remark 9. In other words, Definition 2 states that there is a well-defined sliding vector field on one of the $\Sigma_{1,2}^\pm$ which happens to be also tangent to Σ.

We must emphasize that a first order exit condition is an indicator that Σ loses attractivity, and that there is a Filippov vector field (in the convex hull of all Filippov vector fields) which potentially exits Σ. But, in the present case of a co-dimension 2 singularity surface (and unlike the case of co-dimension 1 singularity surface, see Example 1), a first order exit condition leaves open the possibility of having well-defined Filippov sliding vector fields which keep us on Σ, without leaving it.

A generic first order exit condition is a co-dimension 1 phenomenon, and this is the case in which we are interested.

Lemma 10. *Generic first order exit conditions are equivalent to co-dimension* 1 *losses of attractivity for Σ, in the sense that they are equivalent to having equality in one of the compatibility conditions* $(1_a^\pm)$, $(2_a^\pm)$, *given in Assumptions* 1.

Proof. We need to verify the following four facts:

(a) $f_{\Sigma_1^+}$ tangent to Σ if and only if $w_2^1 w_4^2 = w_4^1 w_2^2$ (i.e., $P(1) = 0$);
(b) $f_{\Sigma_1^-}$ tangent to Σ if and only if $w_3^1 w_1^2 = w_1^1 w_3^2$ (i.e., $P(0) = 0$);
(c) $f_{\Sigma_2^+}$ tangent to Σ if and only if $w_4^1 w_3^2 = w_3^1 w_4^2$ (i.e., $Q(1) = 0$);
(d) $f_{\Sigma_2^-}$ tangent to Σ if and only if $w_1^1 w_2^2 = w_2^1 w_1^2$ (i.e., $Q(0) = 0$).

We verify just (a), the verifications for the other cases being similar. Since $f_{\Sigma_1^+}$ is already tangent to Σ_1, we need to verify that $f_{\Sigma_1^+} \in T_\Sigma \Leftrightarrow (\nabla h_2)^T f_{\Sigma_1^+} = 0$, which is true $\Leftrightarrow (1-\alpha^+) w_2^2 + \alpha^+ w_4^2 = 0$, which (given that $\alpha^+ = w_2^1/(w_2^1 - w_4^1)$) is equivalent to having $w_2^1 w_4^2 = w_4^1 w_2^2$. $\square$

Our interest is to examine what happens to the specific vector field (8). In particular, to see if and when following a trajectory on Σ associated to this vector field, and a generic first order exit condition becomes satisfied, we should expect leaving Σ or remaining on it.

The following fact is useful.

Lemma 11. *In case a first order exit condition is satisfied, then there is a vector field of the type* (8) *which is tangent to* Σ.

Proof. This is because when a first order exit condition is satisfied, there is a solution $(\alpha,\beta)\in[0,1]^2$ of (8) of the type $(0,\beta^-)$ or $(1,\beta^+)$ or $(\alpha^+,1)$ or $(\alpha^-,0)$. □

Accordingly, the solutions identified in Lemma 11 will be called *exit solutions* of (8), the associated values of (α,β) will be called *extremal*, and the vector field will be called an *exit vector field*.

Finally, to arrive at necessary and sufficient conditions telling us when the vector field (8) aligns with an exit vector field, we are going to make use of the following reinterpretation of the solution set of (10) .

Lemma 12. *Solution pairs* $(\alpha,\beta)\in[0,1]^2$ *of the system* (10) *are in one-to-one correspondence with the eigenvalue–eigenvector pairs of either generalized eigenvalue problem below*:

$$\big[(1-\alpha)A+\alpha B\big]\begin{bmatrix}1-\beta\\ \beta\end{bmatrix}=0 \quad or \quad \big[(1-\beta)C+\beta D\big]\begin{bmatrix}1-\alpha\\ \alpha\end{bmatrix}=0, \quad where$$

$$A=\begin{bmatrix}w_1^1 & w_2^1\\ w_1^2 & w_2^2\end{bmatrix},\qquad B=\begin{bmatrix}w_3^1 & w_4^1\\ w_3^2 & w_4^2\end{bmatrix},\qquad C=\begin{bmatrix}w_1^1 & w_3^1\\ w_1^2 & w_3^2\end{bmatrix},\qquad D=\begin{bmatrix}w_2^1 & w_4^1\\ w_2^2 & w_4^2\end{bmatrix}. \tag{30}$$

Proof. The proof is just a rewriting of the system (10) as in (30). □

Remark 13. Consider the polynomials $P(\beta)$ and $Q(\alpha)$ introduced in (16) and (18). Then, with respect to the notation of Lemma 30, we notice that

$$P(0)=-\det(C),\qquad P(1)=-\det(D),\qquad Q(0)=\det(A),\qquad Q(1)=\det(B). \tag{31}$$

We can also rewrite (30) in more standard forms; e.g. at least one of these rewritings is always possible: $(A-\mu B)x=0$, or $(B-\mu A)x=0$, or $(C-\lambda D)y=0$, or $(D-\lambda C)y=0$. Since – see Theorem 14 – for us at least one of these four pencils is a regular pencil (in other words, at least one of A, B, C, D, is nonsingular), then at least one of the above determinantal relations cannot vanish identically. As a consequence, we can never have more than two (possibly identical) solutions of (10) in $[0,1]^2$.

We are now ready to give necessary and sufficient conditions for (10) to have a unique solution, hence for the vector field (8) to be uniquely defined, when a generic first order exit condition is satisfied. By virtue of Lemma 11, if (8) is uniquely defined, then this means that (8) has aligned with an exit vector field, and we should expect to leave Σ (and enter one of $\Sigma_{1,2}^{\pm}$). On the other hand, if – when a generic first order exit condition is satisfied – there are multiple distinct solutions $(\alpha,\beta)\in[0,1]^2$ to the system (10), one (and only one) of which is necessarily extremal, then the vector field (8) will have not aligned with an exit vector field and by following the trajectory determined by (8) we will remain on Σ. This will mean that the extra solution of (10) is "entering $[0,1]^2$ from outside." We emphasize that these cases when solutions to (10) are not unique are occurring when Σ is no longer attracting nearby dynamics according to our first order theory.

