REGULARIZING PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS: CO-DIMENSION 2 DISCONTINUITY SURFACE.

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Abstract. In this paper, we are concerned with numerical solution of piecewise smooth initial value problems. Of specific interest is the case when the discontinuities occur on a smooth manifold of co-dimension 2, intersection of two co-dimension 1 singularity surfaces, and which is *nodally attractive* for nearby dynamics.

In this case of a co-dimension 2 attracting sliding surface, we will give some results relative to two prototypical *time* and *space* regularizations. We will show that, unlike the case of co-dimension 1 discontinuity surface, in the case of co-dimension 2 discontinuity surface the behavior of the regularized problems is strikingly different. On the one hand, the time regularization approach will not select a unique sliding mode on the discontinuity surface, thus maintaining the general ambiguity of how to select a Filippov vector field in this case. On the other hand, the proposed space regularization approach is not ambiguous, and there will always be a unique solution associated to the regularized vector field, which will remain close to the original co-dimension 2 surface. We will further clarify the limiting behavior (as the regularization parameter goes to 0) of the proposed space regularization to the solution associated to the sliding vector field of [8].

Numerical examples will be given to illustrate the different cases and to provide some preliminary exploration in the case of co-dimension 3 discontinuity surface.

1. Introduction

Piecewise smooth differential systems (PWS for short) are receiving an ever increasing attention in mathematics, applied sciences, and engineerings, also because of the possibility to model dynamical systems exhibiting complex behavior in a very compact way. Excellent entry points in the literature on PWS systems are the recent

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books [2] and [3] that present numerical and dynamical overviews of the subject, as well as an extensive list of references on the topic.

The caveat is that –in general– PWS systems fail to have classical solutions, and some form of generalized solution concept becomes necessary. A very successful proposal, and the one appropriate for our present work, is one due to Filippov, see [9], who suggested to replace the PWS system with a certain differential inclusion. The resulting method has come to be known as *Filippov convexification.*

As motivation for the class of problems in which we are interested, let us look at an example arising from delay differential equations with two delays.

Example 1.1. Consider the following system of neutral delay equations (see [12, 13, 14] for such kind of problems and suitable regularizations)

$$\begin{aligned} \dot y_1(t) &= 1 + a\,\dot y_1(y_1(t)-1) + b\,\dot y_2(y_2(t)-1) \\ \dot y_2(t) &= 1 + c\,\dot y_1(y_1(t)-1) + d\,\dot y_2(y_2(t)-1) \end{aligned}$$

with the following choice of parameters: $a=-4$, $b=2$, $c=2$, $d=-4$. Clearly, there are four distinct vector fields in the four different regions $R_1=\{y_1<1, y_2<1\}$, $R_2=\{y_1<1, y_2>1\}$, $R_3=\{y_1>1, y_2<1\}$ and $R_4=\{y_1>1, y_2>1\}$:

Table 1. Vector fields.

Region	R_1	R_2	R_3	R_4
$\dot y_1$	1	3	-3	-1
$\dot y_2$	1	-3	3	-1

The vector field is not defined when either solution component y_1 or y_2 is equal to 1. However, any trajectory with initial conditions in any one of the regions R_i ($i=1,\dots,4$) will eventually reach one (or both) the values $y_1=1$, $y_2=1$, at which point it will cease to exist as a classical solution. □

1.1. **The problem.** Motivated by Example 1.1, we will consider the following class of PWS systems:

$$\dot x = f(x)\ ,\quad f(x)=f_i(x)\ ,\quad x\in R_i\ ,\ i=1,\dots,m\ , \tag{1}$$

to be studied for t in some interval $[0,T]$, subject to initial condition $x(0)=x_0$. In (1), $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, whose closures cover $\mathbb{R}^n$: $\mathbb{R}^n=\overline{\bigcup_i R_i}$; also, f_i is smooth on R_i and $\mathbb{R}^n\setminus\bigcup_i R_i$ has zero (Lebesgue) measure. So, in each region R_i, we have a standard differential equation with smooth vector field f_i, but on the boundary of these regions there is no properly defined differential equation.

