LIMIT CYCLES FOR REGULARIZED DISCONTINUOUS DYNAMICAL SYSTEMS WITH A HYPERPLANE OF DISCONTINUITY.

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Abstract. We consider an n dimensional dynamical system with discontinuous right-hand side (DRHS), whereby the vector field changes discontinuously across a co-dimension 1 hyperplane S. We assume that this DRHS system has an asymptotically stable periodic orbit γ, not fully lying in S. In this paper, we prove that also a regularization of the given system has a unique, asymptotically stable, periodic orbit, converging to γ as the regularization parameter goes to 0.

1. Introduction

Systems with discontinuous right-hand side (also called piecewise smooth, PWS, systems), have been actively investigated during the last 20-30 years, because of their relevance in many applications, such as in control theory, mechanical systems with dry frictions, biological models. See [1], [3] , [9] and [11], for important theoretical and modeling work.

Among many ways in which one can study DRHS systems, the regularization method is very appealing, since it replaces the discontinuous system with a smooth system. The first authors to formally introduce this technique were Sotomayor and Teixeira in 1995, see [19], and recently this method has been exploited to study singularities, Filippov sliding vector fields and dynamical behavior near sliding regions of the PWS vector field. See [12], [13], [14] and [16].

Limit cycles of PWS vector fields also are of considerable interest. Several authors have studied the persistence of limit cycles for regularized **planar** vector fields; e.g., see [18] and [5]. However, these results rely heavily on the planar nature of the problem; in particular, they make use of the Poincaré Bendixson Theorem. Our own interest in this paper is to establish existence (and stability) of limit cycles of regularization of n-dimensional PWS vector fields ($n \geq 2$), having a hyperplane of discontinuity. We will do so in the cases where the discontinuous system has a periodic orbit with sliding and/or crossing segments.

The work [2] is a precursor of our results. In that work, the authors considered the case of a regularized vector field for a discontinuous problem with a limit cycle having a sliding segment. No crossing case was considered in [2]. Moreover, even for the sliding case, our treatment is different from [2] . Rather than relying on the implicit function theorem applied to the Poincaré map, as the

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authors proposed in [2], we will first use Brouwer's fixed point theorem to establish the existence of a periodic orbit for the regularized problem, and then study its stability properties by exploiting the associated monodromy matrix. Moreover, we provide a unified treatment of limit cycles with sliding and/or crossing.

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, and let $h(x) \equiv x_1$. Define the switching manifold as $S = h^{-1}(0)$, and let $R_+ = \{q \in \mathbb{R}^n : h(q) > 0\}$, $R_- = \{q \in \mathbb{R}^n : h(q) < 0\}$. Consider the following system with discontinuous right-hand side:

$$\dot{x} = F_0(x) = \begin{cases} F_+(x), & \text{if} \quad x \in R_+ \\ F_-(x), & \text{if} \quad x \in R_- \ . \end{cases} \tag{1}$$

Here, F_- and F_+ are C^r functions, where $r \geq 1$, which we assume to be well defined in $R_\mp$, on S, and in a neighborhood of S. Write $F_0 = (F_-, F_+)$. A smooth function $\phi : \mathbb{R} \to \mathbb{R}$ is a transition function if $\phi(x) = -1$ for $x \leq -1$, $\phi(x) = 1$ for $x \geq 1$ and $\phi'(x) > 0$ if $x \in (-1, 1)$. To fix ideas, we consider the following C^1 function

$$\phi(z) = \begin{cases} -1 & z < -1 \\ \frac{3}{2}z - \frac{1}{2}z^3 & -1 \leq z \leq 1 \\ 1 & z > 1 \end{cases} . \tag{2}$$

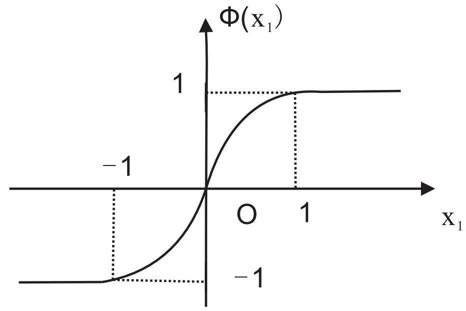

FIGURE 1. Graph of transition function $\phi(x_1)$

The ϕ-regularization of $F_0 = (F_-, F_+)$ is a 1-parameter family of vector fields F_ϵ, connecting F_- and F_+, and giving the following regularized system for (1):

$$\dot{x} = F_\epsilon(x) = \frac{1}{2}\left(1 - \phi\left(\frac{h(x)}{\epsilon}\right)\right) F_-(x) + \frac{1}{2}\left(1 + \phi\left(\frac{h(x)}{\epsilon}\right)\right) F_+(x). \tag{3}$$

When needed, we will use the notation $\phi_\epsilon(z) = \phi(\frac{z}{\epsilon})$. The vector field F_ϵ is an average of F_- and F_+ inside the *boundary layer* $\{x \in \mathbb{R}^n | \ -\epsilon < h(x) < \epsilon\}$, while it is equal to either F_- or F_+ outside the boundary layer.

In [18] and [6], the authors consider discontinuous planar systems. They show that if γ is a hyperbolic periodic orbit of (1) in $\mathbb{R}^2$, then, under suitable assumptions, the regularized vector field (3) has a hyperbolic limit cycle γ_ϵ, converging to γ as $\epsilon \to 0$.

In this paper, we consider vector fields in $\mathbb{R}^n$, with a co-dimension 1 hyperplane of discontinuity, for any $n \geq 2$. Under appropriate assumptions, we will prove that, if the original PWS system has an asymptotically stable periodic orbit, then so will the regularized system. We mention that this is not a trivial generalization. There are two main difficulties. The first is that for non-planar problems we do not have a Poincaré-Bendixson Theorem to help us in establishing existence of the limit cycle (cfr. with [4, 5, 6, 18]); extensions of the Poincaré-Bendixson Theorem for systems in $\mathbb{R}^n$, see [17, 21], require special type of systems (competitive or monotone systems), which do not fit our type of problem. The second difficulty is to establish the stability of the limit cycle of the regularized problem. We will do this by using the monodromy matrices of the discontinuous and regularized problems, and showing that the latter converges (as $\epsilon \to 0$) to the former.

The remainder of this paper is organized as follows. We give some definitions and state our main result in section 2. In section 3, we prove our main result. Conclusions are in section 4. In Appendix A, we give proofs of some technical results needed in section 3.

2. Basic definitions and main result

In this section, we will give definitions and assumptions.

We assume that $F_\pm$ are C^r, $r \geq 1$, in $R_\pm$ and in a neighborhood of S. We denote the flow of (1) as $\varphi_0^t(x)$ and the flow of (3) as $\varphi_\epsilon^t(x)$.

Definition 1. *A subset U of S is said to be an attractive (repulsive) sliding subset if for all $x \in U$ the following occurs*

$$\nabla h^T F_-(x) > 0 \ \text{ and } \ \nabla h^T F_+(x) < 0 \ \ (\nabla h^T F_-(x) < 0 \ \text{ and } \ \nabla h^T F_+(x) > 0). \tag{4}$$

A point x that verifies (4) *is said to be an attractive (repulsive) sliding point. If a solution of* (1) *slides on an attractive (repulsive) subset of S, we say that attractive (repulsive) sliding occurs along the solution.*

If a solution intersects S at an attractive sliding point x then it must remain on S. However the vector field F_0 is not defined on S and a sliding vector field needs to be defined. We follow Filippov (see [9]) and for each $x \in S$ that verifies the first condition in (4) we define the sliding vector field as

$$F_S(x) = \frac{1}{2}\left[(1-\phi^*)F_- + (1+\phi^*)F_+\right](x), \ \ \phi^*(x) = \frac{\nabla h^T(F_- + F_+)}{\nabla h^T(F_- - F_+)}(x), \tag{5}$$

where the value of $\phi^*(x)$ in (5) is such that $\nabla h^T F_S(x) = 0$.

In this paper we will also make use of the following definitions.

Definition 2. *Let $x \in S$. Then*

i) *x is a crossing point if $(\nabla h^T F_-)(\nabla h^T F_+) > 0$;*
ii) *x is a first order tangential exit point into R_- if $\nabla h^T F_-(x) = 0$, $\nabla h^T F_+(x) < 0$, and if, letting $g(x) = \nabla h^T F_-(x)$, $\nabla g^T F_-(x) < 0$;*
iii) *x is a first order tangential exit point into R_+ if $\nabla h^T F_+(x) = 0$, $\nabla h^T F_-(x) > 0$, and if, letting $g(x) = \nabla h^T F_+(x)$, $\nabla g^T F_+(x) > 0$.*

Definition 3. *Assume that a solution of* (3) *reaches S at an attractive sliding point x. Then x is said to be a transversal entry point.*

Let γ be a periodic orbit of (1). In this paper we consider one of the following forms for γ (see [10] for planar systems). See Figure 2.

i) γ has a finite number of sliding arcs on S and contains no crossing points. Then, γ is called a *sliding periodic orbit*. We further exclude the case in which γ is entirely contained in S.
ii) γ meets S only at a finite number of crossing points. Then, γ is called a *crossing periodic orbit*.
iii) γ has a finite number of sliding arcs and a finite number of crossing points. Then, γ is called a *crossing and sliding periodic orbit*.

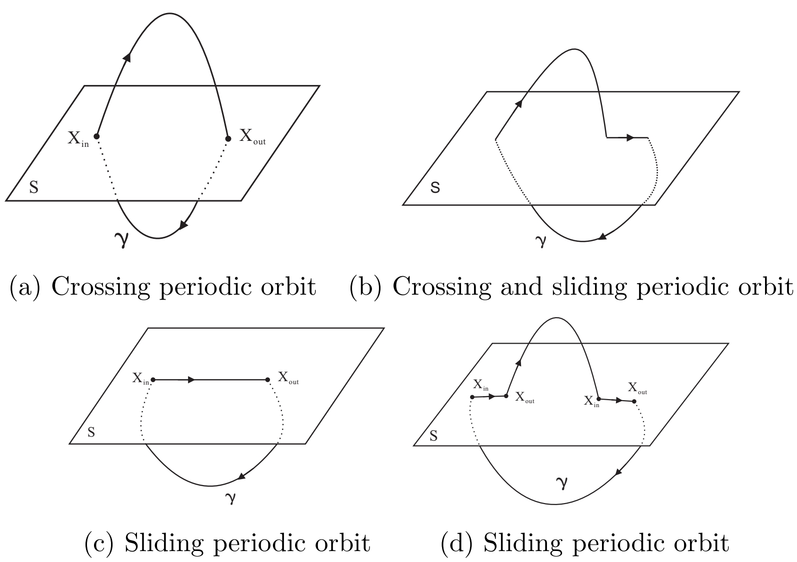

FIGURE 2. Periodic orbits of (1)

As for smooth dynamical systems, stability properties of a periodic orbit γ can be studied via the eigenvalues of the monodromy matrix, the Floquet multipliers. As we will see in Section 3, to take into account the jumps in the derivatives of the solution, the monodromy matrix along γ is defined with the aid of suitable saltation matrices. These matrices are full rank in the case of crossing, but they are singular (of rank $(n-1)$) in case γ has one or more sliding arcs on S.

Definition 4. *Let γ be a periodic orbit of (1). Let $\mu_1, \mu_2, \cdots, \mu_n$ be the corresponding Floquet multipliers. We say that γ is asymptotically stable if one of the multipliers is 1, say $\mu_1 = 1$, and all other μ_i's are less than 1 in modulus.*

When γ is a sliding or sliding and crossing periodic orbit, the associated monodromy matrix has one multiplier equal to 0. This witnesses that there is sliding on a co-dimension 1 region of $\mathbb{R}^n$. Notice that when (1) is planar, a sliding or sliding and crossing periodic orbit has a multiplier at 1 and one at 0. In this case, γ is said to be stable in finite time.

2.1. **Basic assumptions.** Before we state our main result, we make the following basic assumptions.

