AUTHOR QUERY FORM

ELSEVIER	**Journal:** MATCOM **Article Number:** 4028	**Please e-mail or fax your responses and any corrections to:** **E-mail:** corrections.esch@elsevier.thomsondigital.com **Fax:** +353 6170 9272

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list. Note: if you opt to annotate the file with software other than Adobe Reader then please also highlight the appropriate place in the PDF file. To ensure fast publication of your paper please return your corrections within 48 hours.

For correction or revision of any artwork, please consult http://www.elsevier.com/artworkinstructions.

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

Location in article	Query / Remark: click on the Q link to go Please insert your reply or correction at the corresponding line in the proof
Q1	Please confirm that given names and surnames have been identified correctly.
Q2	Please check the telephone number of the corresponding author, and correct if necessary.
Q3	As per the stylesheet of this journal there should be a maximum of five keywords but in this article more than five keywords are given. Please delete extra keyword.
Q4	Please provide captions for Tables 1–5.

Please check this box or indicate your approval if you have no corrections to make to the PDF file ☐

Thank you for your assistance.

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
MATHEMATICS AND COMPUTERS IN SIMULATION
Mathematics and Computers in Simulation xxx (2014) xxx–xxx
www.elsevier.com/locate/matcom

Original Article

Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface☆

Q1 L. Dieci[a], C. Elia[b],*, L. Lopez[b]

[a] School of Mathematics, Georgia Tech, Atlanta, GA 30332, USA
[b] Dipartimento di Matematica, Univ. of Bari, I-70100 Bari, Italy

Received 26 March 2013; received in revised form 18 December 2013; accepted 19 December 2013

Abstract

We consider Filippov sliding motion on a co-dimension 2 discontinuity surface. We give conditions under which Σ is *attractive through sliding* which are sharper than those given in a previous paper of ours. Under these sharper conditions, we show that the sliding vector field considered in the same paper is still uniquely defined and varies smoothly in $x \in \Sigma$. A numerical example illustrates our results.

Q3 *Keywords:* Piecewise smooth systems; Filippov systems; Sliding modes; Discontinuity surface; Co-dimension 2; Attractivity

1. Introduction

An outstanding problem in the study of piecewise smooth differential systems is how to properly define a Filippov sliding vector field when sliding motion has to take place on a co-dimension 2 surface, Σ, intersection of two co-dimension 1 surfaces. In [3], we gave sufficient conditions which guaranteed that Σ attracted nearby trajectories (through sliding), and that a certain sliding vector field, (7) below, was well defined on Σ. Our goal in this work is to sharpen the conditions given in [3], while still obtaining the same conclusions.

The basic problem we consider is the piecewise smooth system

$$\dot{x} = f(x), \quad f(x) = f_i(x), \quad x \in R_i, \quad i = 1, \ldots, 4, \tag{1}$$

with initial condition $x(0) = x_0 \in R_i$, for some i. Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and (locally) $\mathbb{R}^n = \overline{\bigcup_i R_i}$. Each f_i can be assumed smooth in an open neighborhood of the closure of each R_i, $i = 1, \ldots, 4$. Clearly, from (1), the vector field is not properly defined on the boundaries of the R_i's.

Above, we will assume that the R_i's are (locally) separated by two intersecting smooth surfaces of co-dimension 1, $\Sigma_1 = \{x : h_1(x) = 0\}$ and $\Sigma_2 = \{x : h_2(x) = 0\}$, and we let $\Sigma = \Sigma_1 \cap \Sigma_2$. We will always assume that $\nabla h_1(x) \neq 0, x \in \Sigma_1$,

☆ This work was done while the second and third author were visiting the School of Mathematics of Georgia Institute of Technology, whose hospitality is gratefully acknowledged.
Q2 * Corresponding author. Tel.: +39 3333514208.
E-mail addresses: dieci@math.gatech.edu (L. Dieci), cinzia.elia@uniba.it, cinzia.elia@gmail.com (C. Elia), luciano.lopez@uniba.it (L. Lopez).

http://dx.doi.org/10.1016/j.matcom.2013.12.005
0378-4754/

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

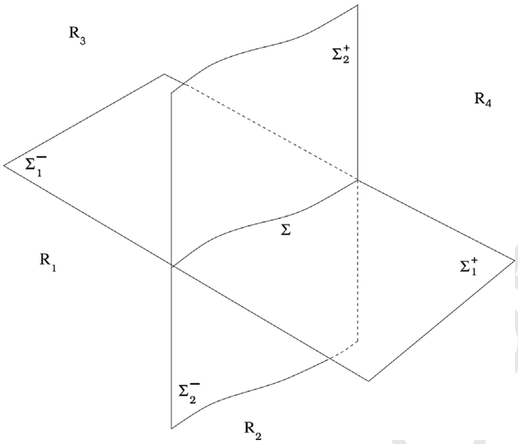

Fig. 1. Regions R_i's, Σ and $\Sigma^{\pm}_{1,2}$.

$\nabla h_2(x) \neq 0, x \in \Sigma_2$, that $h_{1,2}$ are $\mathcal{C}^k$ functions, with $k \geq 2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for x on (and in a neighborhood of) Σ.