In this work, we are interested in the case where the regions R_i's are separated (locally) by an implicitely defined surface Σ of co-dimension p, so that near Σ there are 2^p regions R_i's and 2^p vector fields f_i's. In other words, in (1) $m = 2^p$, the discontinuity surface is

$$\Sigma = \{x \in \mathbb{R}^n \ : \ h(x) = 0\,, \quad h:\ \mathbb{R}^n \to \mathbb{R}^p\}\,,$$

and for all $x \in \Sigma$: $h(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix}$, $\nabla h_j(x) \neq 0$, each $h_j \in \mathcal{C}^k,\ k \geq 2,\ j = 1, \dots, p$, and the vectors $\{\nabla h_1(x), \dots, \nabla h_p(x)\}$ are linearly independent.

Furthermore, we are interested in the case in which Σ is *nodally attractive* (see below), whereby trajectories near Σ are attracted to Σ and reach it in a finite time. Afterwards, trajectories cannot leave Σ and motion must continue on Σ, a situation referred to as having attractive *sliding motion*. The question is: *What is (if any) the vector field associated to this sliding motion*? Indeed, clearly on Σ the vector field is not properly defined.

Filippov proposal consists of considering the differential inclusion

$$\dot{x} \ \in \ F(x) = \sum_{i=1}^{2^p} \lambda_i(x) f_i(x)\ , \qquad \text{where} \quad \lambda_i(x) \geq 0\ ,\ \text{and} \quad \sum_{i=1}^{2^p} \lambda_i(x) = 1\,, \tag{2}$$

subject to the constraint that $F(x)$ lies in $T_x\Sigma$, the tangent plane to Σ at x:

$$(\nabla h_j(x))^T F(x) = 0\ , \ \text{ for all } \ j = 1, \dots, p\,. \tag{3}$$

Example 1.2. The case when Σ is of co-dimension $p = 1$ is well understood. There are two regions, R_1, R_2, with vector fields f_1 and f_2, and $\Sigma = \{x \in \mathbb{R}^n \ : \ h(x) = 0\,, \quad h:\ \mathbb{R}^n \to \mathbb{R}\}$. We can assume that in R_1 we have $h(x) < 0$, and in R_2 we have $h(x) > 0$. In this case, nodal attractivity of Σ is characterized by having $\nabla h(x)^T f_1(x) > 0$ and $\nabla h(x)^T f_2(x) < 0$, and (2)-(3) render a unique sliding vector on Σ. Indeed, we can write $\lambda_2 = \alpha$, $\lambda_1 = 1 - \alpha$, and imposing (3) we immediately get the differential system on Σ:

$$x' = (1-\alpha) f_1 + \alpha f_2\ , \quad \alpha = \frac{\nabla h(x)^T f_1(x)}{\nabla h(x)^T f_1(x) - \nabla h(x)^T f_2(x)}\,. \tag{4}$$

□

However, already when Σ has co-dimension $p = 2$, the construction based on (2)-(3) in general does not select a unique sliding vector field on Σ, since it will give us a system of 3 equations in 4 unknowns. This general ambiguity (which only gets more severe for higher co-dimension) inhibits from having a well defined differential equation (of Filippov type) governing the evolution of the dynamics, a fact which would be clearly desirable from the practical (modeling and numerical) point of view.

To remove the aforementioned general ambiguity, one can in principle proceed according to one of the following alternatives: (i) Enforce extra condition(s) to select a unique vector field from the convex set F of (2) (e.g., this path was followed in [8], and see also references in there), or (ii) Replace the original PWS differential system (1) with a globally regularized one, which for any positive value of the regularization parameter ε allows to obtain a classical solution, and then analyze the behaviour of such solutions for $\varepsilon \to 0$. In this work, we consider this second possibility in the case of a co-dimension 2 discontinuity surface. In particular, we will consider two different, prototypical, regularizations of (1): (a) A time regularization suggested in [10], based on a time average of the right-hand-side over a small interval ε, and (b) A space regularization based on smoothly connecting the different vector fields around Σ.

Remark 1.3. *In the case of a single manifold of co-dimension* 1*, both of the above mentioned regularizations have been previously explored (e.g.,* [10, 18]*), and the limiting behavior (in the regularization parameter) of their respective solutions does converge, in an appropriate topology, to the classical Filippov solution of* (4)*. In the case of a manifold of co-dimension 2, this study does not seem to have been carried out in the literature. Interestingly, and unlike the case of co-dimension 1 discontinuity surface, we will see that the behavior of the two different regularizations we consider is strikingly different. In a nutshell, the time regularization approach will not select a unique sliding mode on the discontinuity surface, thus retaining the general ambiguity present in the selection of a Filippov vector field in this case. However, the proposed space regularization approach will always select a unique solution, which stays close to the original co-dimension 2 surface.*