The result below adopts the same notation – and simplifications up to equivalent configurations – of Theorem 7. It is implicitly understood that when we write that a root is the "only solution in $[0,1]^2$" we mean that the root is a simple root.

Theorem 14. *The following hold.*

$(S_{\Sigma_1^+})$ *Let the signs of the entries of* w^1 *and* w^2 *be as in Table* 1*, case* $(S_{\Sigma_1^+}:1)$*, or as in Table* 2*, case* $(S_{\Sigma_1^+}:2)$*, but let there be equality in* (25)*. Then, the solution* $(\alpha,\beta)=(\alpha^+,1)$ *is the only solution* *of* (10) *in* $[0,1]^2$ *if and only if*

$$w_1^1 w_4^2 - w_4^1 w_1^2 + w_2^1 w_3^2 - w_3^1 w_2^2 > 0. \tag{32}$$

$(S_{\Sigma_1^+,\Sigma_2^+})$ *We have five cases.*

(1) *The signs of the entries of* w^1 *and* w^2 *are as in Table* 3, *but let there be equality in* (26). *Then, the solution* $(\alpha,\beta)=(1,\beta^+)$ *is the only solution of* (10) *in* $[0,1]^2$, *and it is an exit solution.*

(2) *The signs of the entries of* w^1 *and* w^2 *are as in Table* 4, *but there is equality in* (25). *Then, the solution* $(\alpha,\beta)=(\alpha^+,1)$ *is the only solution of* (10) *in* $[0,1]^2$, *and it is an exit solution.*

(3) *The signs of the entries of* w^1 *and* w^2 *are as in Table* 5, *but there is equality in one of* (25) *or* (26), *the other condition being satisfied.*

  (a) *If there is equality in* (25), *the solution* $(\alpha,\beta)=(\alpha^+,1)$ *is the only solution of* (10) *in* $[0,1]^2$ *if and only if* (32) *holds.*

  (b) *If there is equality in* (26), *then the solution* $(\alpha,\beta)=(1,\beta^+)$ *is the only solution of* (10) *in* $[0,1]^2$, *and it is an exit solution.*

(4) *The signs of the entries of* w^1 *and* w^2 *are as in Table* 6, *but there is equality in one of* (25) *or* (26), *the other condition being satisfied.*

  (a) *If there is equality in* (25), *the solution* $(\alpha,\beta)=(\alpha^+,1)$ *is the only solution of* (10) *in* $[0,1]^2$, *if and only if*

$$\begin{gathered} w_1^1 w_2^2 - w_2^1 w_1^2 < 0, \quad \textit{i.e.} \quad Q(0) < 0, \quad \textit{or} \\ w_4^1\big(w_1^1 w_4^2 - w_4^1 w_1^2\big) < w_2^1\big(w_3^1 w_4^2 - w_4^1 w_3^2\big). \end{gathered} \tag{33}$$

  (b) *If there is equality in* (26), *then the solution* $(\alpha,\beta)=(1,\beta^+)$ *is the only solution of* (10) *in* $[0,1]^2$, *if and only if*

$$\begin{gathered} w_1^1 w_3^2 - w_3^1 w_1^2 < 0, \quad \textit{i.e.} \quad P(0) < 0, \quad \textit{or} \\ w_3^2\big(w_2^1 w_4^2 - w_4^1 w_1^2\big) < w_4^2\big(w_1^1 w_4^2 - w_4^1 w_1^2\big). \end{gathered} \tag{34}$$

(5) *The signs of the entries of* w^1 *and* w^2 *are as in Table* 7, *but there is equality in one of* (25) *or* (26), *the other condition being satisfied.*

  (a) *If there is equality in* (25), *the solution* $(\alpha,\beta)=(\alpha^+,1)$ *is the only solution of* (10) *in* $[0,1]^2$, *and* $f_{\Sigma_1^+}$ *is an exit vector field.*

  (b) *If there is equality in* (26), *then the solution* $(\alpha,\beta)=(1,\beta^+)$ *is the only solution of* (10) *in* $[0,1]^2$, *if and only if*

$$w_1^1 w_4^2 - w_4^1 w_1^2 - w_2^1 w_3^2 + w_3^1 w_2^2 < 0. \tag{35}$$

$(S_{\Sigma_1^\pm})$ *Here, we have two cases (see* $(S_{\Sigma_1^\pm}:1)$ *and* $(S_{\Sigma_1^\pm}:2)$ *in Theorem* 7*). The signs of the entries of* w^1 *and* w^2 *are as in Table* 8, *respectively as in Table* 9, *and let there be equality in one of* (27)(a) *or* (27)(b), *the other condition being satisfied.*

(a) *If there is equality in* (27)(a), *the solution* $(\alpha,\beta)=(\alpha^-,0)$ *is the only solution of* (10) *in* $[0,1]^2$, *and* $f_{\Sigma_1^-}$ *is an exit vector field, if and only if*

$$w_1^1 w_4^2 - w_4^1 w_1^2 + w_2^1 w_3^2 - w_3^1 w_2^2 < 0. \tag{36}$$

(b) *If there is equality in* (27)(b), *then the solution* $(\alpha,\beta)=(\alpha^+,1)$ *is the only solution of* (10) *in* $[0,1]^2$, *and* $f_{\Sigma_1^+}$ *is an exit vector field, if and only if* (32) *is satisfied.*

($S_{\Sigma_1^{\pm},\Sigma_2^+}$) *We have three cases.*

(1) *The signs of the entries of w^1 and w^2 are given in Table 10, but there is equality in (27)(a). Then, the solution $(\alpha,\beta)=(\alpha^-,0)$ is the only solution of (10) in $[0,1]^2$, if and only if (36) is satisfied.*

(2) *The signs of the entries of w^1 and w^2 are given in Table 11 but there is equality in (27)(b). Then, the solution $(\alpha,\beta)=(\alpha^+,1)$ is the only solution of (10) in $[0,1]^2$, if and only if (32) is satisfied.*

(3) *The signs of the entries of w^1 and w^2 are given in Table 12, and let there be equality in one of (27)(a) or (27)(b), the other condition being satisfied.*

(a) *If there is equality in (27)(a), then the solution with $(\alpha,\beta)=(\alpha^-,0)$ is the only solution of (10) in $[0,1]^2$ if and only if (36) is satisfied.*

(b) *If there is equality in (27)(b), then the solution with $(\alpha,\beta)=(\alpha^+,1)$ is the only solution of (10) in $[0,1]^2$ if and only if (32) is satisfied.*

Proof. We examine the different cases.