Without loss of generality, we can label the regions as follows:

$$\begin{aligned} &R_1 : f_1 \text{ when } h_1 < 0, h_2 < 0, \quad R_2 : f_2 \text{ when } h_1 < 0, h_2 > 0, \\ &R_3 : f_3 \text{ when } h_1 > 0, h_2 < 0, \quad R_4 : f_4 \text{ when } h_1 > 0, h_2 > 0, \end{aligned} \tag{2}$$

and we will also adopt the notation $\Sigma^{+}_{1,2}$ and $\Sigma^{-}_{1,2}$ to denote the set of points $x \in \Sigma_{1,2}$ for which we also have $h_{2,1}(x) > 0$ or $h_{2,1}(x) < 0$. See Fig. 1.

Finally, we let

$$\begin{aligned} &w^1_1 = \nabla h_1^T f_1, \quad w^1_2 = \nabla h_1^T f_2, \quad w^1_3 = \nabla h_1^T f_3, \quad w^1_4 = \nabla h_1^T f_4, \\ &w^2_1 = \nabla h_2^T f_1, \quad w^2_2 = \nabla h_2^T f_2, \quad w^2_3 = \nabla h_2^T f_3, \quad w^2_4 = \nabla h_2^T f_4, \end{aligned} \tag{3}$$

which we assume to be well defined in a neighborhood of Σ. As it turns out, the signs of the w^i_j's are the key property to monitor.

Remark 1. Looking ahead, let us suppose that we are following a solution trajectory on Σ, $x(t)$. In this case, we will need to consider the w^i_j's along this solution trajectory, and can thus think of the w^i_j's as functions of t.

Remark 2. The classical Filippov theory (see [6]) is concerned with the case of two regions separated by a surface Σ defined as the 0-set of a smooth scalar valued function h:

$$\begin{aligned} &\dot{x} = f_1(x), \quad x \in R_1 = \{x : h(x) < 0\}, \quad \text{and} \quad \dot{x} = f_2(x), \quad x \in R_2 = \{x : h(x) < 0\}, \\ &\Sigma := \{x \in \mathbb{R}^n : h(x) = 0\}, \quad h : \mathbb{R}^n \to \mathbb{R}, \end{aligned} \tag{4}$$

Filippov convexification method allows to define a sliding motion on Σ, in particular when Σ attracts nearby trajectories. Filippov proposal is to take a convex combination of f_1 and f_2 and impose that the vector field is tangent to Σ. That is, take $f_F := (1-\alpha)f_1 + \alpha f_2$, with α chosen so that $f_F \in T_\Sigma$:

$$x' = (1-\alpha) f_1 + \alpha f_2, \quad \alpha = \frac{w^1}{w^1 - w^2}, \quad w^1 = \nabla h(x)^T f_1(x), \quad w^2 = \nabla h(x)^T f_2(x), \tag{5}$$

With the above in mind, we can consider sliding on $\Sigma^{\pm}_{1,2}$ previously defined (if such sliding motion indeed can take place). According to (5), we will call $f_{\Sigma^{\pm}_{1,2}}$ these four vector fields, defined as follows (as long as the denominators are nonzero):

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

$$f_{\Sigma_1^+} = (1-\alpha^+)f_2 + \alpha^+ f_4, \quad f_{\Sigma_1^-} = (1-\alpha^-)f_1 + \alpha^- f_3, \quad f_{\Sigma_2^+} = (1-\beta^+)f_3 + \beta^+ f_4,$$

$$f_{\Sigma_2^-} = (1-\beta^-)f_1 + \beta^- f_2, \quad \text{and} \quad \alpha^+ = \frac{w_2^1}{w_2^1 - w_4^1}, \quad \alpha^- = \frac{w_1^1}{w_1^1 - w_3^1}, \quad \beta^+ = \frac{w_3^2}{w_3^2 - w_4^2},$$

$$\beta^- = \frac{w_1^2}{w_1^2 - w_2^2}. \tag{6}$$

2. Background

When attempting to define a Filippov sliding vector field on $\Sigma = \Sigma_1 \cap \Sigma_2$, one needs to consider a convex combination of the four vector fields $f_1, \ldots, f_4$; $f_F = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4$, $\lambda_i \geq 0$, $i = 1, \ldots, 4$, and $\sum_i \lambda_i = 1$. Imposing that $f_F \in T_\Sigma$, however, is now no longer sufficient (unlike the case of Remark 2) to uniquely determine the coefficients λ_i's.

To resolve the above ambiguity, in [2,5,1] the authors proposed to restrict consideration to the following bilinear vector field

$$f_F = (1-\alpha)(1-\beta)f_1 + (1-\alpha)\beta f_2 + \alpha(1-\beta)f_3 + \alpha\beta f_4, \tag{7}$$

where now $\alpha, \beta \in [0, 1]$ need to be found so to satisfy the following nonlinear system:

$$(1-\alpha)(1-\beta)\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix} + (1-\alpha)\beta\begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix} + \alpha(1-\beta)\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix} + \alpha\beta\begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix} = 0. \tag{8}$$

The question then becomes solvability (unique) of this system. To address this problem, in [3] we considered the case of Σ being reached through sliding on one of the $\Sigma_{1,2}^{\pm}$, and to characterize this situation we worked under the following assumptions.