Remark 1.4. *Naturally, many different means to regularize* (1) *can be contemplated and different points of view lead to different interepretations of a certain regularization (e.g., see* [19, 17] *for an interpretation of regularization via geometric singular perturbation techniques). Indeed, alternatives to those we examine here have been proposed in the case of a co-dimension 1 discontinuity surface, for example the Euler polygonal approach of* [1] *(a type of time regularizations). Whereas of course any new regularization proposal will require a separate study of its limiting behavior, our limited experience leads us to suspect that the distinguished difference we observe in the present study (in co-dimension 2) between time and space regularizations will generally hold true.*

A plan of the paper is as follows. In the remainder of this Introduction we further examine the case of discontinuity surface of co-dimension 2 and recalls the selection process of a unique Filippov vector field made in [8]. Section 2 is concerned with regularization by a certain time averaging process, while Section 3 is concerned with a certain space averaging regularization. Section 4 presents results of numerical simulations in the different cases as well as some experimental evidence of what may happen when the discontinuity surface has co-dimension 3. Section 5 contains some conclusions.

1.2. **Co-dimension 2 case: Setup and a Filippov vector field.** Let us fix the notation relatively to the problem in co-dimension 2. We have $\Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_1 = \{\ x : \ h_1(x) = 0\ ,\ \ h_1 : \mathbb{R}^n \to \mathbb{R}\}$, $\nabla h_1(x) \neq 0, x \in \Sigma_1$, and $\Sigma_2 = \{\ x : \ h_2(x) = 0\ ,\ \ h_2 : \mathbb{R}^n \to \mathbb{R}\}$, $\nabla h_2(x) \neq 0, x \in \Sigma_2$, and further $\nabla h_1(x)$ and $\nabla h_2(x)$ linearly independent for x on and in a neighborhood of Σ. We will henceforth assume (as in [9]) that $h_{1,2}$ are $\mathcal{C}^k$ functions, with $k \geq 2$.

We have four different regions R_1, R_2, R_3 and R_4 with the four different vector fields f_i, $i = 1, \ldots, 4$, in these regions:

$$\dot{x} = f_i(x)\ ,\ \ x \in R_i\ ,\ \ i = 1, \ldots, 4\ . \tag{5}$$

Without loss of generality, we can label the regions as follows:

$$\begin{aligned} &R_1 :\ f_1 \quad \text{when} \quad h_1 < 0\ ,\ h_2 < 0\ , \qquad && R_2 :\ f_2 \quad \text{when} \quad h_1 < 0\ ,\ h_2 > 0\ , \\ &R_3 :\ f_3 \quad \text{when} \quad h_1 > 0\ ,\ h_2 < 0\ , \qquad && R_4 :\ f_4 \quad \text{when} \quad h_1 > 0\ ,\ h_2 > 0\ . \end{aligned} \tag{6}$$

When x is on $\Sigma_{1,2}$ or on their intersection Σ, the vector field is not properly defined. Following Filippov, we rewrite the problem as the set valued differential inclusion

$$\begin{aligned} \dot{x} \in & \frac{1-\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2} f_1(x) + \frac{1-\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2} f_2(x) \\ & + \frac{1+\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2} f_3(x) + \frac{1+\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2} f_4(x)\ , \end{aligned} \tag{7}$$

where $\sigma(\cdot)$ is the set valued sign function: $\sigma(y) = \begin{cases} 1 & y > 0 \\ [-1, 1] & y = 0\ . \\ -1 & y < 0 \end{cases}$ Naturally, when $x \in R_i$, for some $i = 1, \ldots, 4$, we simply have $\dot{x} = f_i$; also, when $x \in \Sigma_{1,2}, x \notin \Sigma$, the rewriting (7) is equivalent to the standard Filippov convexification of Example 1.2.

1.2.1. *A vector field on* Σ. For $x \in \Sigma$, in [8] the authors considered a special convex combination of the type above as a way to define a unique vector field in Σ. Their choice was to select

$$\dot{x} = (1-\alpha)(1-\beta) f_1(x) + (1-\alpha)\beta f_2(x) + \alpha(1-\beta) f_3(x) + \alpha\beta f_4(x) \tag{8}$$

where the functions α, β take values in $[0, 1]$ and must be found so that $\nabla h_1^T(x)\dot{x} = \nabla h_2^T(x)\dot{x} = 0$.