($S_{\Sigma_1^+}$) We know that $f_{\Sigma_1^+}$ is a sliding vector field on Σ. We need to find necessary and sufficient conditions giving that $(\alpha^+,1)$ is the only solution of (10) in $[0,1]^2$. We have two cases.

($S_{\Sigma_1^+}:1$) (We detail all steps in this particular case, so to clarify the general argument of proof; in later cases, we will omit some of the details.) Because of Table 1, the matrix C in (30) is invertible, $\det(C)>0$, and because of equality in (25) $\det(D)=0$. So, we consider the eigenvalues $\mu=(\beta-1)/\beta$, of $C^{-1}D$. One eigenvalue is $\mu_1=0$, and then for the other eigenvalue we have $\mu_2=\mathrm{tr}(C^{-1}D)$. An explicit computation shows that $\mathrm{tr}(C^{-1}D)=\frac{1}{\det(C)}(w_1^1w_4^2-w_4^1w_1^2+w_2^1w_3^2-w_3^1w_2^2)$ and thus $\mu_1=0$ is the only eigenvalue of $C^{-1}D$ less than or equal to 0 (hence, $\beta=1$ is the only root of P in $[0,1]$) if and only if (32) holds. Now, using the expression for $\alpha=g(\beta)$ from (16), and the signs of Table 1, it is immediate to realize that $\alpha\in[0,1]$ for any $\beta\in[0,1]$. Hence, $(\alpha,\beta)=(\alpha^+,1)$ is the only solution of (10) in $[0,1]^2$ if and only if (32) holds.

($S_{\Sigma_1^+}:2$) Because of Table 2, again $\det(C)>0$. The argument is now identical to the previous case of ($S_{\Sigma_1^+}:1$).

($S_{\Sigma_1^+,\Sigma_2^+}$) We have five cases.

(1) According to Table 3, we have that the matrices C and D in (30) are invertible, with $\det(C)>0$ and $\det(D)<0$. So, we have the eigenvalue problem $[(1-\beta)I+\beta C^{-1}D]\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right]=0$, or $[C^{-1}D-\mu I]\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right]=0$, with $\mu=(\beta-1)/\beta$. Since $f_{\Sigma_2^+}=(1-\beta^+)f_3+\beta^+f_4$ is a solution, we have $[C^{-1}D-\mu^+I]\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]=0$, $\mu^+=(\beta^+-1)/\beta^+$, and since $\det(C^{-1}D)<0$, then the other eigenvalue of $C^{-1}D$ is positive, which means that the other root β is outside of $[0,1]$ and thus $f_{\Sigma_2^+}$ is an exit vector field.

(2) According to Table 4, the matrices A and B in (30) are invertible, with $\det(A)<0$ and $\det(B)>0$. So, we have the eigenvalue problem $[A^{-1}D-\mu I]\left[\begin{smallmatrix}1-\beta\\ \beta\end{smallmatrix}\right]=0$, with $\mu=(\alpha-1)/\alpha$. Since $f_{\Sigma_1^+}$ is a sliding vector field (hence $(\alpha^+,1)$ is a solution), and further $\det(A^{-1}B)<0$, then the other eigenvalue of $A^{-1}B$ is positive, which means that the other root α is outside of $[0,1]$ and thus $f_{\Sigma_1^+}$ is an exit vector field.

(3) According to Table 5, we have that both A and C in (30) are invertible, $\det(A)>0$, $\det(C)>0$.

(a) In the present case, we know that $(\alpha^+,1)$ is a solution, and we also have $\det(B)>0$ and $\det(D)=0$, hence Q has two roots in $[0,1]$, one of them being $\alpha_1\equiv\alpha^+$. Hence, we can factor Q in (18) as

$$Q(\alpha)=(\alpha-\alpha^+)\big[c_1+c_2(\alpha-\alpha^+)\big],$$

so that the other root of Q is $\alpha_2 = \alpha^+ - \frac{c_1}{c_2}$ (note that $c_2 \neq 0$, otherwise Q would be linear which is not possible since $Q(0) > 0$, $Q(1) > 0$). Direct comparison with the form of Q in (18) gives

$$c_1 = -2\det(A) + \gamma + 2c_2\alpha^+, \qquad c_2 = \det(A) + \det(B) - \gamma,$$
$$\gamma = \left(w_1^1 w_4^2 - w_4^1 w_1^2 + w_3^1 w_2^2 - w_2^1 w_3^2\right),$$

and because of the equality in (25), and the signs in Table 5, we obtain that: $\gamma < 0$ and $c_2 > 0$.
Finally, using $\beta = h(\alpha)$ from (18), we need to verify if/when $h(\alpha_2) \in [0,1]$. But we have

$$h(\alpha) = \frac{(1-\alpha_2)w_1^1 + \alpha_2 w_3^1}{((1-\alpha_2)w_1^1 + \alpha_2 w_3^1) - \frac{c_1}{c_2}(w_2^1 - w_4^1)},$$

hence $h(\alpha_2) \in [0,1]$ if and only if $c_1 \leqslant 0$. A lengthy, but otherwise simple, computation shows that $c_1 = w_1^1 w_4^2 - w_4^1 w_1^2 + w_2^1 w_3^2 - w_3^1 w_2^2$. Therefore, $(\alpha^+, 1)$ is the unique solution in $[0,1]^2$ of (10) if and only if (32) holds.
[Alternatively, we could also use the rewriting as the eigenvalue problem $[C^{-1}D - \mu I]\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = (\beta-1)/\beta$, for which we know that $\mu = 0$ is eigenvalue, and hence $(\alpha^+, 1)$ is a solution of (10). Therefore, $\mu = 0$ is the only eigenvalue less than or equal to 0 (and $(\alpha^+, 1)$ is the unique solution of (10) in $[0,1]^2$) if and only if $\operatorname{tr}(C^{-1}D) > 0$, that is, if and only if (32) holds.]