Assumptions 1.

(a) $(w_j^1(x), w_j^2(x))$ do not have the same signs as $(h_1(x), h_2(x))$ for $x \in R_j$, $j = 1, 2, 3, 4$.

(b) At least one pair of the relations $[(1^+) and (1_a^+)]$, or $[(1^-) and (1_a^-)]$, or $[(2^+) and (2_a^+)]$, or $[(2^-) and (2_a^-)]$, is satisfied on Σ and in a neighborhood of Σ, where

$$(1^+) w_2^1 > 0,\ w_4^1 < 0, \quad (1_a^+) \frac{w_2^2}{w_2^1} - \frac{w_4^2}{w_4^1} < 0,$$
$$(1^-) w_1^1 > 0,\ w_3^1 < 0, \quad (1_a^-) \frac{w_3^2}{w_3^1} - \frac{w_1^2}{w_1^1} < 0,$$
$$(2^+) w_3^2 > 0,\ w_4^2 < 0, \quad (2_a^+) \frac{w_3^1}{w_3^2} - \frac{w_4^1}{w_4^2} < 0,$$
$$(2^-) w_1^2 > 0,\ w_2^2 < 0, \quad (2_a^-) \frac{w_2^1}{w_2^2} - \frac{w_1^1}{w_1^2} < 0.$$

(c) If any of $(1^{\pm})$ or $(2^{\pm})$ is satisfied, then $(1_a^{\pm})$ or $(2_a^{\pm})$ must be satisfied as well.

Let us clarify the meaning of Assumptions 1 insofar as the dynamics of the system. Assumption 1(a) implies that the vector fields f_j, $j = 1, \ldots, 4$, must point toward at least one of $\Sigma_{1,2}$. Assumption 1(b) guarantees that there is attractive sliding toward Σ along at least one of the $\Sigma_{1,2}^{\pm}$. Assumption 1(c) states that if attractive sliding occurs along $\Sigma_{1,2}^{\pm}$ it must be toward Σ.

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

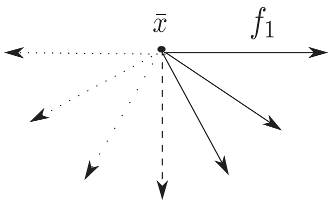

Fig. 2. Admissible f_1 under the assumption $w_1^1(\bar{x}) < 0$.

It must be emphasized that our theory is justified under the assumption that Σ **is attractive in finite time upon sliding on a co-dimension 1 surface**. Hence, Assumption 1(c) are fundamental in this setting.

In [3], we made a simplifying assumption on the w_j^i's, expressed by the following:

Old assumption (see [3]):

$$w_j^i(x) \quad \text{are bounded away from } 0, \quad i = 1, 2, \quad j = 1, 2, 3, 4, \; x \in \Sigma. \tag{9}$$

Note that (9) implies that no trajectory can approach Σ tangentially from a region R_j, $j = 1, 2, 3, 4$.

Under Assumptions 1 and (9), in [3] it was proved that Σ attracted nearby trajectories, which in fact reached Σ in finite time, and moreover that (8) had a unique solution. More precisely, we proved the following result.

Theorem 3. *Let Assumptions 1 be satisfied and let (9) hold.*

(a) *Then, there exists a unique solution $(\bar{\alpha}, \bar{\beta})$ of system (8) in $(0, 1) \times (0, 1)$.*
(b) *Further, let (1_a^+), (1_a^-), (2_a^+), and (2_a^-), hold uniformly; that is (1_a^+) be replaced by $(w_2^2/w_2^1) - (w_4^2/w_4^1) \leq -\lambda_1^+ < 0$, and similarly for the others. Then, Σ is attractive in finite time.*

3. Weaker attractivity assumptions

It was already observed in [3] that (9) was too strong a sufficient condition to guarantee the conclusions of Theorem 3. For this reason, our goal below is to weaken (9) in such a way that: Σ still remains attractive through sliding and reached in finite time, and the vector field (7) still is well defined on Σ.

We restrict ourselves to co-dimension 1 phenomena, as characterized by having just one scalar value among the w_j^i's being 0 at any point in Σ. Higher co-dimension phenomena (such as two of the w_j^i's becoming 0 at the same time) are not necessarily going to preclude the aforementioned conclusions (i.e., attractivity of Σ and well posedness of the vector field (7)), but require a host of different possibilities to be examined, which is beyond our present scope.

Assumptions 2. At most one of the w_j^i's is zero at any given x on Σ.

In this paper we replace condition (9) with Assumptions 2. This means that, while sliding on Σ, one of the w_j^i's can be zero at a point $x \in \Sigma$, as long as Assumptions 1 are still satisfied.

Assumptions 1 and 2 together imply the following:

(i) f_j cannot be tangential to Σ at $x \in \Sigma$;
(ii) f_j cannot be tangential to Σ_2 (respectively, Σ_1), at a point $\bar{x}$ on Σ, whenever f_j points away from Σ_1 (respectively, Σ_2) at $\bar{x}$.