It was further shown that α and β are unique (and smooth in x) whenever Σ satisfies some attractivity conditions, in particular when Σ is nodally attractive. By letting

$$\begin{aligned} w_1^1 = \nabla h_1^T f_1\ ,\ w_2^1 = \nabla h_1^T f_2\ ,\ w_3^1 = \nabla h_1^T f_3\ ,\ w_4^1 = \nabla h_1^T f_4\ , \\ w_1^2 = \nabla h_2^T f_1\ ,\ w_2^2 = \nabla h_2^T f_2\ ,\ w_3^2 = \nabla h_2^T f_3\ ,\ w_4^2 = \nabla h_2^T f_4\ , \end{aligned}$$

the system to be solved for α and β is simply

$$
\begin{aligned}
(1-\alpha)\big[(1-\beta)w_1^1+\beta w_2^1\big]+\alpha\big[(1-\beta)w_3^1+\beta w_4^1\big] &= 0\\
(1-\alpha)\big[(1-\beta)w_1^2+\beta w_2^2\big]+\alpha\big[(1-\beta)w_3^2+\beta w_4^2\big] &= 0\ ,
\end{aligned}
\tag{9}
$$

and nodal attractivity is characterized by the constraints on the sign of w^1 and w^2 expressed in Table 2, which are assumed to be valid on Σ and near it. It is implicitly understood that the writing, say, $w_1^1 > 0$ means $w_1^1 \geq c > 0$, etc., in a neighborhood of Σ, where c is a positive constant. We will henceforth assume that the sign conditions of which in Table 2 hold in a neighborhood of Σ.

TABLE 2. Nodal Attractivity.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1, $i=1:4$	>0	>0	<0	<0
w_i^2, $i=1:4$	>0	<0	>0	<0

For later use, let us recall that the proof that (α,β) in (9) exist, unique, and depend smoothly on the state variables, rest on the following direct solution process for (9) in two steps:

1. From the first equation in (9), let

$$
\begin{aligned}
&\alpha = \frac{L_1(\beta)}{L_1(\beta)-L_2(\beta)}\ , \quad \text{where}\\
&L_1(\beta) = (1-\beta)w_1^1+\beta w_2^1\ , \quad L_2(\beta) = (1-\beta)w_3^1+\beta w_4^2\ ,
\end{aligned}
\tag{10}
$$

   and notice that (see Table 2) $L_1>0,\ L_2<0$, for $\beta\in[0,1]$.
2. Solve for β the quadratic equation resulting from substitution of α from (10) in the second equation of (9):

$$
\begin{aligned}
&P(\beta) \equiv -L_2(\beta)N_1(\beta)+L_1(\beta)N_2(\beta) = 0\ , \quad \text{where}\\
&N_1(\beta) = (1-\beta)w_1^2+\beta w_2^2\ , \quad N_2(\beta) = (1-\beta)w_3^2+\beta w_4^2\ .
\end{aligned}
\tag{11}
$$

   As in [8], $P(0)>0$ and $P(1)<0$, so that there is a unique solution $\beta^*\in[0,1]$, and thus from (10) a unique pair (α^*,β^*) in the square $[0,1]^2$ solution of (9). Further, since P is a parabola, we necessarily have $P'(\beta^*)\neq 0$; although we do not know its sign, nor magnitude, $P'(\beta^*)$ is surely bounded.

We are now ready for the following Lemma, which will be needed below. This result was not explicitly proved in [8], though it was effectively implied in that work.

Lemma 1.5. *Let $F(\alpha,\beta)=0$ be the nonlinear system (9), and let (α^*,β^*) be its unique solution in $[0,1]^2\equiv[0,1]\times[0,1]$, found as above. Then, the Jacobian DF evaluated at this root is nonsingular.*

Proof. The Jacobian is given by the matrix $J = \begin{bmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{bmatrix}$, and $\det(J) = \partial_\alpha F_1 \partial_\beta F_2 - \partial_\alpha F_2 \partial_\beta F_1$, which needs to be evaluated at the root (α^*, β^*).