H_1 Sliding subsets of S are attractive (see Definition 1).
H_2 Entry points on S are transversal (see Definition 3).
H_3 Exit points from S are first order tangential exit points (see Definition 2).
H_4 For each solution of (1), only a finite number of crossings/exits/entries can occur.

Our main result is the following

Theorem 5. *Assume that hypotheses $H_1 - H_4$ hold. Let γ be an asymptotically stable periodic orbit of* (1) *not entirely contained in S. Then, for $\epsilon > 0$ sufficiently small, system* (3) *has a unique asymptotically stable limit cycle γ_ϵ, and $\gamma_\epsilon \to \gamma$ when $\epsilon \to 0$.*

Of course, Theorem 5 above holds true also when (1) is planar and γ is stable in finite time.

3. Proof of Main result

In this section, we prove Theorem 5. We will treat the crossing and sliding cases separately.

3.1. **Crossing periodic orbit.** We consider the case in which γ meets S in just two crossing points, denote them with $\bar{x}_1$ and $\bar{x}_2$. In Remark 15 we discuss the case of a finite number of crossings.

Theorem 6. *Assume that* (1) *has an asymptotically stable periodic orbit γ with two transversal crossing points with S. Then, for ϵ sufficiently small, there exists one and only one periodic orbit γ_ϵ of* (3) *in a neighborhood of γ. Moreover γ_ϵ is asymptotically stable and $\lim_{\epsilon\to 0} \gamma_\epsilon = \gamma$.*

We will prove Theorem 6 according to the following steps.

(1) Prove that (3) has at least one limit cycle. To do this, we will define a Poincaré map P_ϵ, and use Brouwer's fixed point Theorem to show that it has a fixed point. This will give at least one limit cycle γ_ϵ of (3), and we will show that $\gamma_\epsilon \to \gamma_0$ when $\epsilon \to 0$.
(2) Then, we will show that γ_ϵ is asymptotically stable, so γ_ϵ is the unique limit cycle of (3), for ϵ sufficiently small.

Let us define a Poincarè map associated to γ. Without loss of generality, we will assume that the periodic solution associated to γ crosses S at $\bar{x}_1$ coming from R_+ and entering in R_-, and then again (at a later time) crosses S at $\bar{x}_2$ coming from R_- and entering in R_+, as in Figure 3.

With $\varphi^t_\pm$ we denote the flows of $F_\pm$, and with $\varphi^t_0(x_0)$ the solution of (1) at time t, with initial condition x_0 at time $t = 0$. Recalling that in the present crossing case if $x_0 \in S$, then $\nabla h^T F_+(x_0)$ and $\nabla h^T F_-(x_0)$ have the same sign, we note that $\varphi^t_0(x_0)$, for $t \geq 0$ sufficiently small, is the solution of

$$\dot{x} = \begin{cases} F_+(x), & \text{if} \quad \nabla h^T F_+(x_0) > 0 \ , \\ F_-(x), & \text{if} \quad \nabla h^T F_+(x_0) < 0 \ . \end{cases}$$

Let $B_\delta(\bar{x}_1) = \{x \in \mathbb{R}^n : \ \|x - \bar{x}_1\| < \delta\}$ be the open ball centered at $\bar{x}_1$ and of radius δ; here, and later on, the norm is always the Euclidean norm. Denote with $B_\delta(\bar{x}_1, S)$ its intersection with S and with $\overline{B_\delta(\bar{x}_1, S)}$ its closure. Then for δ sufficiently small the Poincarè map $P_-(x) = \varphi_-^{t_-(x)}(x)$, where $t_-(x)$ is the first return time to S, is well defined and smooth in x and it takes a point x in $\overline{B_\delta(\bar{x}_1, S)}$ into a neighborhood of $\bar{x}_2$. Similarly, we can define a Poincarè map $P_+(x)$ that, due to the asymptotic stability of γ, takes a point x in a neighborhood of $\bar{x}_2$ into $\overline{B_\delta(\bar{x}_1, S)}$. Let $P = P_+ \circ P_- : \overline{B_\delta(\bar{x}_1, S)} \to \overline{B_\delta(\bar{x}_1, S)}$ be the Poincarè map of system (1). Then P is well defined and smooth with its inverse in $\overline{B_\delta(\bar{x}_1, S)}$ and since γ is asymptotically stable, P satisfies $P(\overline{B_\delta(\bar{x}_1, S)}) \subset B_\delta(\bar{x}_1, S)$ for δ sufficiently small. Let ψ_δ be the boundary of $\overline{B_\delta(\bar{x}_1, S)}$, then ψ_δ is the intersection of the

$(n-1)$-sphere of center $\bar{x}_1$ and radius δ with S. The set $P(\psi_\delta)$ is a diffeomorphic image of ψ_δ. Let V be the union of all trajectories of (1) with initial point on ψ_δ and endpoint on $P(\psi_\delta)$ together with $\hat{B}_\delta = B_\delta(\bar{x}_1, S)\backslash P(B_\delta(\bar{x}_1, S))$ and let $\hat{V}$ be the compact subset of $\mathbb{R}^n$ whose boundary is V. Then all solution trajectories of (1) that intersect $\hat{B}_\delta$ will do so transversally, will enter $\hat{V}$ and will remain inside it. The periodic orbit γ attracts all trajectories inside $\hat{V}$.

Stability of γ can be studied via the monodromy matrix $X(T)$ at γ. Let T be the period of γ and assume that at time $t = \bar{t}$, $\varphi_0^{\bar{t}}(\bar{x}_1) = \bar{x}_2$. Then, $X(T)$ can be written as the composition of the following matrices (e.g., see [8, 15]):

$$X(T) = X(T, \bar{t})S_{-+}(\bar{x}_2)X(\bar{t}, 0)S_{+-}(\bar{x}_1) \tag{6}$$

where $S_{+-}(\bar{x}_1) = I + \frac{(F_- - F_+)}{\nabla h^T F_+}\nabla h^T(\bar{x}_1)$ and $S_{-+}(\bar{x}_2) = I + \frac{(F_+ - F_-)}{\nabla h^T F_-}\nabla h^T(\bar{x}_2)$ are so-called saltation matrices, while the fundamental matrix solutions $X(t, 0)$ and $X(t, \bar{t})$ satisfy

$$\begin{aligned} \dot{X}(t, 0) &= DF_-(\varphi_0^t(\bar{x}_1))X(t, 0), \quad X(0, 0) = I; \\ \dot{X}(t, \bar{t}) &= DF_+(\varphi_0^t(\bar{x}_1))X(t, \bar{t}), \quad X(0, \bar{t}) = I. \end{aligned}$$

The two saltation matrices in (6) are nonsingular and hence $X(T)$ has an eigenvalue at 1 and all the other eigenvalues are less than 1 in modulus because of asymptotic stability of γ.

To prove the existence of a periodic orbit of (3) in a neighborhood of γ we employ the Poincarè map of (3). In a neighborhood of $\bar{x}_1$ and $\bar{x}_2$ solutions of (3) intersect S transversally and hence we will show that the following Poincarè map, $P_\epsilon : \overline{B_\delta(\bar{x}_1, S)} \to \overline{B_\delta(\bar{x}_1, S)}$ that associates to a point in $\overline{B_\delta(\bar{x}_1, S)}$ its first return to $\overline{B_\delta(\bar{x}_1, S)}$, is well defined. The following proposition establishes the existence of at least one periodic orbit of (3).

Proposition 7. *For ϵ sufficiently small, the map P_ϵ has at least one fixed point in $\overline{B_\delta(\bar{x}_1, S)}$.*

In order the prove this proposition we will need the following Lemma.

Lemma 8. *For each $x_0 \in \overline{B_\delta(\bar{x}_1)}$ the following is satisfied*

$$\lim_{\epsilon \to 0} \varphi_\epsilon^t(x_0) = \varphi_0^t(x_0),$$

uniformly for t in a compact interval.

Proof. Denote with φ_0^t and φ_ϵ^t the flows of (1) and (3) respectively. Together with S, consider also the hyperplanes $S_\epsilon = \{x \in \mathbb{R}^n |\ h(x) = \epsilon\}$ and $S_{-\epsilon} = \{x \in \mathbb{R}^n |\ h(x) = -\epsilon\}$. In what follows, for $x_0 \in \overline{B_\delta(\bar{x}_1)}$, we want to estimate the distance between $\varphi_0^t(x_0)$ and $\varphi_\epsilon^t(x_0)$ at their intersection points with S, S_ϵ and $S_{-\epsilon}$. Without loss of generality assume that $x_0 \in \overline{B_\delta(\bar{x}_1, S)}$. Then for δ and ϵ sufficiently small $\nabla h^T F_\epsilon(x_0) < 0$, $\nabla h^T F_-(x_0) < 0$. Let t_1 be such that $x_1 = \varphi_0^{t_1}(x_0) \in S_{-\epsilon}$ and similarly, let $x_1^\epsilon = \varphi_\epsilon^{t_1^\epsilon}(x_0) \in S_{-\epsilon}$, with x_1, x_1^ϵ in a neighborhood of $\bar{x}_1$. We want to bound $\|\varphi_0^{t_1}(x_0) - \varphi_\epsilon^{t_1^\epsilon}(x_0)\|$ and show that it goes to zero when $\epsilon \to 0$. To fix ideas, assume $t_1^\epsilon > t_1$. Let $L_- = \max\left(\max_{t\in[0,t_1]} \|DF_-(\varphi_0^t(x_0))\|, \max_{t\in[0,t_1]} \|DF_-(\varphi_\epsilon^t(x_0))\|\right)$ and $M_\pm = \max_{t\in[t_1,t_1^\epsilon]} \|F_\pm(\varphi_\epsilon^t(x_0))\|$, then the following inequality holds

$$\begin{aligned} \|\varphi_0^{t_1}(x_0) - \varphi_\epsilon^{t_1^\epsilon}(x_0)\| \quad & \leq \left\| \int_0^{t_1} F_-(\varphi_0^s(x_0)) - F_-(\varphi_\epsilon^s(x_0))ds \right\| + \\ & \left\| \int_0^{t_1^\epsilon} (\tfrac{1}{2} + \tfrac{\phi_\epsilon}{2}(s))(F_+ - F_-)(\varphi_\epsilon^s(x_0))ds \right\| + \left\| \int_{t_1}^{t_1^\epsilon} F_-(\varphi_\epsilon^s(x_0))ds \right\| \\ & \leq L_- \int_0^{t_1} \|\varphi_0^s(x_0) - \varphi_\epsilon^s(x_0)\|ds + t_1^\epsilon(M_+ + M_-) + (t_1^\epsilon - t_1)M_- \\ & \text{and so } \|\varphi_0^{t_1}(x_0) - \varphi_\epsilon^{t_1^\epsilon}(x_0)\| \leq [t_1^\epsilon(M_+ + M_-) + (t_1^\epsilon - t_1)M_-]e^{t_1 L_-}, \end{aligned}$$

where the last inequality follows from Gronwall's Lemma. Moreover using $h(\varphi_0^{t_1}(x_0)) = -\epsilon$, if we consider the Taylor polynomial in Lagrange form of $\varphi_0^{t_1}(x_0)$ at the point $t = 0$, we obtain

$$t_1 = \frac{-\epsilon}{\nabla h(\varphi_0^{t_1}(x_0))^T F_-(\varphi_0^{\eta}(x_0))}, \quad \eta \in (0, t_1). \tag{7}$$

In particular $\lim_{\epsilon\to 0} t_1 = 0$ and, in a similar way, $\lim_{\epsilon\to 0} t_1^\epsilon = 0$. This, together with the bound for $\|\varphi_0^t(x_0) - \varphi_\epsilon^{t_\epsilon}(x_0)\|$, implies $\lim_{\epsilon\to 0} \varphi_\epsilon^{t_1^\epsilon}(x_0) = \varphi_0^t(x_0)$.