Item (i) above is just a rewriting of Assumptions 2. To exemplify the instances in (ii), assume that Σ is attractive and that we are following a trajectory on Σ. Assumptions 1 are satisfied along the trajectory and take for example $w_1^1(x(t)) < 0$ and $w_1^2(x(t)) > 0$ for $t < \bar{t}$. At $\bar{x} = x(\bar{t})$, $w_1^2(\bar{x}) = 0$ while $w_1^1(\bar{x})$ stays negative. Now f_1 is tangent to Σ_2, but points away from Σ_1. Thus, for $t > \bar{t}$, $w_1^2(x(t)) < 0$ and $w_1^1(x(t)) < 0$. This, together with the continuity of f_1, violates Assumptions 1(a). The vector f_1 now points away from Σ so that Σ looses attractivity. The same reasoning as above applies to any other vector $w_j = (w_j^1, w_j^2)$.

Fig. 2 shows the admissible configurations for f_1 at $\bar{x} \in \Sigma$. Here we take $w_1^1(\bar{x}) < 0$ and we show only the component of f_1 in the normal plane at $\bar{x}$ to Σ. The dotted and dashed vectors are not admissible due to Assumptions 1(a), while

Q4 Table 1

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	≤ 0	>0	<0	<0
w_i^2	>0	≥ 0	≤ 0	<0

Table 2

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	≤ 0	>0	≤ 0	≤ 0
w_i^2	>0	≥ 0	≥ 0	<0

the solid vectors are admissible configurations. The dashed vector has first component $w_1^1 < 0$ and second component $w_1^2 = 0$.

On the other hand, if, as before, while sliding on Σ, $w_1^2(\bar{x}) = 0$, but $w_1^1(x) > 0$ along the trajectory, Assumptions 1(a) are not violated, (see the numerical example in Section 4 where $\bar{x} = x_5$) and, as we will show in Lemma 4, Σ retains attractivity in finite time. This shows how Assumptions 1(a) are sharper than condition (9).

Lemma 4. *Let Assumptions 1 and 2 be verified and let* $(1_a^{\pm})$ *and* $(2_a^{\pm})$ *hold uniformly, then* Σ *is attractive in finite time.*

Proof. The proof is analogous to the proof of Lemma 4 in [3]. We only outline the first part of the proof since it is slightly different. Assumptions 1(a) and 2 guarantee that every f_j, $j = 1, 2, 3, 4$ points toward at least one of Σ_1 or Σ_2. This, together with the fact that f_j is never tangent to Σ, guarantees that if we start in R_j, we reach Σ_1 or Σ_2 or Σ in finite time. The rest of the proof is the same as that of [3, Lemma 4]. □

It must be emphasized that a sign change of the w_j^i's, while Assumptions 1 are still satisfied, does not necessarily lead to a loss of attractivity of Σ.

Remark 5. Having one of the f_j's tangent to Σ at a point $\bar{x} \in \Sigma$, is neither a necessary not a sufficient condition for loss of attractivity of Σ. Hence, if, while sliding on Σ, $w_j^1(\bar{x}) = w_j^2(\bar{x}) = 0$, this does not necessarily mean that the trajectory should exit Σ tangentially with vector field f_j, as one may expect according to a first order theory. Contrast this to the case of sliding on a co-dimension 1 surface where, if one of the vector fields is tangent to Σ at a point $\bar{x}$, there is loss of attractivity of Σ and (according to a first order theory) a tangential exit from Σ.

The theorem below shows that Assumptions 1 and 2 are sufficient for (7) to be well defined on Σ.

Theorem 6. *Let Assumptions 1 and 2 hold. Then, there exists a unique solution* $(\bar{\alpha}, \bar{\beta})$ *of system (8) in* $(0, 1) \times (0, 1)$.

Remark 7. Assumptions 1 and 2 are not necessary to have a unique solution $(\bar{\alpha}, \bar{\beta})$ of (8) on $(0, 1)^2$. As a matter of facts, the vector field (7) might exist and be unique even if Σ is not attractive (see Example 10).

Proof. The proof is analogous to the proof of Theorem 3 in [3], and below we will highlight just those cases requiring modifications to the arguments used in [3]. □

In what follows we consider some of the cases that appear in the proof of Theorem 3 in [3], and use the labeling of these cases using the same notation adopted in [3]. These cases are chosen so that each one occurs from one of the others due to a sign change of one of the w_j^i's. In this way, we can visualize the change in dynamics around Σ if one of the w_j^i's changes sign, with Assumptions 1 and 2 holding. We emphasize that under each of these sign changes Σ retains attractivity in finite time. In all the tables below (Tables 1–4) we show the sign of each component w_j^i; in these tables, the writing ≥ 0, ≤ 0, must be understood within the limitations imposed by Assumption 2: at any given point $\bar{x} \in \Sigma$ only one of the w_j^i's is allowed to be zero. In the corresponding figures (Figs. 3–6) we only display the w_j^i's when they are all different from zero.

Table 3

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	≤ 0	>0	≥ 0	<0
w_i^2	>0	≥ 0	>0	<0

Table 4

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	≥ 0	>0	≤ 0	≤ 0
w_i^2	≥ 0	≥ 0	≥ 0	<0

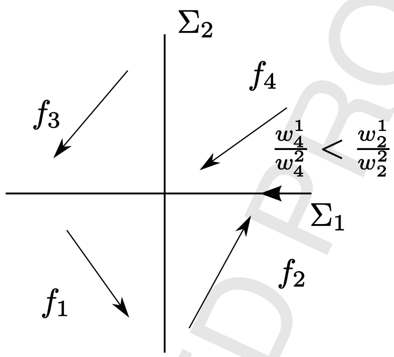

Fig. 3. Case $S_{\Sigma_1^+}: 2$.