Now, from (10)-(11), we have that the polynomial $P(\beta)$ can be also written as

$$P(\beta) \equiv (L_1 - L_2) F_2 \left(\frac{L_1}{L_1 - L_2}, \beta \right)$$

and therefore

$$P'(\beta) = (L_1 - L_2)\big[\partial_\alpha F_2 \frac{d}{d\beta} \frac{L_1}{L_1 - L_2} + \partial_\beta F_2\big] + F_2 \partial_\beta (L_1 - L_2) \ ,$$

which is not zero when evaluated at β^*. Now, the second term in this expression is 0 when evaluated at β^*. Moreover, a simple algebraic computation shows that $\partial_\alpha F_1 = -(L_1 - L_2)$ and that $\partial_\beta F_1 = -\frac{L_1 L_2' - L_1' L_2}{L_1 - L_2}$, with the "prime" indicating differentiation with respect to β. Therefore, we can rewrite

$$\det(J)_{\beta^*} = -P'(\beta^*) \ .$$

As a consequence, J is nonsingular at the root (α^*, β^*). □

Remark 1.6. *We emphasize that the solution of (9) depends on the w_j^i's (hence on x), but for any given collection of the w_j^i's (satisfying the signs of Table 2), the solution is unique and the Jacobian evaluated at the root is nonsingular.*

In a practical simulation, one may have to enter and exit repeatedly the discontinuity surface, and a numerical method based on monitoring when one reaches Σ, when one needs to leave it, and how one needs to modify the vector field in the different regimes, may be somewhat cumbersome if not downright impractical. An appealing alternative is to globally regularize (1) before hand, so that one has a unique (smooth, or at least continuous) differential system which can be integrated by one of a host of numerical methods for solving differential equations. Although a systematic comparison of different possibilities remains to be done, it is precisely this basic appeal which prompted us to perform the present study. Naturally, for these global regularizations to make sense, we will assume that the functions f_i, $i = 1, 2, 3, 4$, extend in a neighborhood of the discontinuity surfaces Σ_1, Σ_2 (hence of Σ).

2. A time regularization

Here we look at the generalization of the approach used in [10] for the case of a co-dimension 1 discontinuity surface. The basic idea is to consider a vector field given by the time average of vector fields evaluated along a solution trajectory sampled from a small time interval in the past.

It is convenient to rewrite (1) by using the characteristic functions associated to the region R_i's. That is, letting

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in R_i \\ 0 & \text{if } x \notin R_i \end{cases} \ , \quad i = 1, \ldots, m \ ,$$

then (1) can be rewritten as (recall that $m = 2^p$ for us)

$$\dot{x}(t) = \sum_{i=1}^{2^p} \chi_i\left(x(t)\right)\ f_i(x(t))\,. \tag{12}$$

We are interested in the case of a trajectory which reaches Σ. Since (12) is autonomous, we will let $t = 0$ be the time at which the solution reaches Σ, that is $x(0) \in \Sigma$. Before $t = 0$, the solution will be some function $\psi(t)$.

Generalizing the approach of [10], we consider the following regularization of (12)

$$\begin{cases} \dot{x}(t) & = & \dfrac{1}{\varepsilon}\displaystyle\int_{t-\varepsilon}^{t} \sum_{i=1}^{2^p} \chi_i\left(x(s)\right)\ f_i\left(x(s)\right)\, ds\ ,\ t > 0 \\ x(t) & = & \psi(t), \qquad t \in [-\varepsilon, 0], \end{cases} \tag{13}$$

where we assume that we cannot have $h_j\left(\psi(t)\right) = 0$ for all $j = 1, \ldots, p$, and all $t \in [-\varepsilon, 0]$. We will indicate with x^ε the solution of (13).

Remark 2.1. *Note that –as long as the trajectory is sampled in some of the regions R_i's, and there are no accumulation points on Σ for the solution of* (13)*– the vector field is in fact a time average of different vector fields around Σ.*

Example 2.2. Consider the case of a discontinuity surface of co-dimension 1 (the case considered in [10]). Using the same notation of Example 1.2, we have $\Sigma := \{x \in \mathbb{R}^n :\ h(x) = 0\}$, separating (locally) $\mathbb{R}^n$ into the two regions R_1 and R_2. In this case, the proposed regularization (13) reads

$$\begin{cases} \dot{x}(t) & = & \dfrac{1}{\varepsilon}\displaystyle\int_{t-\varepsilon}^{t} \Big[H\Big(-h(x(s))\Big) f_1\left(x(s)\right) + H\Big(h(x(s))\Big) f_2\left(x(s)\right)\Big] ds, \\ & & t > 0, \\ x(t) & = & \psi(t), \qquad t \in [-\varepsilon, 0], \end{cases} \tag{14}$$

where we have made use of the more compact notation afforded by the Heaviside function: $H(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$.