For $t > t_1$, $\varphi_0^t(x_0)$ moves away from $S_{-\epsilon}$ in direction opposite to $\nabla h(x)$. But it will eventually change direction and it will meet $S_{-\epsilon}$ again in a neighborhood of $\bar{x}_2$. Let t_2 be such that $x_2 = \varphi_0^{t_2}(x_1) \in S_{-\epsilon}$ and t_2^ϵ be such that $x_2^\epsilon = \varphi_\epsilon^{t_2^\epsilon}(x_1^\epsilon) \in S_{-\epsilon}$. To fix ideas, again we assume $t_2^\epsilon > t_2$. Let now $M_- = \max_{t\in[t_2,t_2^\epsilon]} \|F_-(\varphi_\epsilon^t(x_1^\epsilon))\|$, and $L_- = \max\left(\max_{t\in[0,t_2]} \|DF_-(\varphi_0^t(x_1))\|, \max_{t\in[0,t_2]} \|DF_-(\varphi_\epsilon^t(x_1^\epsilon))\|\right)$ be the local Lipschitz constant for F_-, then the following bound holds

$$\begin{aligned} \|\varphi_0^{t_2}(x_1) - \varphi_\epsilon^{t_2^\epsilon}(x_1^\epsilon)\| &\leq \|x_1 - x_1^\epsilon\| + \left\| \int_0^{t_2} \left(F_-(\varphi_0^s(x_1)) - F_-(\varphi_\epsilon^s(x_1^\epsilon))\right) ds + \int_{t_2}^{t_2^\epsilon} F_-(\varphi_\epsilon^s(x_1^\epsilon)) ds \right\| \\ &\leq \|x_1 - x_1^\epsilon\| + L_- \int_0^{t_2} \|\varphi_0^s(x_1) - \varphi_\epsilon^s(x_1^\epsilon)\| ds + (t_2^\epsilon - t_2) M_- \\ \text{and so } \ \|\varphi_0^{t_2}(x_1) - \varphi_\epsilon^{t_2^\epsilon}(x_1^\epsilon)\| &\leq (\|x_1 - x_1^\epsilon\| + (t_2^\epsilon - t_2)M_-) e^{t_2 L_-}, \end{aligned}$$

where the last inequality follows from Gronwall Lemma. Notice that $\lim_{\epsilon\to 0} t_2^\epsilon = t_2$, since $x_1^\epsilon \to x_1$ and $F_\epsilon = F_-$ for $h(x) < -\epsilon$. Then $\lim_{\epsilon\to 0} \varphi_\epsilon^{t_2^\epsilon}(x_1^\epsilon) = \varphi_0^{t_2}(x_1)$. In a similar way we can show that $\|\varphi_0^t(x_0) - \varphi_\epsilon^{t_\epsilon}(x_0)\| \to 0$ up to returning in a neighborhood of $\bar{x}_1$. This proves the Lemma. □

As a consequence of Lemma 8, this Corollary holds.

Corollary 9. *As $\epsilon \to 0$, P_ϵ converges pointwise to P in $\overline{B_\delta(\bar{x}_1, S)}$.*

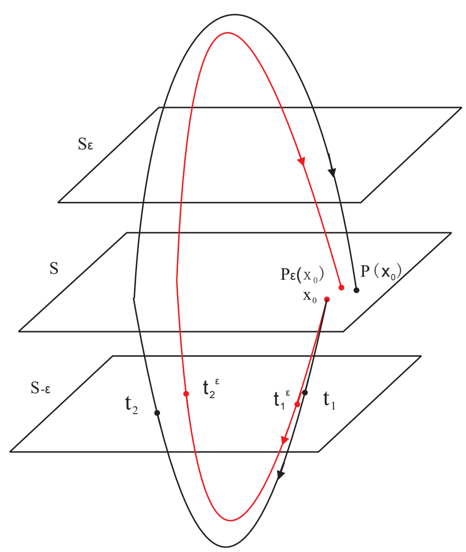

Figure 3. P and P_ϵ

Proof of Proposition 7. We will prove that $P_\epsilon(\overline{B_\delta(\bar{x}_1, S)}) \subset \overline{B_\delta(\bar{x}_1, S)}$. Then the statement will follow from Brouwer's fixed point Theorem. Consider the following relation between ψ_δ and $P(\psi_\delta)$: $\beta(\psi_\delta, P(\psi_\delta)) = \min_{a\in\psi_\delta, b\in P(\psi_\delta)} \|a - b\|$. Then $\beta(\psi_\delta, P(\psi_\delta)) > \bar{\eta} > 0$ and $\bar{\eta}$ is bounded away from 0 since $P(\overline{B_\delta(\bar{x}_1, S)}) \subset B_\delta(\bar{x}_1, S)$. Let η, $0 < \eta < \bar{\eta}$, be fixed, and for every $x \in \overline{B_\delta(\bar{x}_1, S)}$ denote with

$B_\eta(P(x), S)$ the intersection of the η-ball centered at $P(x)$ with S. Then $B_\eta(P(x), S) \subset B_\delta(\bar{x}_1, S)$. Corollary 9 implies that for every $x \in \overline{B_\delta(\bar{x}_1, S)}$ there exists $\epsilon_{\eta,x} > 0$ such that, for $\epsilon \in (0, \epsilon_{\eta,x})$, $P_\epsilon(x) \in B_\eta(P(x), S) \subset B_\delta(\bar{x}_1, S)$. The proof follows upon noticing that for every x, $\epsilon_{\eta,x}$ is bounded away from 0. □

Proposition 10. *System* (1) *has at least a periodic orbit.*

Proof. Proposition 7 ensures existence of a fixed point of P_ϵ in $B_\delta(\bar{x}_1, S)$. We need to exclude the possibility that the fixed point is an equilibrium point for F_ϵ. Assume by contradiction that there exists $\bar{x} \in B_\delta(\bar{x}_1, S)$ such that $F_\epsilon(\bar{x}) = 0$. Then $\nabla h(\bar{x})^T F_\epsilon(\bar{x}) = (\frac{1}{2}(1-\phi_\epsilon)\nabla h^T F_- + \frac{1}{2}(1+\phi_\epsilon)\nabla h^T F_+)(\bar{x}) = 0$. But $(\nabla h^T F_-)(\bar{x}) > 0$, $\nabla h^T F_+(\bar{x}) > 0$ and $\frac{1}{2}(1+\phi_\epsilon), \frac{1}{2}(1-\phi_\epsilon) > 0$, hence we reach a contradiction. □

An invariant region V_ϵ for F_ϵ can be built by considering the union of all trajectories with initial points on ψ_δ and endpoints on $P_\epsilon(\psi_\delta)$ together with $B_\delta(\bar{x}_1, S) \backslash P_\epsilon(B_\delta(\bar{x}_1, S)) = \hat{B}^\epsilon_\delta$. Let $\hat{V}_\epsilon$ be the compact subset of $\mathbb{R}^n$ with boundary V_ϵ. Then all trajectories of (1) that cross $\hat{B}^\epsilon_\delta$ will do so transversally, will enter $\hat{V}_\epsilon$ and will remain inside it. See Figure 4. We will show that (3) has a unique periodic orbit γ_ϵ in $\hat{V}_\epsilon$ and γ_ϵ attracts all the solutions inside $\hat{V}_\epsilon$.

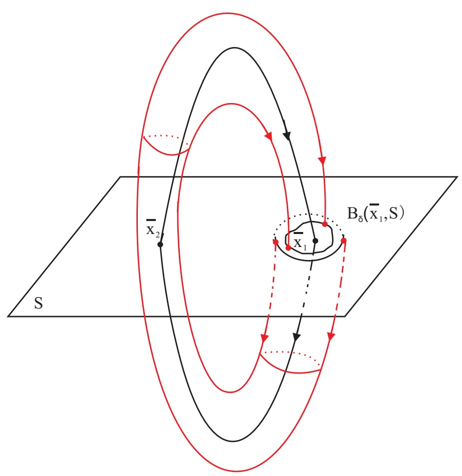

FIGURE 4. invariant region V_ϵ

Next, for each ϵ, select a fixed point of P_ϵ and denote it with x_ϵ. Let $\gamma_\epsilon = \{x \in \mathbb{R}^n |\ x = \varphi^t_\epsilon(x_\epsilon),\ \ t \in \mathbb{R}\}$, be the corresponding periodic orbit. In order to prove Theorem 6 we will show

Main steps

i) as $\epsilon \to 0$, $x_\epsilon \to \bar{x}_1$, which in turn will imply $\gamma_\epsilon \to \gamma$;
ii) for ϵ sufficiently small, γ_ϵ is asymptotically stable and this allows us to exclude that (3) has a family of periodic orbits that converges to γ.

Let $x \in \overline{B_\delta(\bar{x}_1, S)}$ and let $\varphi^t_0(x)$ be the solution of (1) with initial condition x. Denote with x_2 its intersection with S in a neighborhood of $\bar{x}_2$ and let $x_3 = P(x)$. Notice that $x_3 \neq x$ unless $x = \bar{x}_1$. Let t_2 be such that $\varphi^{t_2}_0(x) = x_2$ and $T(x)$ be such that $\varphi^{T(x)}_0(x) = x_3$. The following Lemma holds.

Lemma 11. *Denote with $X(t,0,x)$ and $X_\epsilon(t,0,x)$ respectively the fundamental matrix solution of* (1) *along the solution $\varphi_0^t(x)$ and the fundamental matrix solution of* (3) *along $\varphi_\epsilon^t(x)$, $x \in B_\delta(\bar{x}_1, S)$ Then*

$$\lim_{\epsilon\to 0} X_\epsilon(T_\epsilon(x),0,x) = X(T(x),0,x),$$

where with $T_\epsilon(x)$ we denote the first return time of $\varphi_\epsilon^t(x)$ to $\overline{B_\delta(\bar{x}_1, S)}$.

Proof. In what follows we will omit the explicit dependence of $X(t,0,x)$ and $X_\epsilon(t,0,x)$ on x. The principal matrix solution of (3) along $\varphi_\epsilon^t(x)$ satisfies

$$\dot{X}_\epsilon = DF_\epsilon(\varphi_\epsilon^t(x))X_\epsilon, \qquad X_\epsilon(0) = I, \tag{8}$$

where I is the identity matrix and $DF_\epsilon(x)$ is the Jacobian of F_ϵ and it is given by

$$DF_\epsilon(x) = \frac{1}{2}[(1-\phi_\epsilon)DF_- + (1+\phi_\epsilon)DF_+ + \frac{1}{\epsilon}\phi_\epsilon'(F_+ - F_-)\nabla h^T](x). \tag{9}$$

Similarly to what we have done in the proof of Lemma 8, we consider the intersections of $\varphi_\epsilon^t(x)$ with S, S_ϵ and $S_{-\epsilon}$ and use these intersection points to rewrite $X_\epsilon(T_\epsilon(x),0)$ as product of transition matrices. It will be handy to take the initial condition on S_ϵ instead of S. So, let $x_0^\epsilon \in S_\epsilon$ be such that $\varphi_\epsilon^t(x_0^\epsilon)$ meets S at time t_1^ϵ at the point x, i.e. $\varphi_\epsilon^{t_1^\epsilon}(x_0^\epsilon) = x$, and $\nabla h^T F_\epsilon(x_0^\epsilon) < 0$. We evaluate the monodromy matrix $X_\epsilon(T_\epsilon(x),0)$ along the shifted solution $\varphi_\epsilon^t(x_0^\epsilon)$. Notice that in a whole neighborhood of x the following inequality is satisfied $\nabla h^T F_\epsilon(x) < 0$ so that $\varphi^t(x_0^\epsilon)$ intersects S and then $S_{-\epsilon}$ at two isolated points: $x = \varphi_\epsilon^{t_1^\epsilon}(x_0^\epsilon)$ and $x_2^\epsilon = \varphi_\epsilon^{t_2^\epsilon}(x_0^\epsilon)$. Then the trajectory enters the set $\{x \in \mathbb{R}^n | h(x) < -\epsilon\}$ and remains in this set until, at time t_3^ϵ, it meets $S_{-\epsilon}$ again in x_3^ϵ. In a neighborhood of x_3^ϵ the following inequality is satisfied $\nabla h^T F_\epsilon(x) > 0$ so that, at time t_4^ϵ, $\varphi_\epsilon^t(x_0^\epsilon)$ meets S in x_4^ϵ and then S_ϵ in x_5^ϵ at time t_5^ϵ. At $t = T_\epsilon(x)$, $\varphi_\epsilon^t(x_0^\epsilon)$ returns to S_ϵ. With these notations, we can rewrite X_ϵ as follows

$$X_\epsilon(T_\epsilon(x),0) = X_\epsilon(T_\epsilon(x),t_5^\epsilon)X_\epsilon(t_5^\epsilon,t_3^\epsilon)X_\epsilon(t_3^\epsilon,t_2^\epsilon)X_\epsilon(t_2^\epsilon,0). \tag{10}$$

We want to show that all the factors making up X_ϵ in (10) converge to the corresponding factors in $X(t,0,x)$ (see (6) as well).