(b) Now we have $\det(B) = 0$ and $\det(D) < 0$, hence $P(0) < 0$ and $P(1) > 0$, and thus P has only one root in $[0,1]$. Since we know that $(1, \beta^+)$ is a solution, then this is the only solution.

(4) According to Table 6, we cannot say a priori if any of A, B, C, D in (30) is invertible.

(a) Here we have equality in (25), hence $\det(D) = 0$, but (26) is satisfied, hence $\det(B) > 0$. This means that B in (30) is invertible, hence we can consider the eigenvalue problem $[B^{-1}A - \mu I]\left[\begin{smallmatrix}1-\beta\\ \beta\end{smallmatrix}\right] = 0$, $\mu = (\alpha-1)/\alpha$, for which we know that $\mu_1 = (\alpha^+ - 1)/\alpha^+$ is eigenvalue (given that $f_{\Sigma_1^+}$ is a sliding vector field). So, for the two eigenvalues we have $\mu_1 = \mu^+ = (\alpha^+ - 1)/\alpha^+$, and $\mu_2 = \frac{1}{\mu_1}\frac{\det(A)}{\det(B)}$ and thus (since $\mu_1 < 0$ and $\det(B) > 0$) μ^+ is the only eigenvalue less than (or equal to) 0 if and only if $\det(A) < 0$, that is, if and only if the first relation in (33) holds. However, in case $Q(0) \geqslant 0$, we still cannot say that to the two roots of $Q(\alpha)$ there correspond two values of $\beta \in [0,1]$. So, consider this case, and let $\alpha_1 = \alpha^+$ and α_2 be the two roots of Q. Let $\beta = h(\alpha)$ from (18), for which obviously $1 = h(\alpha^+)$. A straightforward computation shows that the value of β associated to α_2 is in $[0,1]$ if and only if $\alpha_2 \geqslant \alpha_1$, that is, if and only if $\mu_2 \leqslant \mu_1$. Now, $\mu_2 = \operatorname{tr}(B^{-1}A) - \mu_1$, and $\mu_1 = w_2^1/w_4^1$. With a little algebra, $\operatorname{tr}(B^{-1}A) - \mu_1 = \frac{w_1^1 w_4^2 - w_4^1 w_1^2}{\det B}$. Hence, the requirement of $\mu_2 \leqslant \mu_1$ translates into

$$w_3^2\left(w_2^1 w_4^2 - w_4^1 w_1^2\right) \geqslant w_4^2\left(w_1^1 w_4^2 - w_4^1 w_1^2\right).$$

In conclusion, $(\alpha^+, 1)$ is the only solution of (10) if and only if (33) holds.

(b) Now we have equality in (26), hence $\det B = 0$, but (25) holds, hence $\det D < 0$. Therefore, we consider the eigenvalue problem $[D^{-1}C - \mu I]\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = (\beta - 1)/\beta$, for which we know that $\mu_1 = (\beta^+ - 1)/\beta^+$ is eigenvalue (given that $f_{\Sigma_2^+}$

is a sliding vector field). So, for the two eigenvalues we have $\mu_1 = \mu^+ = (\beta^+ - 1)/\beta^+$, and $\mu_2 = \frac{1}{\mu_1}\frac{\det(C)}{\det(D)}$ and thus (since $\mu_1 < 0$ and $\det(D) < 0$) μ^+ is the only eigenvalue less than (or equal to) 0 if and only if $\det(C) > 0$, that is, if and only if the first relation in (34) holds.

Similarly to case (a), in case $P(0) \geqslant 0$, we still need to see when/if to the two roots of $P(\beta)$ there correspond two values of $\alpha \in [0,1]$. So, suppose $P(0) \geqslant 0$ and let $\beta_1 = \beta^+$ and β_2 be the two roots of P. Working with $\alpha = g(\beta)$ from (16), a straightforward computation shows that the value of α associated to β_2 is in $[0,1]$ if and only if $\alpha_2 \geqslant \alpha_1$, that is, if and only if $\mu_2 \leqslant \mu_1$. Proceeding in a similar way to case (a), now the requirement of $\mu_2 \leqslant \mu_1$ translates into

$$w_3^2\left(w_2^1 w_4^2 - w_4^1 w_1^2\right) \geqslant w_4^2\left(w_1^1 w_4^2 - w_4^1 w_1^2\right).$$

In conclusion, $(1, \beta^+)$ is the only solution of (10) if and only if (34) holds.

(5) This case is effectively much the same as case $(S_{\Sigma_1^+,\Sigma_2^+})$(3) above. From Table 7, we have that $\det A < 0$ and $\det C < 0$.
   (a) When there is equality in (25), but (26) holds, then $\det B > 0$. Thus, $Q(\alpha)$ has only one root in $[0,1]$, and since $f_{\Sigma_1^+}$ is a sliding vector field, then $(\alpha^+, 1)$ is the only solution of (10) in $[0,1]^2$.
   (b) Now there is equality in (26), but (25) holds, so that $\det D < 0$. Hence, there are two roots in $[0,1]$ for $P(\beta)$. To ascertain if there are two corresponding values of α, we look at the eigenvalue problem $(A^{-1}B - \mu I)\left[\begin{smallmatrix}1-\beta\\ \beta\end{smallmatrix}\right] = 0$, $\mu = (\alpha - 1)/\alpha$. Since $f_{\Sigma_2^+}$ is a sliding vector field, then $\mu_1 = 0$ is an eigenvalue. For the other eigenvalue we have $\mu_2 = \operatorname{tr}(A^{-1}B) = \frac{1}{\det(A)}(w_2^2 w_3^1 - w_2^1 w_3^2 - w_1^2 w_4^1 + w_1^1 w_4^2)$ and thus $\mu_2 > 0$ (hence $(\alpha, \beta) = (1, \beta^+)$ is the only solution of (10) in $[0,1]^2$) if and only if (35) is satisfied.