Case ($S_{\Sigma_1^+}: 2$) The signs of the entries of w^1 and w^2 are as in Table 1, and the following condition is satisfied

$$\frac{w_2^2}{w_2^1} < \frac{w_4^2}{w_4^1} \tag{10}$$

Condition (10) ensures sliding along Σ_1^+ toward Σ.

According to Assumptions 1, w_1^1 can be zero on Σ since $w_1^2 > 0$, and similarly for w_2^2 and w_3^2. Notice, instead, that w_4^1 must be bounded away from zero even though $w_4^2 < 0$. This is in order to ensure sliding on at least one of the co-dimension 1 surfaces. Indeed assume that, while following

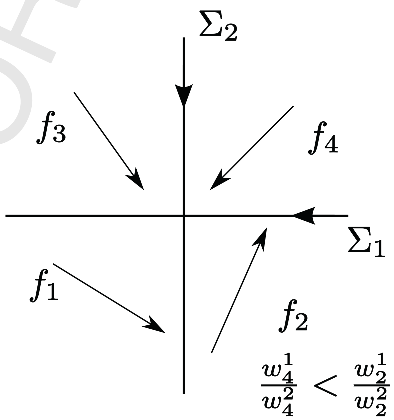

Fig. 4. Case $S_{\Sigma_1^+,\Sigma_2^+}: 2$.

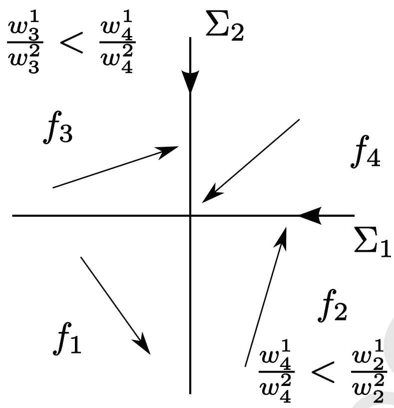

Fig. 5. Case $S_{\Sigma_1^+,\Sigma_2^+}:5$.

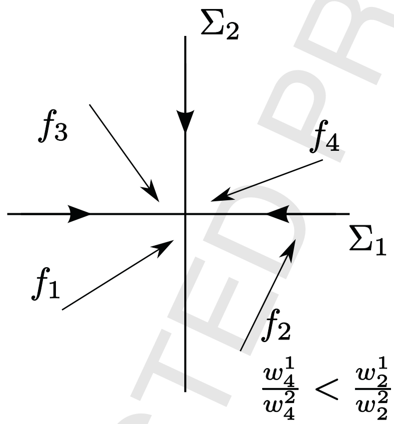

Fig. 6. Case $S_{\Sigma_1^\pm,\Sigma_2^+}:2$.

a trajectory on Σ, at $t=T$, $w_4^1(x(T))=0$. Then for $t>T$ and t sufficiently close to T, $w_4^1(x(t))>0$ and there is no sliding on a co-dimension 1 surface. This is against Assumptions 1(b).

Case $(S_{\Sigma_1^+,\Sigma_2^+}:2)$ This case follows from Case $(S_{\Sigma_1^+}:2)$ above, here w_3^2 has undergone a sign change.

The signs of the entries of w^1 and w^2 are as in Table 2 and (10) is satisfied. See Fig. 4.

Case $(S_{\Sigma_1^+,\Sigma_2^+}:5)$ This case follows from Case $(S_{\Sigma_1^+,\Sigma_2^+}:2)$ above, here w_3^1 has undergone a sign change.

The signs of the entries of w^1 and w^2 are as in Table 3 and condition (10) is satisfied together with the following:

$$\frac{w_3^1}{w_3^2} < \frac{w_4^1}{w_4^2}, \tag{11}$$

see Fig. 5.

Here (1_a^+) and (2_a^+) imply that w_4^1 and w_4^2 must be different from zero.

Case $(S_{\Sigma_1^\pm,\Sigma_2^+}:2)$ This case follows from Case $S_{\Sigma_1^+,\Sigma_2^+}:2$ above after w_1^2 has undergone a sign change.

The signs of the entries of w^1 and w^2 are given in Table 4 and (11) is satisfied; see Fig. 6. For this configuration (1_a^+) implies that w_4^2 must be different from zero.

Example 8. Here we illustrate all the changes in dynamics that might occur while sliding on Σ under Case $(S_{\Sigma_1^+,\Sigma_2^+}:2)$, when one of the components allowed to be zero in Table 2 goes to zero. Suppose that, while following a trajectory

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

Table 5

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	−1	1	−1	−0.5
w_i^2	0	1	−1	−1

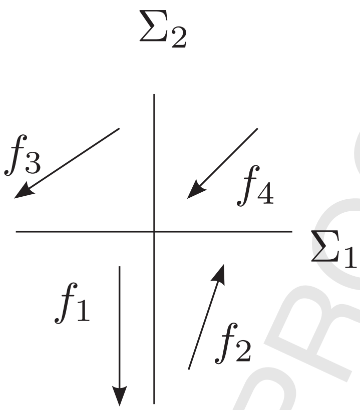

Fig. 7. Example 10, w_j^i's at $t=T$.