We first look at the factor $X_\epsilon(t_2^\epsilon,0)$, rewritten as

$$X_\epsilon(t_2^\epsilon,0) = X_\epsilon(t_2^\epsilon,t_1^\epsilon)X_\epsilon(t_1^\epsilon,0) \ ,$$

so that the limit of $X_\epsilon(t_2^\epsilon,0)$, as $\epsilon \to 0$, will exist if the limits of the other two factors do. We have

$$\begin{aligned}X_\epsilon(t_1^\epsilon,0) = I + \int_0^{t_1^\epsilon} DF_\epsilon(x(t))X_\epsilon(t,0)dt = \\ I + \frac{1}{2}\int_0^{t_1^\epsilon}\left[(1-\phi_\epsilon)DF_- + (1+\phi_\epsilon)DF_+\right](x(t))X_\epsilon(t,0)dt + \\ \frac{1}{\epsilon}\int_0^{t_1^\epsilon}\frac{1}{2}\phi'\left(\frac{h(x(t)}{\epsilon}\right)(F_+ - F_-)\nabla h(x(t))^T X_\epsilon(t,0)dt.\end{aligned}$$

The first integral goes to 0 as $\epsilon \to 0$, since $t_1^\epsilon \to 0$ and the integrand is bounded. The second integral is dealt with by noticing that $\phi'(\frac{h}{\epsilon})$ is positive and thus we can consider the change of variable $t \to x_1$, along with $\frac{1}{2\epsilon}\int_\epsilon^0 \phi'(\frac{x_1}{\epsilon})dx_1 = -\frac{1}{2}$, and the mean value Theorem for integrals to obtain

$$\begin{aligned}\frac{1}{2\epsilon}\int_\epsilon^0 \frac{1}{\nabla h^T F_\epsilon(x)}\phi'\left(\frac{x_1}{\epsilon}\right)(F_+ - F_-)(x)\nabla h^T X_\epsilon dx_1 = \\ -\frac{1}{2}\frac{(F_- - F_+)\nabla h^T X_\epsilon}{\nabla h^T F_\epsilon},\end{aligned} \tag{11}$$

where the entries of the matrix on the right-hand side are evaluated at some value $\bar{t}$, $0 \le \bar{t} < t_1^\epsilon$, possibly different for each matrix entry. Since all objects on the right-hand side are smooth in $[0, t_1^\epsilon)$, $t_1^\epsilon \to 0$, F_ϵ at $t = 0$ is F_+, and $X_\epsilon(t_1^\epsilon, 0) = I + \mathcal{O}(t_1^\epsilon)$, by expanding around $t = 0$ (or $x_1 = \epsilon$), we conclude that the limit as $\epsilon \to 0$ of $X_\epsilon(t_1^\epsilon, 0)$ exists and it is:

$$\lim_{\epsilon\to 0} X_\epsilon(t_1^\epsilon, 0) = I + \frac{1}{2}\frac{(F_- - F_+)\nabla h^T}{\nabla h^T F_+}(x)\,. \tag{12}$$

For $X_\epsilon(t_2^\epsilon, t_1^\epsilon)$, we have

$$\begin{aligned}X_\epsilon(t_2^\epsilon, t_1^\epsilon) = I + \int_{t_1^\epsilon}^{t_2^\epsilon} DF_\epsilon(x(t))X_\epsilon(t, t_1^\epsilon)dt = \\ I + \frac{1}{2}\int_{t_1^\epsilon}^{t_2^\epsilon}\left[(1-\phi_\epsilon)DF_- + (1+\phi_\epsilon)DF_+\right](x(t))X_\epsilon(t, t_1^\epsilon)dt + \\ \frac{1}{\epsilon}\int_{t_1^\epsilon}^{t_2^\epsilon}\frac{1}{2}\phi'\left(\frac{h(x(t)}{\epsilon}\right)(F_+ - F_-)\nabla h(x(t))^T X_\epsilon(t, t_1^\epsilon)dt.\end{aligned}$$

Again, the first integral goes to 0 as $\epsilon \to 0$, since t_2^ϵ, and $t_1^\epsilon \to 0$ and the integrand is bounded. For the second integral, since $\phi'(\frac{h}{\epsilon})$ is positive, we can consider the change of variable $t \to x_1$, along with $\frac{1}{2\epsilon}\int_0^{-\epsilon}\phi'(\frac{x_1}{\epsilon})dx_1 = -\frac{1}{2}$, and the mean value Theorem for integrals to obtain

$$\begin{aligned}\frac{1}{2\epsilon}\int_0^{-\epsilon} \frac{1}{\nabla h^T F_\epsilon(x)}\phi'\left(\frac{x_1}{\epsilon}\right)(F_+ - F_-)(x)\nabla h^T X_\epsilon dx_1 = \\ -\frac{1}{2}\frac{(F_+ - F_-)\nabla h^T X_\epsilon}{\nabla h^T F_\epsilon},\end{aligned} \tag{13}$$

where the entries of the matrix on the right-hand side are evaluated at some value $\bar{t}$, $t_1^\epsilon < \bar{t} \le t_2^\epsilon$, possibly different for each matrix entry. Since all objects on the right-hand side are smooth in $(t_1^\epsilon, t_2^\epsilon]$, $t_{1,2}^\epsilon \to 0$, F_ϵ at $t = t_2^\epsilon$ is F_-, then by expanding around $t = t_2^\epsilon$ (or $x_1 = -\epsilon$), we conclude that the limit as $\epsilon \to 0$ of $X_\epsilon(t_2^\epsilon, t_1^\epsilon)$ exists, call it M, and it is defined by the relation:

$$M \equiv \left[I + \frac{1}{2}\frac{(F_+ - F_-)\nabla h^T}{\nabla h^T F_-}(x)\right]^{-1}\,. \tag{14}$$

Note that the matrix defining M is invertible, and we have

$$M = I - \frac{(F_+ - F_-)\nabla h^T}{\nabla h^T(F_+ + F_-)}(x)\,.$$

Finally, using (11) and (13), we can then conclude that $\lim_{\epsilon\to 0} X_\epsilon(t_2^\epsilon, 0)$ exists, and it is given by

$$\left(I - \frac{(F_+ - F_-)\nabla h^T}{\nabla h^T(F_+ + F_-)}\right)\left(I + \frac{1}{2}\frac{(F_- - F_+)\nabla h^T}{\nabla h^T F_+}\right)(x)\,.$$

Doing the algebra, this simplifies to

$$\lim_{\epsilon \to 0} X_\epsilon(t_2^\epsilon, 0) = I + \left[\frac{F_- - F_+}{\nabla h^T F_+} \nabla h^T\right]_x = S_{+-}(x_1). \tag{15}$$

Similarly to (15), we can prove $\lim_{\epsilon\to 0} X_\epsilon(t_5^\epsilon, t_3^\epsilon) = S_{-+}(x_2)$.

Finally, $F_\epsilon = F_-$ in $[t_2^\epsilon, t_3^\epsilon]$ and $F_\epsilon = F_+$ in $[t_5^\epsilon, T_\epsilon]$. This, together with the following limits: $x_3^\epsilon, x_4^\epsilon, x_5^\epsilon \to x_2$, and $t_3^\epsilon, t_4^\epsilon, t_5^\epsilon \to t_2$, and $T_\epsilon(x) \to T(x)$, imply:

$$\lim_{\epsilon\to 0} X_\epsilon(t_3^\epsilon, t_2^\epsilon) = X(t_2, 0), \quad \text{and} \quad \lim_{\epsilon \to 0} X_\epsilon(T_\epsilon(x), 0) = X(T(x), 0).$$

□

Lemma 11 implies in particular that $X_\epsilon(T_\epsilon(x), 0, x)$ is bounded for all ϵ. At first, this might come as a surprise, since the derivative with respect to x_1 of F_ϵ is multiplied by the factor $\frac{1}{\epsilon}$ in the boundary layer, (see also the first column of DF_ϵ in (9)). However, the boundary layer and the time spent inside the boundary layer go to zero linearly with ϵ, so that the derivative of the vector field in the boundary layer remains finite in the limit. Moreover, boundedness of $X_\epsilon(T_\epsilon(x), 0, x)$ implies Lipschitzianity of the solution for each $\epsilon > 0$, as the following corollary states.

Corollary 12. *There exists $\beta > 0$ such that for each $t \geq 0$ and for each $\epsilon > 0$*

$$\|\varphi_\epsilon^t(x) - \varphi_\epsilon^t(y)\| \leq \beta \|x - y\|, \ \ x, y \in B_\delta(\bar{x}_1) \ .$$

Proof. The proof follows from the mean value Theorem for integrals applied to $\|\varphi_\epsilon^t(x) - \varphi_\epsilon^t(y)\| = \| \int_0^1 X_\epsilon(t, 0, sx + (1-s)y)(x-y)ds\|$. □

Corollary 13. *There exists $\alpha > 0$ such that for each $\epsilon > 0$*

$$\|P_\epsilon(x) - P_\epsilon(y)\| \leq \alpha\|x - y\|, \ \ x, y \in B_\delta(\bar{x}_1, S)$$

Proof. The result follows from the equality

$$DP_\epsilon(x) = E^T X_\epsilon(T_\epsilon(x), 0, x)E,$$

where $x \in B_\delta(\bar{x}_1, S)$, $E = (e_2, \ldots, e_n)$, e_j is the j-th canonical vector and $T_\epsilon(x)$ is the first return time of $\varphi_\epsilon^t(x)$ to $B_\delta(\bar{x}_1, S)$. Then $\|DP_\epsilon(x)\| \leq \|X_\epsilon(T_\epsilon(x), 0, x)\|$ and the norm on the right is bounded as $\epsilon \to 0$ because of Lemma 11. □

The following Lemma is the first part of Main step i) in the proof of Theorem 6.

Lemma 14.

$$\lim_{\epsilon\to 0} x_\epsilon = \bar{x}_1.$$

Proof. Let us denote with x_ϵ^k and $\bar{x}_1^k$ the k-th component of x_ϵ and $\bar{x}_1$, $k \geq 2$. (The first components are 0.) Let

$$\liminf_{\epsilon\to 0} x_\epsilon^2 = \underline{x}^2, \qquad \limsup_{\epsilon\to 0} x_\epsilon^2 = \overline{x}^2$$

and let x_{ϵ_i} and x_{ϵ_s} be two sequences such that $\lim_{\epsilon_i\to 0} x^2_{\epsilon_i} = \underline{x}^2$ and $\lim_{\epsilon_s\to 0} x^2_{\epsilon_i} = \overline{x}^2$. From x_{ϵ_i} and x_{ϵ_s} we can extract convergent subsequences which, with abuse of notation, we sill denote as x_{ϵ_i} and x_{ϵ_s}. Let

$$\lim_{\epsilon_i\to 0} x_{\epsilon_i} = \underline{x}, \quad \lim_{\epsilon_s\to 0} x_{\epsilon_s} = \overline{x}. \tag{16}$$

Notice that $\lim_{\epsilon_i\to 0} P_{\epsilon_i}(x_{\epsilon_i}) = \lim_{\epsilon_i\to 0} x_{\epsilon_i} = \underline{x}$, and $\lim_{\epsilon_s\to 0} P_{\epsilon_s}(x_{\epsilon_s}) = \lim_{\epsilon_s\to 0} x_{\epsilon_s} = \overline{x}$.