The analysis provided in [10] shows that under standard assumptions and for ε sufficiently small, the problem (14) has a C^1 solution x^ε and the sequence of such solutions converges (in the C^0-topology) as $\varepsilon \to 0$ to a C^1-function which identifies with the classical Filippov solution. To be precise, relatively to (14), the following theorem holds.

Theorem 2.3 ([10]). *Let f_1, f_2, h and ψ be as above. Then there exist $\varepsilon_0 > 0$, $T > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$ problem* (14) *has a C^1-solution $x^\varepsilon : [0,T] \to \mathbb{R}^n$. Moreover:*

(1) $|h(x^\varepsilon(t))| \leq C\varepsilon \ , \forall t \in [0,T]$.

There is a C^1 function x^0 such that

(2) $h(x^0(t)) = 0 \ , \forall t \in [0,T]$;
(3) $\lim_{\varepsilon \to 0} \|x^\varepsilon - x^0\|_{C^0[0,T]} = 0$;
(4) x^0 *solves the differential system* (4). □

2.1. **The case of a co-dimension 2 discontinuity surface.** In the case of a co-dimension 2 manifold separating regions of continuity of the vector field, the problem can also be regularized using an approach similar to the previous co-dimension 1 case.

The problem at hand is that given in (5)-(6), and we consider the regularization (13) with $p = 2$, where of course the vector fields satisfy the assumptions of Table 2.

Next, we show by a simple example (suggested in [8] and also considered in [10]) that the results in this case are substantially different from the co-dimension 1 case.

In fact, now there does not exist in general a unique limit function associated to the sequence $\{x^\varepsilon\}$ of solutions of (13).

Example 2.4. Let $h_1(x) = x_1$ and $h_2(x) = x_2$, so that $\Sigma = \{x \in \mathbb{R}^3 \ : \ x_1 = x_2 = 0\}$. With $a, b, c > 0$, let

$$f_1(x) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad f_3(x) = \begin{pmatrix} -a \\ b \\ -c \end{pmatrix}, \quad f_4(x) = \begin{pmatrix} -a \\ -b \\ c \end{pmatrix}.$$

It is simple to see that the Filippov convexification technique (2-3) gives the following family of vector fields on Σ

$$\dot{x} \in \begin{bmatrix} 0 \\ 0 \\ (4\lambda - 1)c \end{bmatrix}, \quad 0 \leq \lambda \leq 1/2 \ . \tag{15}$$

Next, let us analyze the behavior of (13) for this problem.

Analysis of Example 2.4 with $a = b = c = 1$.

Given the simplicity of the problem, it is easy to determine that (13) is equivalent to the following regularized equations

$$
\begin{aligned}
\dot{x}_i(t) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \Big(2H\big(x_i(s)\big)-1\Big)\, ds\ , \quad i=1,2, \\
\dot{x}_3(t) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \Big(2H\big(x_1(s)\big)-1\Big)\ \Big(2H\big(x_2(s)\big)-1\Big)\, ds\ . \qquad (16)
\end{aligned}
$$

The first two equations are decoupled so that we can make use of the results available for the co-dimension 1 case (see [10]).

Take an initial condition $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ with $\bar{x}_1 \neq 0$ and $\bar{x}_2 \neq 0$, and consider first the equations for x_i, x_i, $i = 1, 2$. No matter whether $\bar{x}_1$ and $\bar{x}_2$ are positive or negative, there will be a first time when x_i^{ε} $(i = 1, 2)$ reaches the value 0. Let $t = t_i$ be the time instants when $x_i^{\varepsilon}(t) = 0$, $i = 1, 2$.