$(S_{\Sigma_1^\pm})$ Regardless of whether we have the coefficients as in Table 8, or as in Table 9, we have the following situation.
   (a) When there is equality in (27)(a), while (27)(b) is satisfied, then the matrix D in (30) is invertible (and $\det(D) < 0$), while C is singular. We look at the eigenvalues of $(D^{-1}C - \mu I)\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = \beta/(\beta - 1)$. Clearly $\mu = 0$ is an eigenvalue (corresponding to the solution $(\alpha^-, 0)$ since $f_{\Sigma_1^-}$ is a sliding vector field). The other eigenvalue is $\operatorname{tr}(D^{-1}C) = \frac{1}{\det(D)}(w_1^1 w_4^2 - w_1^2 w_4^1 + w_2^1 w_3^2 - w_2^2 w_3^1)$ and thus it is positive if and only if (36) is satisfied. Finally, observe that using $\alpha = g(\beta)$ from (16), we have that $\alpha \in [0,1]$ for any $\beta \in [0,1]$. Hence, $(\alpha, \beta) = (\alpha^-, 0)$ is the only solution of (10) in $[0,1]^2$ if and only if (36) holds.
   (b) When there is equality in (27)(b), while (27)(a) is satisfied, then the matrix C in (30) is invertible (and $\det(C) > 0$), while D is singular. We look at the eigenvalues of $(C^{-1}D - \mu I)\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = (\beta - 1)/\beta$. Clearly $\mu = 0$ is an eigenvalue (corresponding to the solution $(\alpha^+, 1)$ since $f_{\Sigma_1^+}$ is a sliding vector field). The other eigenvalue is $\operatorname{tr}(C^{-1}D) = \frac{1}{\det(C)}(w_3^2 w_2^1 - w_3^1 w_2^2 - w_1^2 w_4^1 + w_1^1 w_4^2)$ and thus it is positive if and only if (32) is satisfied. Again, using $\alpha = g(\beta)$ from (16), we have that $\alpha \in [0,1]$ for any $\beta \in [0,1]$. Hence, $(\alpha, \beta) = (\alpha^+, 1)$ is the only solution of (10) in $[0,1]^2$ if and only if (32) holds.

$(S_{\Sigma_1^\pm,\Sigma_2^+})$ We have three cases.
   (1) Because of Table 10, and equality in (27)(a), we have D in (30) invertible (and $\det(D) < 0$), while C is singular. The eigenvalues of $(D^{-1}C - \mu I)\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = \beta/(\beta - 1)$, are $\mu = 0$ (corresponding to the solution $(\alpha^-, 0)$), and $\operatorname{tr}(D^{-1}C)$. Thus, $\mu = 0$ is the only eigenvalue in $[0,1]$ if and only if (36) is satisfied. To complete the proof, observe that using $\alpha = \hat{g}(\beta)$ from (17) shows that $\alpha \in [0,1]$ for any $\beta \in [0,1]$.

Table 14
Example (1): $(S_{\Sigma_1^+} : 2)$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	$-1/8$	2	-1	-2
w_i^2	$1/8$	1	-1	-1

Table 15
Example (2): $(S_{\Sigma_1^+,\Sigma_2^+} : 4-a)$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	3	2	$1/8$	-2
w_i^2	1	1	$1/8$	-1

Hence, in conclusion, $(\alpha^-, 0)$ is the only solution of (10) in $[0,1]^2$ if and only if (36) is satisfied.

(2) Because of Table 11, and equality in (27)(b), we have C in (30) invertible (and $\det(C) > 0$), while D is singular. The eigenvalues of $(C^{-1}D - \mu I)$, $\mu = (\beta - 1)/\beta$, are $\mu = 0$ (corresponding to the solution $(\alpha^+, 1)$), and $\operatorname{tr}(C^{-1}D)$. As in case (1) above, using $\alpha = \hat{g}(\beta)$ from (17) shows that $(\alpha^+, 1)$ is the only solution of (10) in $[0,1]^2$ if and only if (32) is satisfied.

(3) Because of Table 12, we have that $\det A > 0$ and $\det B > 0$.
   (a) If there is equality in (27)(a), but (27)(b) is satisfied, then $\det C = 0$, $\det D < 0$, and we consider the eigenvalue problem $(D^{-1}C - \mu I)\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = \beta/(\beta - 1)$. Obviously, $\mu_1 = 0$ is an eigenvalue (corresponding to the solution $(\alpha^-, 0)$ of (10)). Since the other eigenvalue is $\operatorname{tr}(D^{-1}C)$, $\mu_1 = 0$ is the only eigenvalue less than or equal to 0 if and only if (36) is satisfied. To complete the proof, observe that using $\alpha = \hat{g}(\beta)$ from (17) shows that if there are two values of $\beta \in [0,1]$, then the corresponding values of α would be as well. In conclusion, $(\alpha^-, 0)$ is the only solution of (10) in $[0,1]^2$ if and only if (36) is satisfied.
   (b) If there is equality in (27)(b), but (27)(a) is satisfied, then $\det C > 0$, $\det D = 0$, and we consider the eigenvalue problem $(C^{-1}D - \mu I)\left[\begin{smallmatrix}1-\alpha\\ \alpha\end{smallmatrix}\right] = 0$, $\mu = \beta/(\beta - 1)$. Reasoning as in case (a) above shows that the solution $(\alpha^+, 1)$ is the only solution of (10) in $[0,1]^2$ if and only if (32) is satisfied. □

3.1. Examples: Multiple solutions

Here we give a few examples to illustrate having multiple solutions of (10) in $[0,1]^2$, when (some of) the conditions of Theorem 14 are violated.

Example 15. The following five situations exemplifies the general case. Below, with $(\bar{\alpha}, \bar{\beta})$ we indicate the solution associated to the continuation of the solution of (10).

(1) Case $(S_{\Sigma_1^+} : 2)$. Consider Table 14 (satisfying the signs as in Table 2). Here, (32) is violated.
   We have the two solutions $(\alpha^+ = 1/2, \beta = 1)$ and $(\bar{\alpha} = 3/11, \bar{\beta} = 2/7)$.

(2) Case $(S_{\Sigma_1^+,\Sigma_2^+} : 4-a)$. Take Table 15, see Table 6. There is equality in (25), (26) holds, but both relations in (33) are violated.
   We have the two admissible values for α, $\alpha^+ = 1/2$ and $\bar{\alpha} = 8/9$, and associated values of β given by $\beta = 1$ and $\bar{\beta} = 2/9$.

(3) Let us again consider case $(S_{\Sigma_1^+,\Sigma_2^+} : 4-a)$, with Table 16, see Table 6. Now there is equality in (25), (26) holds, the first relation in (33) is violated, but the second relation in (33) holds true.