Then

$$\|P(\underline{x}) - \underline{x}\| \le \|P(\underline{x}) - P_{\epsilon_i}(\underline{x})\| + \|P_{\epsilon_i}(\underline{x}) - P_{\epsilon_i}(x_{\epsilon_i})\| + \|P_{\epsilon_i}(x_{\epsilon_i}) - \underline{x}\|.$$

The three terms to the right of the inequality go to zero for $\epsilon_i \to 0$. Indeed we have $P_{\epsilon_i}(\underline{x}) \to P(\underline{x})$, $P_{\epsilon_i}(x_{\epsilon_i}) \to \underline{x}$ and $\|P_{\epsilon_i}(\underline{x}) - P_{\epsilon_i}(x_{\epsilon_i})\| \le \alpha\|\underline{x} - x_{\epsilon_i}\|$ because of Corollary 13. It then follows that $\underline{x} = P(\underline{x})$, so that $\underline{x} = \bar{x}_1$. Similarly, $\overline{x} = \bar{x}_1$. As a consequence of this reasoning, $\lim_{\epsilon\to 0} x^2_\epsilon = \bar{x}^2_1$. To show convergence of x^k_ϵ, $k = 3, \dots, n$, we use reasonings analogous to the ones used for (16) together with the reasonings in this proof. □

Lemma 14 and Corollary 12 imply the following inequality

$$\begin{aligned} \|\varphi^t_\epsilon(x_\epsilon) - \varphi^t_0(\bar{x}_1)\| &\le \|\varphi^t_\epsilon(x_\epsilon) - \varphi^t_\epsilon(\bar{x}_1)\| + \|\varphi^t_\epsilon(\bar{x}_1) - \varphi^t_0(\bar{x}_1)\| \\ &\le \beta\|x_\epsilon - \bar{x}_1\| + \|\varphi^t_\epsilon(\bar{x}_1) - \varphi^t_0(\bar{x}_1)\|. \end{aligned} \tag{17}$$

In particular, (17) and Lemma 8 insure $\gamma_\epsilon \to \gamma$ and this proves Main step i) above.

Lemma 11 and Lemma 14 together with continuity of X_ϵ with respect to $x \in B_\delta(\bar{x}_1, S)$, imply that for all $\mu > 0$ there exists ϵ_μ sufficiently small so that for $\epsilon < \epsilon_\mu$ the following inequality is satisfied

$$\begin{aligned} \|X_\epsilon(T_\epsilon, 0, x_\epsilon) - X(T, 0, \bar{x}_1)\| \le \|X_\epsilon(T_\epsilon, 0, x_\epsilon) - X_\epsilon(T_\epsilon(\bar{x}_1), 0, \bar{x}_1)\| + \\ \|X_\epsilon(T_\epsilon(\bar{x}_1), 0, \bar{x}_1) - X(T, 0, \bar{x}_1)\| < \mu. \end{aligned}$$

As a consequence, for ϵ sufficiently small, $X_\epsilon(T_\epsilon, 0, x_\epsilon)$ has all eigenvalues less than 1 and one equal to 1 (since γ_ϵ is periodic). This implies Main step ii), i.e., γ_ϵ is asymptotically stable and hence isolated. This completes the proof of Theorem 6. □

Remark 15. *The proofs of the results in this section do not change in case of a periodic orbit γ with a finite number of transversal crossings. The Poincarè map P can be defined in the neighborhood of one of the transversal crossing points and viewed as composition of two or more smooth maps and it retains its smoothness. The attractivity of γ insures contractivity of P. Pointwise convergence of P_ϵ to P can be proved as in Corollary 9 and all related results can as well, in particular obtaining contractivity of P_ϵ for ϵ sufficiently small. The monodromy matrix is given by the composition of saltation matrices of full rank with fundamental matrix solutions and the proof of convergence for the monodromy matrix remains essentially the same as we have given.*

Theorem 6 together with Remark 15 suffice to prove Theorem 5 for a crossing periodic orbit γ with a finite number of transversal crossings.

3.2. **Sliding periodic orbit.** Let γ be the periodic orbit of (1). We will first assume that γ is the union of two arcs, one in R_-, or R_+, and one in S. In Remark 26 we will discuss how to generalize the proof to the case of an orbit with a finite number of sliding arcs and a finite number of crossing transversal points.

Theorem 16. *Assume that system* (1) *has an asymptotically stable sliding periodic orbit* γ *given by two arcs: one arc in the region* R_+ *or* R_- *and one arc on* S*. Assume moreover that the entry point in* S *is a transversal entry point and that the exit point from* S *is a first order tangential exit point. Then, for* ϵ *sufficiently small, there exists one and only one periodic orbit* γ_ϵ *of problem* (3) *in a neighborhood of* γ*. Moreover* γ_ϵ *is asymptotically stable and* $\lim_{\epsilon\to 0}\gamma_\epsilon = \gamma$*.*

Without loss of generality, we assume that γ has an arc in R_-, and the other on S. Further, we let x_{in} and x_{out} be respectively the entry point in S, from R_-, and the exit point from S, onto R_-, of γ. The point x_{out} is a first order tangential exit point, then in particular $\nabla h(x_{out})^T F_-(x_{out}) = 0$. Assume that the point x_{in} is a transversal entry point. See Figure 5; although the time arrow is for convenience only, we will work with this figure in mind when defining the Poincaré map.

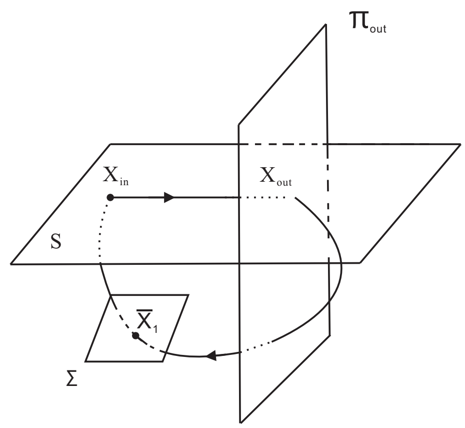

FIGURE 5. Sliding periodic orbit

We first define a Poincarè map for the nonsmooth system in a neighborhood of γ. To do so, let $\bar{x}_1 \in \gamma \cap R^-$ and take a cross section Σ to γ at $\bar{x}_1$. Let $B_\delta(\bar{x}_1, \Sigma) = B_\delta(\bar{x}_1) \cap \Sigma$. Then, for δ sufficiently small, all solutions with initial condition in $\overline{B_\delta(\bar{x}_1, \Sigma)}$ reach S in a neighborhood of x_{in} and they will start sliding on S since x_{in} is a transversal entry point. Let $g(x) = \nabla h^T F_-(x)$ and consider the set $\Pi_{out} = \{x \in B_\eta(x_{out}) |\ g(x) = 0\}$, with $\eta > 0$ and small; note that this is a small neighborhood of x_{out}. Then $\nabla g^T F_-(x_{out}) < 0$ (see Definition 2) implies that all solutions that slide on S in a neighborhood of γ will cross Π_{out} transversally and enter R_-. Then they will reach Σ again. We define the map $P : \overline{B_\delta(\bar{x}_1, \Sigma)} \to \overline{B_\delta(\bar{x}_1, \Sigma)}$ as $P(x) = \varphi_0^{t(x)}(x)$, with $t(x)$ first return time to Σ. P is given by the composition of three smooth maps:

1) $P_1 : \Sigma \to S$, $P_1(x) = \varphi_-^{t_-(x)}(x)$, where $t_-(x)$ is the first time at which $\varphi_-^t(x)$ meets S;
2) $P_S : S \to S \cap \Pi_{out}$, $P_S(x) = \varphi_S^{t_S(x)}(x)$, where with φ_S^t we denote the flow of the sliding Filippov vector field (5) and $t_S(x)$ is the time at which $\varphi_S^t(x)$ meets Π_{out};
3) $P_2 : S \cap \Pi_{out} \to \Sigma$, $P_2(x) = \varphi_-^{t_-^2(x)}(x)$, where $t_-^2(x)$ is the time at which $\varphi_-^t(x)$ reaches Σ.

Remark 17. *The Poincarè map* P *defined in this way is smooth. The cross section in* R_- *allows us to consider only solutions of* (1) *with orbits in* $R_- \cup S$*. [We remark that taking any section along the sliding arc (for example* Π_{out}*), forces us to consider also orbits with an arc in* R_+ *and this has two drawbacks: i) the corresponding Poincarè map is continuous but not smooth (it is defined in a different way for points in* R_- *or in* R_+*); ii) the orbits of* (1) *might cross the section more than once (if we consider the section* Π_{out} *for example, this happens if* $\nabla g^T F_-(x) < 0$ *while* $\nabla g^T F_+(x) > 0$*)*

and a definition of the Poincarè map via the first return time to the section is not possible. Also, notice that considering the cross section $S \cap B_\delta(x_{in})$ would remove the concerns i) and ii) above for the non-smooth problem. However, a section that lies on S cannot be used for the regularized problem because solutions of (3) *might never reach it.]*

P is a contraction so that $P(\overline{B_\delta(\bar{x}_1, \Sigma)})$ is a proper subset of $\overline{B_\delta(\bar{x}_1, \Sigma)}$. We study the stability of γ using the monodromy matrix of (1) along γ. Let t_{in} be such that $\varphi_0^{t_{in}}(\bar{x}_1) = x_{in}$, t_{out} be such that $\varphi_0^{t_{out}}(\bar{x}_1) = x_{out}$, and T be such that $\varphi_0^T(\bar{x}_1) = \bar{x}_1$. Then the monodromy matrix $X(T, 0)$ along γ is given by the following expression (e.g., see [8])

$$X(T,0) = X(T, t_{out})X(t_{out}, t_{in})S_{-S}(x_{in})X(t_{in}, 0), \tag{18}$$

where $S_{-S}(x_{in}) = [I + \frac{F_S - F_-}{\nabla h^T F_-(x_{in})}]\nabla h(x_{in})^T$ is the saltation matrix that satisfies $S_{-S}(x_{in})F_-(x_{in}) = F_S(x_{in})$, and the three fundamental matrices in (18) solve the following Cauchy problems:

$$\begin{aligned} \dot{X}(t,0) &= DF_-(\varphi_-^t(\bar{x}_1))X(t,0), \qquad X(0,0) = I, \\ \dot{X}(t,t_{in}) &= DF_S(\varphi_S^{(t-t_{in})}(x_{in}))X(t,t_{in}), \quad X(t_{in},t_{in}) = I, \\ \dot{X}(t,t_{out}) &= DF_-(\varphi_-^{t-t_{out}}(x_{out}))X(t,t_{out}), \quad X(t_{out},t_{out}) = I. \end{aligned}$$

The matrix $S_{-S}(x_{in})$ has rank $(n-1)$ and hence $X(T,0)$ has an eigenvalue at 1, one at 0, and, due to the asymptotic stability of γ, all the other eigenvalues are less than 1 in modulus. The eigenvalue at 0 is a direct consequence of the fact that solution trajectories reach S in finite time and then slide on S for some time.

To define the Poincaré map, the next result is needed.

Lemma 18. *For all $x \in B_\delta(\bar{x}_1)$,*

$$\lim_{\epsilon \to 0} \varphi_\epsilon^t(x) = \varphi_0^t(x),$$

uniformly for t in a compact interval.

Proof. The proof of this Lemma is in Appendix A, and it relies on a singular perturbation analysis based on [20, pp. 249-260]. □

We now define the Poincarè map P_ϵ for (3) as the map $P_\epsilon : \overline{B_\delta(\bar{x}_1, \Sigma)} \to \Sigma$ so that for every point $x \in \overline{B_\delta(\bar{x}_1, \Sigma)}$, $P_\epsilon(x)$ is the first return point of $\varphi_\epsilon^{t_\epsilon(x)}(x)$ to Σ.

Lemma 18 insures that P_ϵ is well defined and it implies the following Proposition.