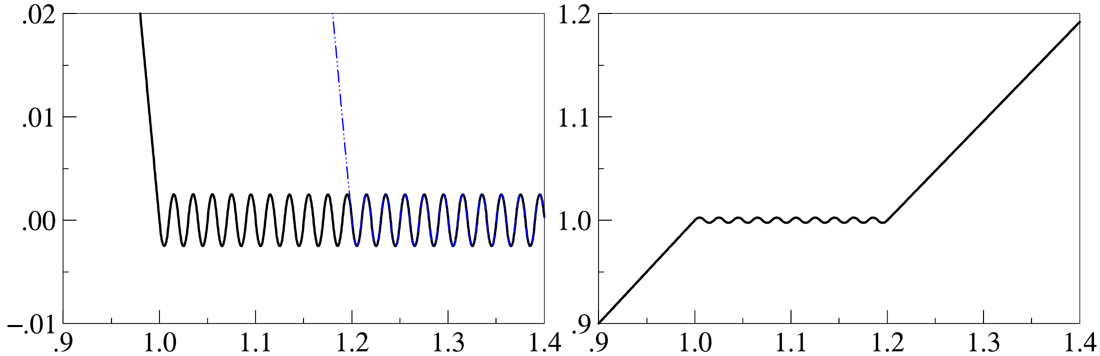


Figure 1. On the left is the graph of $x_1^{\varepsilon}(t)$ and $x_2^{\varepsilon}(t)$ (blue) for an initial value $\bar{x}_1 = 1.0, \bar{x}_2 = 1.2$, $\varepsilon = 10^{-2}$; the abscissas represent time. Note that the wrinkles x_1^{ε} and x_2^{ε} are perfectly in phase so that x_3^{ε} (right picture) is linear in the sliding regime with slope 1.

According to the analysis given in [10], we obtain that the solution x_i^{ε}, for $t \geq t_i$, is 2ε-periodic and $C^1([0, +\infty))$. In the first period $[t_i, t_i + 2\varepsilon]$, for $i = 1, 2$, it is given by

$$
x_i^{\varepsilon}(t) = \begin{cases} -\dfrac{1}{\varepsilon}(t-t_i)(t-t_i-\varepsilon) & \text{for} \quad t_i \leq t \leq t_i+\varepsilon \\ \dfrac{1}{\varepsilon}(t-t_i-\varepsilon)(t-t_i-2\varepsilon) & \text{for} \quad t_i+\varepsilon \leq t \leq t_i+2\varepsilon. \end{cases}
$$

Observe that t_1, respectively t_2, depend continuously on $\bar{x}_1$, respectively $\bar{x}_2$, and on ε. Since the equations for x_1 and x_2 are decoupled, without loss of generality we

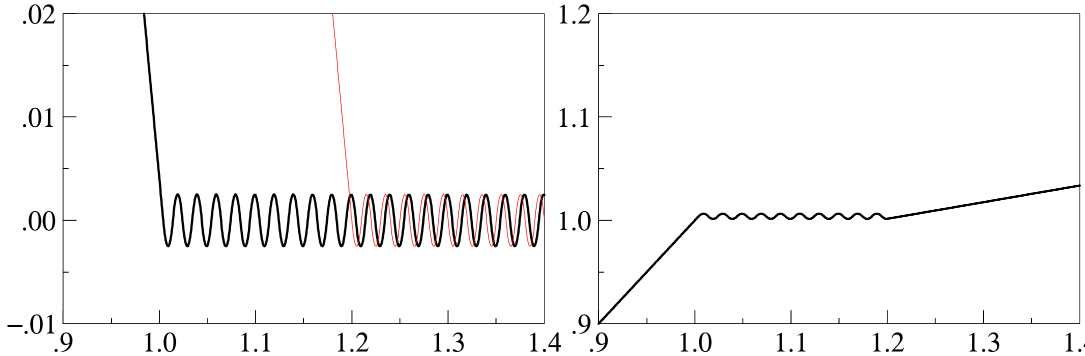


FIGURE 2. On the left is the graph of $x_1^\varepsilon(t)$ and $x_2^\varepsilon(t)$ (red) for an initial value $\bar{x}_1 = 1.004, \bar{x}_2 = 1.2$, $\varepsilon = 10^{-2}$; the abscissas represent time. Now, clearly the wrinkles are not in phase and as a result the slope of x_3^ε (right picture) in the sliding phase is 0.1608.

can assume that $t_2 \geq t_1$ and we can choose $\bar{x}_1$ and $\bar{x}_2$ so that $t_2 = t_1 + 2\,r\,\varepsilon + 2\,\delta\,\varepsilon$ where r is a nonnegative integer and $\delta \in [0, 2]$.

Consider now the equation (16), and observe that the right-hand side only depends on the sign of x_1^ε and x_2^ε. By routine algebraic manipulations we get from (16)

$$\dot{x}_3(t) \;=\; 2\,|\delta - 1| - 1. \tag{17}$$

This implies

$$x_3^\varepsilon(t) = x_3^\varepsilon(t_2) + (2\,|\delta - 1| - 1)\,(t - t_2)\ , \qquad \text{for } t \geq t_2\ ,$$

where $\delta = \delta(\bar{x}_1, \bar{x}_2, \varepsilon)$ is a continuous function of its arguments. This means that simply changing the initial values $\bar{x}_1, \bar{x}_2$ or ε we are able to change the solution $x_3(t)$ and to generate indeed a bunch of solutions. Since this feature persists in the limit $\varepsilon \to 0$ we have a phenomenon of fattening, that is in the singular limit there appear a whole set of weak solutions. By comparison of (15) and (17), we observe that this set of weak solutions coincides with the set of Filippov solutions for this particular problem. □