$x(t)$ on Σ, one of the w_j^i's is zero at time $t=T$. We will list here all the possible changes in dynamic that occur after time T.

(1) If $w_1^1(T)=0$, then the dynamic after time T is the one in Case $(S_{\Sigma_1^\pm,\Sigma_2^+}:2)$.
(2) If $w_3^1(T)=0$, then the dynamic after time T is the one in Case $(S_{\Sigma_1^+,\Sigma_2^+}:5)$.
(3) If $w_3^2(T)=0$, then the dynamic after time T is the one in Case $(S_{\Sigma_1^+}:2)$.
(4) If $w_4^1(T)=0$, then after time T there is attractive sliding only along Σ_2^+ and there is no sliding along Σ_1^+. Trajectories in a neighborhood of Σ will cross Σ_1^+ in the direction of R_4. This case mirrors to Case $(S_{\Sigma_1^+}:2)$.
(5) If $w_2^2(T)=0$, then after time T there is attractive sliding toward Σ along Σ_1^+ and $\Sigma_2^\pm$. This case mirrors Case $(S_{\Sigma_1^\pm,\Sigma_2^+}:2)$.

We stress that Σ is attractive in a neighborhood of $x(T)$, and that (7) is well defined.

Just like in [3, Theorem 8], and with the same proof, the following holds.

Theorem 9. *Under Assumptions 1 and 2, the unique solution* $(\overline{\alpha},\overline{\beta})\in(0,1)\times(0,1)$ *of system (8) varies smoothly with respect to* $x\in\Sigma$.

3.1. *Loss of attractivity*

In [3] we showed that violating any of $(1^\pm)$ or $(2^\pm)$ in Assumptions 1 leads to a loss of attractivity of Σ. We further identified when/how this loss of attractivity conduced to an exit from Σ to slide on one of $\Sigma_{1,2}^\mp$: first order exit condition.

Now, when only one of the w_j^i's is zero, and Assumptions 1 are not satisfied, then Σ looses attractivity, but the vector field (7) might still be defined on Σ as showed in Example 10.

Example 10. Assume that we are following a trajectory on Σ with the w_j^i's as in Table 1. Moreover, assume that all the w_j^i's are bounded away from zero for $t<T$ and that at time $t=T$ they are as in Table 5 and Fig. 7. As it is clear from Fig. 7, at $x(T)$ Assumptions 1 are not satisfied and Σ loses attractivity at time $t=T$. Nonetheless, system 8 still admits a unique solution $(\overline{\alpha},\overline{\beta})\simeq(0.4226, 0.7321)$, hence the vector field (7) is still well defined on Σ.

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

In Remark 5, we noticed how, while sliding on Σ, f_j might be tangent to Σ without this implying a loss of attractivity of Σ. Here, in Remark 11, we emphasize how a co-dimension 2 sliding surface might lose attractivity at a point x even though there is no potential tangential exit vector field at that point.

Remark 11. Consider again Table 5. Note that Σ has lost attractivity, but there is no tangential vector field exiting Σ. This is in distinct contrast with sliding on a co-dimension 1 surface. In the latter case, indeed, when the sliding surface Σ looses attractivity, Filippov theory will predict (at first order) exiting Σ tangentially.

4. Numerical example

Here we result of numerical experiments on an example where one of the w_j^i's (namely, w_2^1) becomes 0 along the sliding trajectory, still satisfying Assumptions 1. Aside from the modification due to w_2^1 becoming 0, the example below is actually the one we meant to use in [3].

All computations have been made with an event driven technique, and event points (when a different regime is reached) have been computed by the secant method. Integration of all relevant differential equations was made using the classical explicit Runge–Kutta (RK) scheme of order four, a projected RK method in case of sliding motion to ensure that all evaluations are made on the constraints' surfaces (e.g., see [4]). The stepsize τ was held constant and equal to $\tau = 0.0025$, and of course adjusted when using the secant method to locate event points. Solution of the system (8) was done by Newton's method.

Example 12. We have the discontinuity surfaces

$$\Sigma_1 = \{x \in \mathbb{R}^3 : h_1(x) = x_2 - p\}, \quad \Sigma_2 = \{x \in \mathbb{R}^3 : h_2(x) = x_3 - q\}, \quad \Sigma = \Sigma_1 \cap \Sigma_2$$

with $p = 0.5$ and $q = 1$. Thus, we have the following four vector fields, at least continuous in their respective regions of definition:

$$R_1(h_1 < 0, h_2 < 0) : f_1(x) = \begin{pmatrix} x_2 \\ -x_1 + \dfrac{1}{(1+p) - x_2} \\ -x_1 + \dfrac{32}{-}\dfrac{1}{(1+q) - x_3} \end{pmatrix};$$

$$R_2(h_1 < 0, h_2 > 0) : f_2(x) = \begin{pmatrix} -(x_2 + x_3) \\ -x_1 + \dfrac{1}{(1+p) - x_2} \\ -x_1 - \dfrac{1}{(1-q) + x_3} \end{pmatrix};$$