Table 16
Example (3): $(S_{\Sigma_1^+,\Sigma_2^+}:4-a)$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	$21/10$	2	$1/8$	-2
w_i^2	1	1	$1/8$	-1

Table 17
Example (4): $(S_{\Sigma_1^\pm}:a)$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	1	1	-1	$-1/4$
w_i^2	1	2	-1	-1

Table 18
Example (5): $(S_{\Sigma_1^\pm,\Sigma_2^+}:3-b)$.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	1	2	-1	-1
w_i^2	-1	1	1	-2

Again, we have two admissible values for α, as expected: $\alpha^+=1/2$ and $\alpha=8/9$. However, we now have only one admissible value of β: $\bar\beta=1$ (the other value being $\beta=11/9$), so there is only one admissible solution to (10).

(4) Case $(S_{\Sigma_1^\pm}:a)$. We have Table 17, see Table 8. There is equality in (27)(a), (27)(b) holds, but (36) is violated.
We have the two solutions ($\alpha^-=1/2$, $\beta=0$) and ($\bar\alpha=4/7$, $\bar\beta=1/3$).

(5) Case $(S_{\Sigma_1^\pm,\Sigma_2^+}:3-b)$. Consider Table 18, see Table 12. There is equality in (27)(a), and (27)(b) holds.
Here, we have only the solution $(\alpha,\beta)=(\alpha^-,0)$, with $\alpha^-=\bar\alpha=1/2$ and it has multiplicity 2. □

Remark 16. As previously remarked, when situations such as those of Example 15 occur, Σ is no longer attractive, so trajectories from outside Σ will not reach it. At the same time, **if** we are **on** Σ, and are following the trajectory determined by (8) when these changes in stability take place, then when there are multiple solutions to (10) there is a well-defined vector field of the type (8) which keeps us on Σ. However, we also note that – when Σ loses attractivity at some point x, as above – then there is always a vector field of the type (8) which is an exit vector field. This means that we can still exit Σ, in case (8) is not uniquely defined, but we will have to do so continuously (rather than smoothly). This is a key difference with respect to the situation of a co-dimension 1 discontinuity surface, where all exits at first order are tangential, hence smooth.

Remark 17. A final remark pertains to the fact that we have assumed that the w_j^i's ($i=1,2$, $j=1,\dots,4$) are never 0, see (11). Violating this assumption, at co-dimension 1, means that one (but not more) of the w_j^i's is 0. Except for the fact that it may (but does not have to) lead to a loss of attractivity of Σ, the eventuality that one of the $w_j^i=0$ does not produce anything particular insofar as solvability of (10). For example, suppose we have the following two modifications of the situation of case $(S_{\Sigma_1^+}:1)$ (see Theorem 7 and cf. with Table 1):

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	$=0$	>0	>0	<0
w_i^2	<0	<0	>0	>0

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	$=0$	>0	<0
w_i^2	<0	<0	>0	>0

Note that (25) holds for both tables above. However, for the signs of the table on the left Σ is not attractive (but $f_{\Sigma_1^-}$ is not an exit vector field), while for the signs of the table on the right Σ is still attractive.

3.2. Co-dimension 2

There are a multitude of possible co-dimension 2 losses of attractivity situations, and we have not even attempted to classify them. We simply remark that a possible one takes place when a pair w_j^1, w_j^2, for a $j = 1, 2, 3, 4$, goes through 0 at some $x \in \Sigma$. At that point, the vector field f_j is itself in T_Σ, it is an exit vector field, and there is a solution (α, β) of (10) at the vertices of the unit square $[0, 1]^2$. For example, if $w_1^1 = w_1^2 = 0$, then f_1 is an exit vector field, obtained with $\alpha = \beta = 0$ in (8). In general, there may be another solution of (10) in the unit square. Indeed, $(0, 0)$ is the only solution of (10) in the unit square if and only if $(w_3^1 w_4^2 - w_2^2 w_4^1)(w_3^1 w_2^2 - w_3^2 w_2^1) > 0$. If this is the case, then the vector field (8) according to first order theory will smoothly leave Σ and enter in the region R_1. [To verify the just stated necessary and sufficient condition, we use the interpretation based on Lemma 12. Namely, letting $\mu = \alpha/(\alpha - 1)$, we consider the eigenvalue problem $\det(A - \mu B) = 0$, which gives $\mu[\mu(w_3^1 w_4^2 - w_2^2 w_4^1) - (w_3^1 w_2^2 - w_3^2 w_2^1)] = 0$.]

4. Numerical example

In this section we report on results of numerical simulation to highlight (some of) the behaviors previously examined. The example we consider is a generalization of the classical stick-slip system (e.g., see [12]). We have the discontinuity surfaces

$$\Sigma_1 = \{x \in \mathbb{R}^3 \colon h_1(x) = x_2 - p\}, \qquad \Sigma_2 = \{x \in \mathbb{R}^3 \colon h_2(x) = x_3 - q\}, \qquad \Sigma = \Sigma_1 \cap \Sigma_2,$$

and in our experiments we fix $p = 0.5$ and $q = 1$. The system consists of the following four vector fields, all at least continuous in their respective regions of definition:

$$R_1\ (h_1 < 0,\ h_2 < 0)\colon \quad f_1(x) = \begin{pmatrix} -(x_2 + x_3) \\ -x_1 + \frac{1}{(1+p)-x_2} \\ -x_1 - \frac{1}{(1+q)-x_3} \end{pmatrix};$$

$$R_2\ (h_1 < 0,\ h_2 > 0)\colon \quad f_2(x) = \begin{pmatrix} -(x_2 + x_3) \\ -x_1 + \frac{1}{(1+p)-x_2} \\ -x_1 - \frac{1}{(1-q)+x_3} \end{pmatrix};$$

$$R_3\ (h_1 > 0,\ h_2 < 0)\colon \quad f_3(x) = \begin{cases} \begin{pmatrix} -(x_2 + x_3) \\ -x_1 + \frac{1}{(1-p)+x_2} \\ -x_1 + \frac{1}{(1+q)-x_3} \end{pmatrix} & \text{when } x_1 \geqslant -1.3, \\ \begin{pmatrix} -(x_2 + x_3) \\ 6 + 1.3 + 6\frac{x_1}{1.3} + \frac{1}{(1-p)+x_2} \\ -x_1 + \frac{1}{(1+q)-x_3} \end{pmatrix} & \text{when } x_1 < -1.3; \end{cases}$$