Proposition 19. *As $\epsilon \to 0$, P_ϵ converges pointwise to P.*

Propositions 20 and 21 below will allow us to view the Poincaré map as the composition of three maps, similarly to what we did for P. This is desirable since it will lead to a decomposition of the fundamental matrix solution of (3) into four different factors, as for the discontinuous case (see (18)).

Proposition 20. *For ϵ sufficiently small, orbits of* (3) *corresponding to solutions with initial conditions in a neighborhood of γ must intersect Π_{out} transversally.*

Proof. Let $S_{-\epsilon}^{out} = \Pi_{out} \cap S_{-\epsilon}$, and note that for all $x \in S_{-\epsilon}^{out}$, $\nabla h^T F_\epsilon(x) = \nabla h^T F_-(x) = 0$. The set $S_{-\epsilon}^{out}$ divides $S_{-\epsilon}$ in two regions, denote them as $S_{-\epsilon}^-$ and $S_{-\epsilon}^+$, such that for all $x \in S_{-\epsilon}^\pm$,

$\nabla h^T F_\epsilon(x) = \nabla h^T F_-(x) \gtrless 0$. Lemma 18 implies that an orbit Γ_ϵ of (3) in a neighborhood of γ must intersect $S^-_{-\epsilon}$ in a point x^ϵ_{out} in a neighborhood of x_{out}. Moreover, since $\nabla h^T F_\epsilon(x^\epsilon_{out}) = \nabla h^T F_-(x^\epsilon_{out}) < 0$, x^ϵ_{out} must be isolated. Uniform convergence of solutions of (3) to solutions of (1) implies that there are points of Γ_ϵ in a neighborhood of x_{out} that satisfy $\nabla h^T F_-(x) > 0$, while $\nabla h^T F_-(x^\epsilon_{out}) < 0$. Continuity of solutions with respect to x imply that there must be a point $\bar{x}_\epsilon$ of Γ_ϵ so that $g(\bar{x}_\epsilon) = \nabla h^T F_-(\bar{x}_\epsilon) = 0$ and $\nabla g^T F_\epsilon(\bar{x}_\epsilon) < 0$. The statement of the proposition follows. □

Proposition 21. *Solution trajectories of* (3) *corresponding to solutions with initial conditions in* $\overline{B_\delta(\bar{x}_1, \Sigma)}$ *must intersect* Π_{out} *only in one point, before returning to* Σ.

Proof. The proof uses some of the tools needed for the proof of Lemma 18 and hence is included in Appendix A. □

Remark 22. *The condition that* $x \in \overline{B_\delta(\bar{x}_1, \Sigma)}$ *is essential in Proposition 21. It guarantees that the corresponding solutions of* (3) *can not cross the boundary layer to enter* $\{x \in \mathbb{R}^n | h(x) > \epsilon\}$. *Without this property, the statement of the proposition might be false. Consider for example the case* $\nabla g^T F_+(x_{out}) > 0$ *and take an initial condition* $x \in \Pi_{out} \cap S_\epsilon$.

The following is a consequence of Brouwer fixed point Theorem and of Lemma 19.

Proposition 23. *For* ϵ *sufficiently small,* P_ϵ *has at least a fixed point in* $\overline{B_\delta(\bar{x}_1, \Sigma)}$.

Proof. See the proof of Proposition 7. □

Moreover, (3) cannot have equilibria in a neighborhood of γ. Indeed outside the boundary layer $F_\epsilon = F_\pm$, while if there is an equilibrium $\bar{x}$ inside the boundary layer, then $F_\epsilon(\bar{x}) = 0$ and $\phi(\bar{x})$ is such that $\frac{1}{2}(1 - \phi(\bar{x}))F_-(\bar{x}) + \frac{1}{2}(1 + \phi(\bar{x}))F_+(\bar{x}) = 0$. In particular $\nabla h^T F_\epsilon(\bar{x}) = 0$ so that $\phi(\bar{x}) = \phi^*(\bar{x})$ in (5) and hence $F_S(\bar{x}) = F_\epsilon(\bar{x}) = 0$, a contradiction. Then, to each fixed point of P_ϵ, there corresponds a periodic orbit of (3).

For each ϵ, we select a fixed point of P_ϵ and we denote it as x_ϵ. Let γ_ϵ be the corresponding periodic orbit. What follows mimics the reasonings employed for the case of a crossing periodic orbit. We will show

Main steps

i) as $\epsilon \to 0$, $x_\epsilon \to \bar{x}_1$, which in turn will imply $\gamma_\epsilon \to \gamma$;
ii) for ϵ sufficiently small, γ_ϵ is asymptotically stable so that it is the unique periodic orbit of (3) in a neighborhood of γ.

Let $x \in B_\delta(\bar{x}_1, \Sigma)$ and let $\varphi^t_0(x)$ be the corresponding solution of (1). Let $\hat{x}_{in} = \varphi^{\hat{t}_{in}}(x)$ be its intersection point with S and let $\varphi^{\hat{t}_{out}}(x) = \hat{x}_{out} \in \Pi_{out}$. At time $T(x)$ the solution reaches Σ at a point different from x, unless $x = \bar{x}_1$.

The following Lemma holds.

Lemma 24. *Let $X(t,0,x)$ be the fundamental matrix solution of* (1) *along $\varphi_0^t(x)$ and $X_\epsilon(t,0,x)$ be the fundamental matrix solution of* (3) *along $\varphi_\epsilon^t(x)$. Then*

$$\lim_{\epsilon\to 0} X_\epsilon(T_\epsilon(x),0,x) = X(T(x),0,x),$$

where $T_\epsilon(x)$ denotes the first return time of $\varphi_\epsilon^t(x)$ to Σ.

Proof. Below, we omit the dependence of X and X_ϵ upon x. We will adopt the notation used in Appendix A.

Together with S, consider the hyperplane $S_{-\epsilon} = \{x \in \mathbb{R}^n | h(x) = -\epsilon\}$. For ϵ sufficiently small, uniform convergence of $\varphi_\epsilon^t(x)$ to $\varphi_0^t(x)$ implies that $\varphi_\epsilon^t(x)$ meets $S_{-\epsilon}$ in two isolated points: x_{in}^ϵ in a neighborhood of $\hat{x}_{in}$ and x_{out}^ϵ in a neighborhood of $\hat{x}_{out}$. Let t_{in}^ϵ be such that $\varphi_\epsilon^{t_{in}^\epsilon}(x) = x_{in}^\epsilon$. Then $\lim_{\epsilon\to 0} x_{in}^\epsilon = \hat{x}_{in}$ and $\lim_{\epsilon\to 0} t_{in}^\epsilon = \hat{t}_{in}$.

Let $\phi^*(\hat{x}_{in})$ be as in (5) and let $\tau = \frac{t}{\epsilon}$. For $0 < \mu < \epsilon$, let τ_μ be such that, for $\tau > \tau_\mu$, $|\tilde{\phi}(\tau) - \phi^*(\hat{x}_{in})| < \frac{\mu}{3}$, where $\tilde{\phi}$ is defined in (23). Then τ_μ satisfies (27) and $t_\mu = \epsilon\tau_\mu \to 0^+$ as $\epsilon \to 0$. Let $x_1^{\mu,\epsilon} = e_1^T \varphi_\epsilon^{t_{in}+\epsilon\tau_\mu}(x), y^{\mu,\epsilon} = E^T \varphi_\epsilon^{t_{in}+\epsilon\tau_\mu}(x)$ and denote with $\phi^{\mu,\epsilon}$ the corresponding value of ϕ_ϵ evaluated at the point $\frac{x_1^{\mu,\epsilon}}{\epsilon}$. Then $|\phi^{\mu,\epsilon} - \phi^*((0,y^{\mu,\epsilon})^T)| < \mu < \epsilon$, see (28).

We write the fundamental matrix solution $X_\epsilon(T_\epsilon(x),0)$ as product of different transition matrices

$$X_\epsilon(T_\epsilon(x),0) = X_\epsilon(T_\epsilon(x),t_{out}^\epsilon)X_\epsilon(t_{out}^\epsilon, t_{in}^\epsilon + t_\mu)X_\epsilon(t_{in}^\epsilon + t_\mu, t_{in}^\epsilon)X_\epsilon(t_{in}^\epsilon, 0). \tag{19}$$

We want to show that each transition matrix in (19) converges to the corresponding transition matrix in $X(T(x),0,x)$ (see also (18)).

Lemma 18 and Proposition 21 imply $\lim_{\epsilon\to 0} x_{out}^\epsilon = \hat{x}_{out}$ and $\lim_{\epsilon\to 0} t_{out}^\epsilon = \hat{t}_{out}$. Moreover $\lim_{\epsilon\to 0} \phi^{\mu,\epsilon} = \phi^*(\hat{x}_{in})$ since $\phi_\epsilon(x(t))$ converges uniformly to $\phi^*(x(t))$ in $[t_{in} + t_\mu, \hat{t}_{out}]$ (see the last steps of the proof of Lemma 18 in Appendix A).
From the reasoning above it follows that

$$\lim_{\epsilon\to 0} X_\epsilon(t_{in}^\epsilon, 0) = X(\hat{t}_{in}, 0).$$

For the second piece, we write

$$X_\epsilon(t_{in}^\epsilon + t_\mu, t_{in}^\epsilon) = I + \int_{t_{in}^\epsilon}^{t_{in}^\epsilon + t_\mu} DF_\epsilon(\varphi_\epsilon(t))X_\epsilon(t, t_{in}^\epsilon)dt =$$

$$I + \frac{1}{2}\int_{t_{in}^\epsilon}^{t_{in}^\epsilon + t_\mu} \left((1-\phi_\epsilon)DF_-(\varphi_\epsilon(t)) + (1+\phi_\epsilon)DF_+(\varphi_\epsilon(t))\right) X_\epsilon(t, t_{in}^\epsilon)dt \tag{20}$$

$$+\frac{1}{2\epsilon}\int_{t_{in}^\epsilon}^{t_{in}^\epsilon + t_\mu} \phi'\left(\frac{x_1}{\epsilon}\right)(F_+ - F_-)\nabla h(x)^T X_\epsilon(t, t_{in}^\epsilon)dt.$$

In what follows we reason similarly to the proof of Lemma 11. The fundamental matrix solution X_ϵ must satisfy $X_\epsilon(t_{in}^\epsilon + t_\mu, t_{in}^\epsilon)F_\epsilon(x_{in}^\epsilon) = F_\epsilon(\varphi_\epsilon^{t_{in}^\epsilon + t_\mu}(x))$. When we take the limit as $\epsilon \to 0$, then $\varphi_\epsilon^{t_{in}^\epsilon + t_\mu}(x) = \hat{x}_{in}$, $F_\epsilon(x_{in}^\epsilon) \to F_-(\hat{x}_{in})$, and $\lim_{\epsilon\to 0} F_\epsilon(\varphi_\epsilon^{t_{in}^\epsilon + t_\mu}(x)) = F_S(\hat{x}_{in})$. In particular, we have that $\lim_{t\to\hat{t}_{in}^+} \lim_{\epsilon\to 0} F_\epsilon(\varphi_\epsilon^t(x)) \neq \lim_{t\to\hat{t}_{in}^-} \lim_{\epsilon\to 0} F_\epsilon(\varphi_\epsilon^t(x))$ and this discontinuity is reflected also in the limit of the fundamental matrix solution. Indeed, if $L = \lim_{\epsilon\to 0} X_\epsilon(t_{in}^\epsilon + t_\mu, t_{in}^\epsilon)$ exists, then it must satisfy

$$LF_-(\hat{x}_{in}) = F_S(\hat{x}_{in}). \tag{21}$$

Now, the first integral in (20) goes to zero as $\epsilon \to 0$ since $t_\mu = \epsilon\tau_\mu \to 0$ and the integrand is bounded.

For the second integral, inside the interval $(t_{in}^\epsilon, t_{in}^\epsilon + t_\mu)$, we can write

$$X_\epsilon(t, t_{in}^\epsilon) = I + R_\epsilon(x_{in}^\epsilon, t),$$
$$F_\pm(x) = F_\pm(x_{in}^\epsilon) + (t - t_{in}^\epsilon) DF_\pm(x_{in}^\epsilon) F_\pm(x_{in}^\epsilon) + \text{ h.o.t.},$$

where R_ϵ is bounded, and $\|R_\epsilon\| \to 0$ as $t \to t_{in}^\epsilon$.