$$R_3(h_1 > 0, h_2 < 0) : f_3(x) = \begin{cases} \begin{pmatrix} -(x_2 + x_3) \\ -x_1 + \dfrac{1}{(1-p) + x_2} \\ -x_1 + \dfrac{1}{(1+q) - x_3} \end{pmatrix} & \text{when } x_1 \geq -1.3, \\ \begin{pmatrix} -(x_2 + x_3) \\ 6 + 1.3 + 6\dfrac{x_1}{1.3} + \dfrac{1}{(1-p) + x_2} \\ -x_1 + \dfrac{1}{(1+q) - x_3} \end{pmatrix} & \text{when } x_1 < -1.3, \end{cases}$$

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

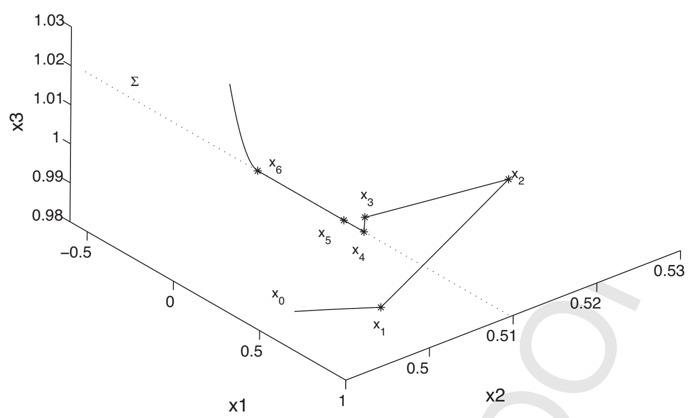

Fig. 8. Solution trajectory: the solution spirals around Σ, starts sliding on Σ_1^+, enters Σ and leaves it to slide on Σ_1^+.

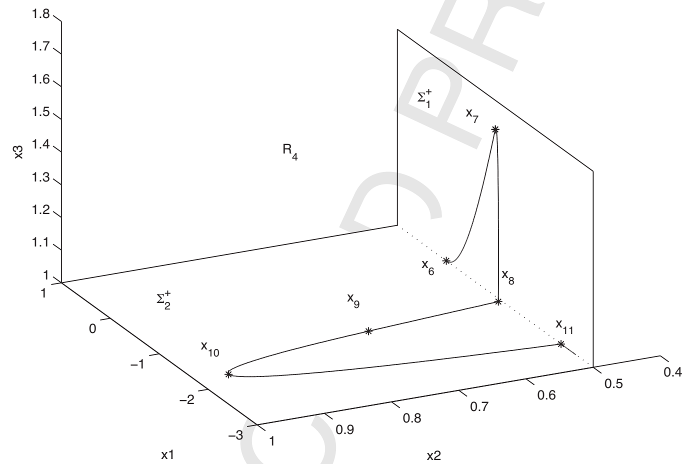

Fig. 9. Solution trajectory: the solution slides on Σ_1^+ and leaves it to enter R_4, hits Σ_2^+ and starts sliding on it, then hits Σ and starts sliding on it.

$$R_4 (h_1 > 0, h_2 > 0) : f_4(x) = \begin{cases} \begin{pmatrix} -(x_2 + x_3) \\ -x_1 - \dfrac{1}{(1-p) + x_2} \\ -x_1 + \dfrac{1}{(1-q) + x_3} \end{pmatrix} & \text{when } x_1 \geq -1, \\ \begin{pmatrix} -(x_2 + x_3) \\ -x_1 - \dfrac{1}{(1-p) + x_2} \\ 130 + 129x_1 + \dfrac{1}{(1-q) + x_3} \end{pmatrix} & \text{when } x_1 < -1. \end{cases}$$

Results below are for initial condition $x_0 = [0.7, 0.49, 0.99]$. We can distinguish several different dynamics of the solution with respect to the two discontinuity surfaces. Indeed, there are several event points, that is values where the solution reaches a different regime: a different region and/or sliding surface. We will assign a time value t_j, and with abuse of notation indicate each event point with x_j. The initial part of the trajectory is plotted in Fig. 8, and Fig. 9 shows the entire trajectory; event points are marked by asterisks: x_j, $j = 1, \ldots, 11$.

Table 6
w_j^i's at x_4

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	>0	<0
w_i^2	<0	<0	>0	>0

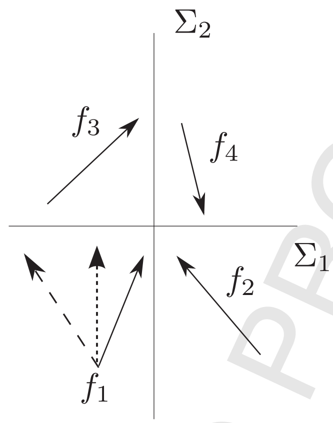

Fig. 10. $w_1 = (w_1^1, w_1^2)$ at $t < t_5$ (left), at t_5 (center), and at $t > t_5$ (right).