$$R_4\ (h_1 > 0,\ h_2 > 0)\colon \quad f_4(x) = \begin{cases} \begin{pmatrix} -(x_2 + x_3) \\ -x_1 - \frac{1}{(1-p)+x_2} \\ -x_1 + \frac{1}{(1-q)+x_3} \end{pmatrix} & \text{when } x_1 \geqslant -1, \\ \begin{pmatrix} -(x_2 + x_3) \\ -x_1 - \frac{1}{(1-p)+x_2} \\ 130 + 129x_1 + \frac{1}{(1-q)+x_3} \end{pmatrix} & \text{when } x_1 < -1. \end{cases}$$

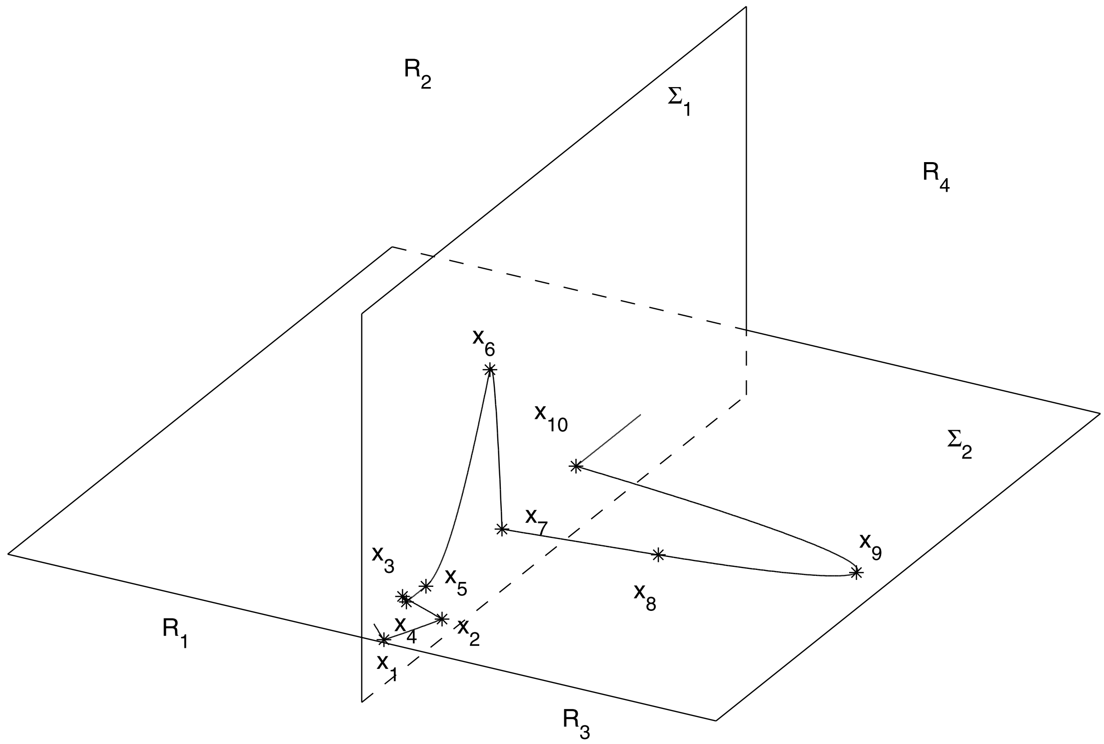


Fig. 14. Solution trajectory.

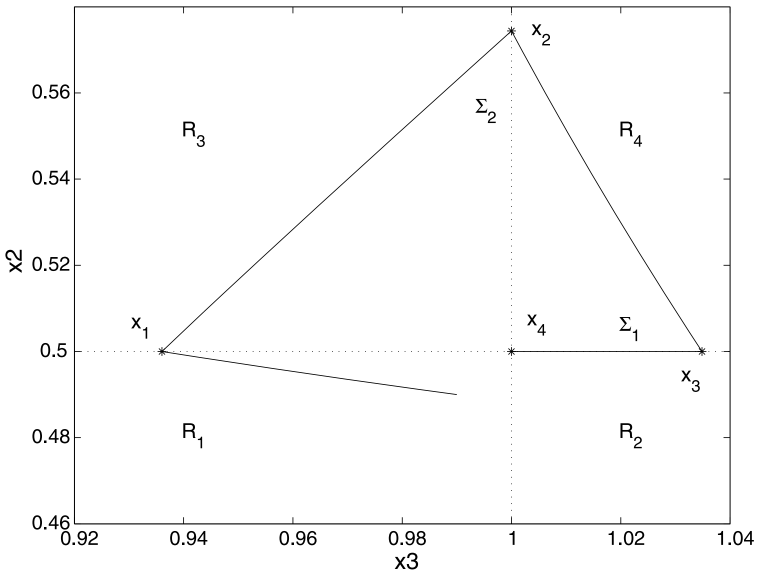


Fig. 15. Initial part of solution trajectory.

In our experiments, we have taken initial condition $x_0 = [0.7, 0.49, 0.99]$, but initial conditions in a neighborhood of x_0 of course lead to similar behavior. We can distinguish several different dynamics of the solution with respect to the two discontinuity surfaces. Indeed, there are several event points, that is, values where the solution reaches a different regime: a different region and/or sliding surface. We will assign a time value t_j to each event point x_j. The trajectory in the time interval $[0, 3]$ is plotted in Fig. 14: the event points are marked by asterisks and are labeled x_j, $j = 1, \ldots, 10$.

The initial condition is in region R_1 and the trajectory crosses Σ_1^- at t_1 and enters R_3 (transversal intersection), at t_2 crosses Σ_2^+ and enters R_4 (transversal intersection), at t_3 hits Σ_1^+ and starts sliding on it in the direction of Σ. In this regime, the vector field $f_{\Sigma_1^+}$ is the one given by Filippov's theory as in (6). In Fig. 15 we plot the trajectory up to time t_4 in the (x_3, x_2)-plane. The plot shows how the

Table 19
w^i_j's at $x = x_5$.