Using this in the integral above, since $\phi'_\epsilon > 0$, we get

$$\begin{aligned}
&\frac{1}{2\epsilon}\int_{t_{in}^\epsilon}^{t_{in}^\epsilon + t_\mu} \phi'\left(\frac{x_1}{\epsilon}\right)(F_+ - F_-)\nabla h(x)^T X_\epsilon(t, t_{in}^\epsilon) dt = \\
&\frac{1}{2\epsilon}\int_{t_{in}^\epsilon}^{t_{in}^\epsilon + t_\mu} \phi'(\frac{x_1}{\epsilon}) dt \left[(F_+ - F_-)\nabla h^T(x_{in}^\epsilon) + \bar{E}\right] = \\
&\left[(F_+ - F_-)\nabla h^T(x_{in}^\epsilon) + \bar{E}\right] c(\epsilon) \ ,
\end{aligned}$$

where $c(\epsilon) = \frac{1}{2\epsilon}\int_{t_{in}^\epsilon}^{t_{in}^\epsilon + t_\mu} \phi'\left(\frac{x_1(t)}{\epsilon}\right) dt$, and $\bar{E}$ is the error matrix whose components are each computed at possibly different values of $\bar{t} \in (t_{in}^\epsilon, t_{in}^\epsilon + t_\mu)$.

Now, for $\epsilon \to 0$, $\bar{E} \to 0$ and $x_{in}^\epsilon \to \hat{x}_1$, and $t_{in}^\epsilon \to \hat{t}_{in}$ from the right. Thus, for $\epsilon \to 0$, the second integral gives

$$\left[(F_+ - F_-)\nabla h^T(\hat{x}_{in})\right] \lim_{\epsilon\to 0} c(\epsilon) \ ,$$

and so would have

$$\lim_{\epsilon\to 0} X_\epsilon(t_{in}^\epsilon + t_\mu, t_{in}^\epsilon) = I + \left[(F_+ - F_-)\nabla h^T(\hat{x}_{in})\right] \lim_{\epsilon\to 0} c(\epsilon)$$

and since (21) holds, we must have that

$$\lim_{\epsilon\to 0} c(\epsilon)(F_+ - F_-)\nabla h^T F_-(\hat{x}_{in}) = (F_S - F_-)(\hat{x}_{in}) = \frac{1 + \phi^*(\hat{x}_{in})}{2}(F_+ - F_-)(\hat{x}_{in}),$$

and hence $\lim_{\epsilon\to 0} c(\epsilon) = \frac{1+\phi^*(\hat{x}_{in})}{2}\frac{1}{\nabla h^T F_-(\hat{x}_{in})}$. Thus, finally we get

$$\lim_{\epsilon\to 0} X_\epsilon(t_{in}^\epsilon, t_{in}^\epsilon + t_\mu) = S_{-S}(\hat{x}_{in}),$$

where S_{-S} is the saltation matrix from R_- to S.

For the third transition matrix in (19), we have

$$X_\epsilon(t_{out}^\epsilon, t_{in}^\epsilon + t_\mu) = I + \int_{t_{in}^\epsilon + t_\mu}^{t_{out}^\epsilon} DF_\epsilon(\varphi_\epsilon) X(t, t_{in}^\epsilon + t_\mu) dt.$$

Lemma 18 implies that $\varphi_\epsilon^t(x) \to \varphi_0^t(x)$ uniformly and $t_\epsilon^{out} \to \hat{t}_{out}$. Moreover, $\phi_\epsilon(t) \to \phi^*(y(t))$ uniformly for $t \in [t_{in}^\epsilon + t_\mu, t_{out}^\epsilon]$, with $y(t) = E^T\varphi(t)$ so that $F_\epsilon(\varphi_\epsilon^t(x))$ converges to $F_S(\varphi_0^t(x))$ for $t \in [t_{in}^\epsilon + t_\mu, t_{out}^\epsilon]$. Hence

$$\begin{aligned}
\lim_{\epsilon\to 0} DF_\epsilon(\varphi_\epsilon(t)) = \lim_{\epsilon\to 0}(1 - \varphi_\epsilon(t)) DF_-(\varphi_\epsilon(t)) + \varphi_\epsilon(t) DF_+(\varphi_\epsilon(t)) + \frac{1}{\epsilon}\varphi'(\frac{x_1}{\epsilon})((F_+ - F_-)(\varphi_\epsilon(t)) = \\
(1 - \varphi^*(y(t)) DF_-(\varphi^*(y(t)) + \varphi^*(y(t)) DF_+(\varphi^*(y(t)) = DF_S(t).
\end{aligned}$$

It follows that

$$\lim_{\epsilon\to 0} X(t_\epsilon^{out}, t_{in}^\epsilon + t_\mu) = X(\hat{t}_{out}, \hat{t}_{in}).$$

Finally, the convergence of the last transition matrix in (19), $X_\epsilon(T_\epsilon(x), t^\epsilon_{out})$, to the corresponding term of $X(T(x), 0)$ (see also (18)), follows from $t^\epsilon_{out} \to \hat{t}_{out}$ and $x^\epsilon_{out} \to \hat{x}_{out}$. □

As in Section 3.1, Lemma 24 implies that φ^t_ϵ and P_ϵ are Lipschitz for all ϵ. Then the following result follows.

Lemma 25.

$$\lim_{\epsilon \to 0} x_\epsilon = \bar{x}_1.$$

Proof. Let us denote with x^k_ϵ the k-th component of x_ϵ. Let

$$\liminf_{\epsilon \to 0} x^1_\epsilon = \underline{x}^1, \qquad \limsup_{\epsilon \to 0} x^1_\epsilon = \overline{x}^1$$

and let x_{ϵ_i} and x_{ϵ_s} be two sequences such that $\lim_{\epsilon_i \to 0} x^1_{\epsilon_i} = \underline{x}^1$ and $\lim_{\epsilon_s \to 0} x^1_{\epsilon_i} = \overline{x}^1$. From x_{ϵ_i} and x_{ϵ_s} we can extract convergent subsequences which we sill denote as x_{ϵ_i} and x_{ϵ_s}. Let

$$\lim_{\epsilon_i \to 0} x_{\epsilon_i} = \underline{x}, \quad \lim_{\epsilon_s \to 0} x_{\epsilon_s} = \overline{x}. \tag{22}$$

As in Lemma 14, using the fact that P_ϵ is Lipschitz for all ϵ, we obtain that $P(\underline{x}) = \underline{x}$ and $P(\overline{x}) = \overline{x}$, so that $\underline{x} = \overline{x} = \bar{x}_1$. This proves convergence of the first component of x_ϵ to the first component of $\bar{x}_1$. The proof for the other components is done in a similar way. □

Lemma 24 and Lemma 25 together with continuity of X_ϵ with respect to x implies that $X_\epsilon(T_\epsilon, 0, x_\epsilon)$ has all eigenvalues less than 1 and one equal to 1 (since γ_ϵ is periodic). This implies that γ_ϵ is asymptotically stable and hence isolated, and Theorem 16 follows.

Remark 26. *In case there are multiple sliding segments, and possibly also crossing, combining together the results of Theorems 16 and 6, then Theorem 5 holds for these scenarios. It should be appreciated that, see* [8, Theorem 2.8]*, for an asymptotically stable periodic orbit of* (1) *with multiple sliding segments on the plane S, the multipliers are still 1,0, all other multipliers being less than 1 in absolute value and different from zero.*

4. Conclusions

In this paper we have considered n dimensional DRHS systems with a discontinuous co-dimension one plane S separating two regions R_- and R_+. We have further assumed that this discontinuous system has an asymptotically stable periodic orbit γ. Our main result shows that if γ consists of arcs on S and/or on R_- and R_+, but does not lie entirely on S, then a regularization of the discontinuous system also has a unique asymptotically stable limit cycle, converging to γ as the regularization parameter goes to 0. The case of γ lying entirely on S remains to be considered.

Finally, we stress that, our switching manifold is a codimension one plane. The case of higher co-dimension switching manifold (say, the intersection of two planes) is considerably more complex (e.g., see [7]), and remains to be considered as well.

Appendix A

Proof of Lemma 18. Let $x \in \overline{B}_\delta(\bar{x}_1)$ and denote with $\varphi_0^t(x)$ the solution of (1) and with $\varphi_\epsilon^t(x)$ the solution of (3). Since in R_-, $F_- = F_\epsilon$, at time $t = t_{in}^\epsilon$, $\varphi_0^t(x)$ and $\varphi_\epsilon^t(x)$ meet $S_{-\epsilon}$ transversally in a point x_{in}^ϵ so that $\nabla h^T F_-(x_{in}^\epsilon) = \nabla h^T F_\epsilon(x_{in}^\epsilon) > 0$. Then both $\varphi_\epsilon^t(x)$ and $\varphi_0^t(x)$ enter the boundary layer $\tilde{S} = \{x \in \mathbb{R}^n \ \ s.t. -\epsilon < h(x) < \epsilon\}$. In particular, there exist $\hat{x}_{in} \in S$ and $\hat{t}_{in} > 0$ such that $\varphi_0^{\hat{t}_{in}}(x) = \hat{x}_{in}$ and $\lim_{\epsilon\to 0} x_{in}^\epsilon = \hat{x}_{in}$. For $t > \hat{t}_{in}$, $\varphi_0^t(x)$ starts sliding on S until it meets Π_{out} at time $\hat{t}_{out}$: $\varphi_0^{\hat{t}_{out}}(x) = \hat{x}_{out} \in S \cap \Pi_{out}$. In order to show uniform convergence of φ_ϵ^t to φ_0^t, we need to study the limiting behavior of φ_ϵ^t inside $\tilde{S}$ and we do this via singular perturbation theory. Let $e_1 = (1, 0, 0, \ldots)^T$, then $h(x) = e_1^T x = x_1$. Let $y = E^T x$ with $E = (e_2, \ldots, e_n) \in \mathbb{R}^{n\times(n-1)}$, where e_i is the i-th canonical vector of $\mathbb{R}^n$. When x is inside the boundary layer the function $\phi = \phi(\frac{x_1}{\epsilon})$ is strictly monotone, so that x_1 in $[-\epsilon, \epsilon]$ can be expressed as a function of ϕ in $[-1, 1]$. Then we can consider the new variables (ϕ, y) and split (3) into fast and slow motion as follows

$$\begin{aligned} \epsilon\dot{\phi} =&\phi'(z) e_1^T F_\epsilon(x) \\ \dot{y} =&E^T F_\epsilon(x) \end{aligned} \tag{23}$$

with $z = \frac{x_1}{\epsilon}$ and with initial conditions $\phi(0) = \phi_0$ and $y(0) = y_0$. The transformation is done strictly inside the boundary layer, hence the initial condition for the fast variable ϕ satisfies $-1 < \phi_0 < 1$, while the initial condition for the slow variable y_0 is arbitrarily close to $E^T x_{in}^\epsilon$ and it converges to $E^T \hat{x}_{in}$ for $\epsilon \to 0$. Since $\phi(z)$ is monotone increasing in $(-1, 1)$, then z can be rewritten in function of ϕ, so $\phi'(z)$ is a function of ϕ as $\phi'(z) = g(\phi)$. Denote the solution of (23) as $(\phi_\epsilon(t), y_\epsilon(t))$. If we set $\epsilon = 0$ in (23), using the fact that $x_1 \to 0$ (while $-1 < z < 1$), we obtain the *reduced problem*

$$\begin{aligned} 0 =&g(\phi) e_1^T F_\epsilon((0, y)^T) \\ \dot{y} =&E^T F_\epsilon((0, y)^T), \end{aligned} \tag{24}$$