The initial condition is in region R_1 and the trajectory crosses Σ_1^- at $x_1 \approx (0.71728, 0.5, 0.98318)$ and enters R_3 (transversal intersection). At $x_2 \approx (0.63696, 0.51686, 1)$, it crosses Σ_2^+ and enters R_4 (transversal intersection). At $x_3 \approx (0.62125, 0.5, 1.00384)$, it hits Σ_1^+ and starts sliding on it in the direction of Σ with vector field $f_{\Sigma_1^+}$. Then, while sliding on Σ_1^+ the solution reaches Σ at time t_4 at the point $x_4 \approx (0.61659, 0.5, 1)$.

At x_4, the vector fields f_j, $j=1, 2, 3, 4$, have the signs given in Table 6 and condition (10) is satisfied, so that Assumptions 1 are satisfied, Σ is attractive, f_Σ as in (7) is well defined and the solution starts sliding on Σ.

At time t_5, the solution is at $x_5 = (0.5, 0.5, 1)$, $w_1^2(x_5) = 0$, and $w_1^1 \neq 0$ for values on Σ in a neighborhood of x_5. Assumptions 1 and 2 are satisfied at x_5, so the solution keeps sliding on Σ. The configuration along the solution path in a neighborhood of t_5 is the one showed in Fig. 10 (it mirrors Case $(S_{\Sigma_1^+, \Sigma_2^+} : 1)$ in [3]). Here the dashed vector is $w_1 = (w_1^1, w_1^2)$ at a specific time $t < t_5$, the dotted vector is $w_1(x(t_5))$ and the solid vector is w_1 at a specific time $t > t_5$.

At $t = t_6 \approx 0.62925$, the solution is at $x_6 = (0, 0.5, 1)$, and there is equality in (10). Moreover, the $w_j^i(x)$'s for $x = x_6$ are as in Table 7 and the trajectory leaves Σ smoothly to enter Σ_1^+.

So, at $t = t_6$, f_Σ aligns to $f_{\Sigma_1^+}$ and the solution exits Σ smoothly to slide on Σ_1^+. At time $t_7 \approx 1.22698$, the solution reaches $x_7 \approx (-1, 0.5, 1.5111)$, $f_{\Sigma_1^+}$ aligns to f_4 and the solution exits Σ_1^+ smoothly to enter in region R_4. At time t_8, it reaches Σ_2^+ at $x_8 \approx (-1.1322, 0.5050, 1)$, and here $w_3^2 > 0$ while $w_4^2 < 0$ so that sliding begins on Σ_2^+ away from Σ_1. At time $t_9 \approx 1.40259$, the solution reaches the surface $x_1 = -1.3$ at $x_9 \approx (-1.3, 0.7105, 1)$; here, f_3 is continuous

Table 7
w_j^i's at x_6

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	1	1	1	−1
w_i^2	$\frac{1}{2}$	−1	1	1

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005

Table 8
w_j^i's at x_{11}

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	<0	>0
w_i^2	>0	>0	>0	<0

but not differentiable. At time t_{10}, we reach the value $x_{10} \approx (-1.7377, 0.9507, 1)$. For $t > t_{10}$ the trajectory continues sliding on Σ_2^+ but now in the direction of Σ_1, since the following condition is satisfied:

$$\frac{w_3^1}{w_3^2} < \frac{w_4^1}{w_4^2}. \tag{12}$$

At time t_{11}, the solution reaches the point $x_{11} \approx (-2.3430, 0.5, 1)$ on Σ. The vector fields $f_j(x_{10}), j = 1, \ldots, 4$, satisfy the conditions of Table 8 and the behavior on Σ is analogous to the one of Case $(S_{\Sigma_1^+} : 2)$. The solution now starts sliding on Σ with vector field f_Σ as in (7), and remains on Σ.

5. Conclusions

In this paper we weakened the assumptions given in [3] for attractivity of a sliding co-dimension 2 surface Σ and for the existence and uniqueness of the Filippov sliding vector field (7) on Σ. We reported on a numerical experiment to show the behavior of a piecewise smooth system that satisfies our new assumptions.

References

[1] V. Acary, B. Brogliato, Numerical Methods for Nonsmooth Dynamical Systems. Applications in Mechanics and Electonics, Lecture Notes in Applied and Computational Mechanics, Springer-Verlag, Berlin, 2008.
[2] J. Alexander, T. Seidman, Sliding modes in intersecting switching surfaces, I: Blending, Houston J. Math. 24 (1998) 545–569.
[3] L. Dieci, C. Elia, L. Lopez, A Filippov sliding vector field on an attracting co-dimension 2 discontinuity surface, and a limited loss-of-attractivity analysis, J. Differ. Equat. 254 (2013) 1800–1832.
[4] L. Dieci, L. Lopez, Sliding motion in Filippov differential systems: theoretical results and a computational approach, SIAM J. Numer. Anal. 47 (2009) 2023–2051.
[5] L. Dieci, L. Lopez, Sliding motion on discontinuity surfaces of high codimension. A construction for selecting a Filippov vector field, Numer. Math. 117 (2011) 779–811.
[6] A. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Mathematics and its Applications, Kluwer Academic, Dordrecht, 1988.

Please cite this article in press as: L. Dieci, et al. Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface, Math. Comput. Simul. (2014), http://dx.doi.org/10.1016/j.matcom.2013.12.005