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
w^1_i	1	1	1	−1
w^2_i	−1	−1	1	1

Table 20
w^i_j's at $x = x_{10}$.

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
w^1_i	> 0	> 0	< 0	> 0
w^2_i	> 0	> 0	> 0	< 0

solution spirals around Σ before reaching Σ_1^+ at x_3. Then, while sliding on Σ_1^+ the solution reaches Σ at time t_4 at the point $x_4 \approx (0.3041, 0.5, 1)$.

At x_4, the vector fields f_j, $j = 1, 2, 3, 4$, satisfy the conditions of Table 1 (see Section 2) as well as condition (25), so that Σ is attractive, f_Σ as in (8) is well defined and the solution starts sliding on Σ. At time t_5 the solution is at $x_5 = (0, 0.5, 1)$ and at this point there is equality in (25) and the exit condition (32) is satisfied. Indeed the $w^i_j(x)$'s for $x = x_5$ are as in Table 19 and $w^1_1 w^2_4 - w^1_4 w^2_1 + w^1_2 w^2_3 - w^1_3 w^2_2 = 2 > 0$.

Remark 18. Notice that if we impose the conditions for a general Filippov vector field $F(x) = \sum_{j=1}^4 \lambda_j(x) f_j(x)$ to be tangent to Σ at $x = x_5$, we have the following system (see Eq. (4)):

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 &= 0, \\ -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 &= 0, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 1, \end{aligned} \tag{37}$$

so there is the one parameter family of solutions $\lambda_2 = 1/2 - \lambda_1$, $\lambda_3 = 0$, $\lambda_4 = 1/2$, and $0 \leqslant \lambda_1 \leqslant 1/2$. This means that not all possible Filippov vector fields on Σ would exit Σ at $x = x_5$. The Filippov solution with $\lambda_1 = 0$, gives our choice $f_\Sigma(x_5)$ and is an exit solution.

So, at $t = t_5$, f_Σ aligns to $f_{\Sigma_1^+}$ and the solution exits Σ smoothly to slide on Σ_1^+. At time t_6 the solution reaches $x_6 \approx (-1, 0.5, 1.5111)$, $f_{\Sigma_1^+}$ aligns to f_4 and the solution exits Σ_1^+ smoothly to enter in R_4. At time t_7 it reaches Σ_2^+ at $x_7 \approx (-1.1322, 0.5050, 1)$. At this point $w^2_3 > 0$ while $w^2_4 < 0$ and sliding starts taking place on Σ_2^+ away from Σ_1. At time t_8, the solution reaches the surface $x_1 = -1.3$ at the point $x_8 \approx (-1.3, 0.7105, 1)$. At this point f_3 is continuous but not differentiable and we locate the exact intersection of the numerical trajectory with the surface $x_1 = -1.3$ to preserve the order of the numerical integrator. At time t_9 we reach $x_9 \approx (-1.7377, 0.9507, 1)$. For time $t > t_9$ the trajectory continues sliding on Σ_2^+ but now in the direction of Σ_1; in this case, the following condition is satisfied:

$$\frac{w^1_3}{w^2_3} < \frac{w^1_4}{w^2_4}. \tag{38}$$

At time t_{10} the solution reaches the point $x_{10} \approx (-2.3430, 0.5, 1)$ on Σ. The vector fields $f_j(x_{10})$, $j = 1, \dots, 4$, satisfy the conditions of Table 20 and the behavior on Σ is analogous to the one of Case $(S_{\Sigma_1^+} : 2)$. This, together with (38), ensures attractivity of Σ. The solution now starts sliding on Σ with vector field f_Σ as in (8) and no further exit conditions are encountered.

For completeness, we remark that our computations have been made with an event driven technique. All event points (when a different regime is reached) have been computed by the secant method. Integration of all relevant differential equations was made using the classical explicit Runge–Kutta (RK) scheme of order four, a projected RK method in case of sliding motion to ensure that all evaluations are made on the constraints' surfaces (e.g., see [9]). The stepsize τ was held constant and equal to $\tau = 0.0025$, and of course adjusted when using the secant method to locate event points. Solution of the system (10) was done by Newton's method with divided difference Jacobian.

5. Conclusions

In this work, we have considered sliding motion – in the sense of Filippov – on a discontinuity surface Σ of co-dimension 2, intersection of two co-dimension 1 discontinuity surfaces. We have shown that a certain Filippov sliding vector field f_F (originally suggested in [2,6,10]) is well defined whenever Σ is attracting nearby dynamics, with attracting sliding motion occurring on at least one of the sub-surfaces given by Σ_1^+, Σ_1^-, Σ_2^+, Σ_2^-. We have further proposed a first order theory for generic co-dimension 1 losses of attractivity of Σ, exit conditions, and clarified when the sliding vector field f_F will (or will not) smoothly leave Σ. A simple (but rigorous) numerical simulation illustrated our theory.

Albeit it is in principle possible to give other choices of Filippov vector field on Σ, we believe that any other possibility would have to pass the test of being well defined for attractive Σ of the type we considered in this work. At present, we know of no other choice beside the one we examined herein. Furthermore, we believe that our effort will also serve as useful benchmark for future works on this topic. In particular, we (i) created a realistic mathematical framework within which to validate possible choices of sliding vector fields on Σ, (ii) clarified that it is the dynamics near Σ that must be used to justify the selection of a sliding Filippov vector field on Σ, and (iii) discussed exit conditions.

There are many directions in which the present work could be taken. We expect to tackle some of the following ones.

(i) In [8], a natural spatial regularization was shown to converge (in the limit of the regularization parameter) to the above mentioned choice of sliding vector field, whenever Σ is nodally attractive, and for constant vector fields. It will be interesting to show (or disprove) that the same holds true for the presently proposed broader class of attractive Σ.
(ii) We have not even begun analyzing co-dimension 2 losses of attractivity for Σ; there are a multitude of these, and their complete classification appears to be a daunting task.
(iii) Finally, the case of discontinuity surfaces of co-dimension 3 or higher is still quite open. In this case, there is not yet a general construction on how to select a unique Filippov sliding vector field even under nodal attractivity assumptions.

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