with initial conditions $\phi(0) = \phi_0$, $y(0) = E^T \hat{x}_{in}$, and with $g(\phi) = \phi'(z)$. The algebraic equation in (24) has a unique solution in $(-1, 1)$ for each y and this is given by the following smooth function of y: $\phi^*(y) = \frac{e_1^T(F_- + F_+)}{e_1^T(F_+ - F_-)}((0, y)^T)$. Then, (24) is just the Filippov sliding differential equation on S with vector field (5) and we denote its solution as $(\phi^*(y(t)), y(t))$. We claim that $\lim_{\epsilon\to 0}(\phi_\epsilon(t), y_\epsilon(t)) = (\phi^*(y(t)), y(t))$ (while $\lim_{\epsilon\to 0} x_1(t) = 0$) uniformly in time in $[\hat{t}_{in} + \eta, \hat{t}_{out}]$ where $\hat{t}_{out}$ is such that $\phi^*(y(\hat{t}_{out})) = \phi^*(E^T \hat{x}_{out}) = -1$ and $\eta > 0$. The uniform convergence is not immediate to verify since we cannot use continuity of solutions with respect to the parameter ϵ. We consider the fast time $\tau = \frac{t}{\epsilon}$ and the derivative with respect to τ. We obtain the following *fast system*

$$\begin{aligned} \frac{d\phi}{d\tau} =&g(\phi) e_1^T F_\epsilon(x) \\ \frac{dy}{d\tau} =&\epsilon E^T F_\epsilon(x), \end{aligned} \tag{25}$$

with initial condition $\phi(0) = \phi_0$, $y(0) = y_0$. Notice that if $(\phi_\epsilon(t), y_\epsilon(t))$ solves (23), then $(\phi_\epsilon(\epsilon\tau), y_\epsilon(\epsilon\tau))$ solves (25). If we set $\epsilon = 0$ we have the *reduced fast system*

$$\begin{aligned} \frac{d\phi}{d\tau} =&g(\phi) e_1^T F_\epsilon((0, y(0))^T), \\ \frac{dy}{d\tau} =&0, \end{aligned} \tag{26}$$

with initial condition $\phi(0) = \phi_0$ and $y(0) = E^T \hat{x}_{in}$ and with y as parameter. We denote the solution of the reduced fast system as $(\tilde{\phi}(\tau, y(0)), y(0))$. Notice that for $\phi = 1, -1$, $g(\phi) = \phi'(z) = 0$.

Lemma 27. *Assume that S is attractive at the point $(0,y)$, i.e., $\nabla h^T F_{\pm}((0,y)) \lessgtr 0$. Then $\phi^*(y) = \frac{e_1^T(F_- + F_+)}{e_1^T(F_+ - F_-)}(y)$ is the unique equilibrium of system (26) in $(-1,1)$. Moreover $\phi^*(y)$ is globally exponentially stable in $(-1,1)$.*

Proof. Let y be fixed and rewrite the first of (26) as

$$\frac{d\phi}{d\tau} = \phi'(z(\tau))(-a\phi + b), \quad -a = \frac{1}{2}e_1^T(F_+ - F_-)(0,y) < 0, \quad b = \frac{1}{2}e_1^T(F_+ + F_-)(0,y),$$

so that $\frac{d\phi}{-a\phi+b} = \phi'(z(\tau))d\tau$. The statement follows from the mean value Theorem for integrals upon noticing that $\phi'(z(\tau))$ is strictly positive in $z \in (-1,1)$ and it is zero only for $z = 1, -1$. $\square$

The proof of Lemma 27 applies also to obtain the following Corollary.

Corollary 28. *Let $(0, y_{out}) = x_{out}$. Then $\phi^*(y_{out}) = -1$ is the unique equilibrium of (26) in $[-1,1)$. Moreover all solutions in $(-1,1)$ converge exponentially fast to -1.* $\square$

Definition 29. *Let $\delta > 0$. The δ-cube is the set of points (ϕ, y) such that*

$$|\phi - \phi^*(y)| < \delta,$$

with $\phi^*(y)$ as in Lemma 27.

For $\mu < \epsilon$, because of Lemma 27, there exists an $\alpha > 0$ such that at time

$$\tau_\mu = -\alpha \log(\mu), \tag{27}$$

the solution of (26) satisfies $|\tilde{\phi}(\tau) - \phi^*(E^T\hat{x}_{in})| < \frac{\mu}{3}$ for $\tau > \tau_\mu$. Moreover, since the solution of (25) depends continuously on ϵ, it converges to the solution of (26) for $\epsilon \to 0$:

$$\lim_{\epsilon \to 0}(\phi_\epsilon(\epsilon\tau), y_\epsilon(\epsilon\tau)) = (\tilde{\phi}(\tau, E^T\hat{x}_{in}), E^T\hat{x}_{in}).$$

Since $\hat{x}_{in}$ is a transversal crossing point then, for ϵ sufficiently small and $\tau > \tau_\mu$, S is attractive at the point $(0, y_\epsilon(\epsilon\tau))$ and $\phi^*(y_\epsilon(\epsilon\tau))$ is in $(-1,1)$. Using this and the fact that ϕ^* is smooth with respect to y, we conclude that there exists an $\epsilon_\mu > 0$ such that for $\epsilon < \epsilon_\mu$ and $\tau > \tau_\mu$, the following is satisfied

$$\begin{aligned} |\phi_\epsilon(\epsilon\tau) - \phi^*(y_\epsilon(\epsilon\tau))| \leq |\phi_\epsilon(\epsilon\tau) - \tilde{\phi}(\tau, E^T\hat{x}_{in})| + |\tilde{\phi}(\tau, E^T\hat{x}_{in}) - \phi^*(E^T\hat{x}_{in})| + \\ |\phi^*(E^T\hat{x}_{in}) - \phi^*(y_\epsilon(\epsilon\tau))| < 3\frac{\mu}{3} = \mu. \end{aligned} \tag{28}$$

Then $(\phi_\epsilon(\epsilon\tau), y_\epsilon(\epsilon\tau))$ is in the μ-cube. The following Lemma is in [20].

Lemma 30. *Let $\delta > 0$ small. There exist $\mu_\delta > 0$ and $\epsilon_\delta > 0$ so that for $\epsilon < \epsilon_\delta$ if the solution of (23) enters the μ_δ cube at $t = \bar{t}$, it remains strictly inside the δ-cube as long as $-1 \leq \phi^*(y_\epsilon(t)) \leq 1$.*

Proof. The proof of the Lemma is by contradiction and can be found in [20, Lemma 39.1]. It relies on continuity of solutions of (25) with respect to the parameter ϵ and on the asymptotic convergence of solutions of $\frac{d\phi}{d\tau} = \phi'(z)e_1^T F_\epsilon(0,y)$ to $\phi^*(y)$. $\square$

Lemma 30 together with (28) implies that, for $\eta > 0$, the solution of (23) remains inside the η-cube, as long as $-1 \leq \phi^*(y_\epsilon(t)) \leq 1$. Let T_ϵ be such that $\phi^*(y_\epsilon(T_\epsilon)) = -1$. Then, for $\epsilon\tau_\mu \leq t \leq T_\epsilon$, we can write

$$\phi_\epsilon(t) = \phi^*(y_\epsilon(t)) + \omega_\epsilon(t), \tag{29}$$

with $\lim_{\epsilon\to 0}\omega_\epsilon(t)=0$. Then $y_\epsilon(t)$ satisfies the following differential equation in $[\epsilon t_\mu, T_\epsilon]$

$$\frac{dy_\epsilon}{dt}=E^T F_\epsilon\left(\phi^*(y_\epsilon(t))+\omega_\epsilon(t), y_\epsilon(t)\right),$$

and using continuity of the solution of the differential equation with respect to the state variable and the fact that $\lim_{\epsilon\to 0}\omega_\epsilon(t)=0$, we can conclude that $y_\epsilon(\epsilon t)\to y(t)$ uniformly in time in $[\epsilon\tau_\mu, t_{out}(x)]$, with t_{out} such that $\phi^*(y(t_{out}))=-1$. Moreover, since $y_\epsilon(0)\to E^T\hat{x}_{in}$, for $\epsilon\to 0$, the convergence is uniform also in $[0,\epsilon\tau_\mu]$. Hence $\lim_{\epsilon\to 0} y_\epsilon(t)=y(t)$ uniformly in $[0,T]$. This together with (29) implies uniform convergence of $\phi_\epsilon(t)$ to $\phi^*(y(t))$ in $[\epsilon\tau_\mu, T]$. Notice that the convergence is not uniform in $[0,\epsilon\tau_\mu]$ since $\phi_\epsilon(0)\neq\phi^*(E^T\hat{x}_{in})$. However if we consider the first component of the solution of (3) and we denote it as $x^\epsilon_{1,in}(t)$, then $x^\epsilon_{1,in}(t)\to 0$ ad $\epsilon\to 0$ uniformly in time in $[0,T]$. Hence $(x_{1,\epsilon}(t), y_\epsilon(t))\to(0,y^*(t))$ uniformly in $[0,T]$. This completes the proof of Lemma 18. □

Proof of Proposition 21. We will reason by contradiction. Let $x_0\in\overline{B_\delta(\bar{x}_1,\Sigma)}$ and consider the two flows $\varphi^t_\epsilon(x_0)$ and $\varphi^t_0(x_0)$. Let t_1 be such that $\varphi^{t_1}_0(x_0)$ meets S in x_1 in a neighborhood of x_{in}, starts sliding on S and at $t_2>t_1$ intersects Π_{out} transversally at a point x_2 in a neighborhood of x_{out}: $\varphi^{t_2}_0(x_0)=x_2\in\Pi_{out}\cap S$, and $\nabla g^T F_S(x_2)<0$, with $g(x)=\nabla h^T F_-(x)$. Similarly, $\varphi^t_\epsilon(x_0)$ intersects $S_{-\epsilon}$ at an isolated point x^ϵ_1 so that $\nabla h^T F_\epsilon(x^\epsilon_1)=\nabla h^T F_-(x^\epsilon_1)>0$. Let t^ϵ_1 be such that $x^\epsilon_1=\varphi^{t^\epsilon_1}_\epsilon(x_0)$. Then the solution remains inside the boundary layer and for $\hat{t}>t^\epsilon_1$ it crosses Π_{out} transversally at a point $\hat{x}$ such that $\nabla g^T F_\epsilon(\hat{x})<0$, see Proposition 20. At time $t^2_\epsilon>\hat{t}$, the solution crosses $S_{-\epsilon}$ and leaves the boundary layer. Assume by contradiction that there are $\hat{t}_1,\hat{t}_2\in(\hat{t},t^\epsilon_2)$, with $\hat{t}_1<\hat{t}_2$, so that $\hat{x}_{1,2}=\varphi^{\hat{t}_{1,2}}_\epsilon(x_0)$ are two other intersection point with Π_{out}. Then it must be

$$g(\hat{x}_{1,2})=0,\ \ \nabla g^T F_\epsilon(\hat{x}_1)>0,\ \ \nabla g^T F_\epsilon(\hat{x}_2)<0. \tag{30}$$

Because of uniform convergence of $\varphi^t_\epsilon(x_0)$ to $\varphi^t_0(x_0)$, we know that $t^\epsilon_1\to t_1$ and $\hat{t},\hat{t}_1,\hat{t}_2,t^\epsilon_2\to t_2$. Using same notations as in the proof of Lemma 18, if we fix $\mu>0$ small, there there exists ϵ_μ small such that for every $\epsilon<\epsilon_\mu$ the following must be satisfied

$$|\phi(y_\epsilon(\hat{t}_1))-\phi^*(E^T x_2)|\leq|\phi(y_\epsilon(\hat{t}_1))-\phi^*(y_\epsilon(\hat{t}_1))|+|\phi^*(y_\epsilon(\hat{t}_1))-\phi^*(E^T x_2)|<\mu.$$

Then $F_\epsilon(\hat{x}_1)$ is arbitrarily close to $F_S(0,E^T x_2)$, where with F_S we have denoted the sliding vector field on S. But at the exit point $(0,E^T x_2)$ it must be $\nabla g^T F_S(E^T x_2)<0$ and this is in contradiction with (30). □

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