3. Space regularization

Here we consider replacing the problem (7) by regularizing the sign-function. Namely, given (a small value of) $\varepsilon \geq \varepsilon_0 > 0$, we consider the differential equation

$$\begin{aligned}\dot{x} = {} & \frac{1-\sigma_\varepsilon(h_1(x))}{2}\frac{1-\sigma_\varepsilon(h_2(x))}{2}f_1(x) + \frac{1-\sigma_\varepsilon(h_1(x))}{2}\frac{1+\sigma_\varepsilon(h_2(x))}{2}f_2(x) \\ & + \frac{1+\sigma_\varepsilon(h_1(x))}{2}\frac{1-\sigma_\varepsilon(h_2(x))}{2}f_3(x) + \frac{1+\sigma_\varepsilon(h_1(x))}{2}\frac{1+\sigma_\varepsilon(h_2(x))}{2}f_4(x)\ ,\end{aligned} \tag{18}$$

where the smoothed sign function σ_ε is defined as follows:

$$\sigma_\varepsilon(y) = \begin{cases} 1 & y > \varepsilon \\ g(y) & y \in [-\varepsilon, \varepsilon] \ , \\ -1 & y < -\varepsilon \end{cases} \tag{19}$$

and the function g will be required to satisfy the following conditions:

(a) *Smooth and Increasing*: g must be at least $\mathcal{C}^1$, and monotone increasing (hence $g'(y) > 0$) for $y \in (-\varepsilon, \varepsilon)$;
(b) *Interpolate*: g must satisfy the interpolatory conditions $g(-\varepsilon) = -1, g(\varepsilon) = 1$;
(c) *Odd*: We will also select g as an odd function, $g(-y) = -g(y)$.

Remark 3.1. *Of course, there are many choices for the function g, and one could select g so to make σ_ε become as smooth as desired. Still, in our experiments (and in section 3.2 as well), we used the simple linear function*

$$g(y) = y/\varepsilon \ , \quad y \in [-\varepsilon, \varepsilon] \ . \tag{20}$$

However, other choices are certainly possible. For example, adopting the choice $g(y) = \frac{y}{2\varepsilon}\big(3 - (y/\varepsilon)^2\big)$, $y \in [-\varepsilon, \varepsilon]$, will give a globally $\mathcal{C}^1$ function for all y and require just minimal modifications to the arguments of section 3.2.

We are interested in studying the dynamics of the regularized system (18) when x is in a neighborhood Σ_ε of the co-dimension 2 discontinuity surface Σ:

$$\Sigma_\varepsilon = \{x \in \mathbb{R}^n \ : \ |h_1(x)| \le \varepsilon \ , \ |h_2(x)| \le \varepsilon \ \} \ ,$$

so that for $x \in \Sigma_\varepsilon$ (18) rewrites as

$$\begin{aligned} \dot{x} =& \frac{1 - g(h_1(x))}{2} \frac{1 - g(h_2(x))}{2} f_1(x) + \frac{1 - g(h_1(x))}{2} \frac{1 + g(h_2(x))}{2} f_2(x) \\ &+ \frac{1 + g(h_1(x))}{2} \frac{1 - g(h_2(x))}{2} f_3(x) + \frac{1 + g(h_1(x))}{2} \frac{1 + g(h_2(x))}{2} f_4(x) \ . \end{aligned} \tag{21}$$

We will further assume that Σ is nodally attractive and that the signs of which in Table 2 hold (uniformly) in Σ_ε.

Now, since ∇h_1 and ∇h_2 are smooth and linearly independent in Σ_ε, we can locally change coordinates[1]: $x \to y$, so that $y_1 = h_1(x)$, $y_2 = h_2(x)$. So doing, Σ_ε rewrites simply as the region

$$\Sigma_\varepsilon = \{y \in \mathbb{R}^n \ : \ y_1 \in [-\varepsilon, \varepsilon] \ , \ y_2 \in [-\varepsilon, \varepsilon]\} \ .$$

[1]We are using the fact that a basis for the tangent plane to Σ can be chosen so to vary smoothly in $x \in \Sigma$. This is a consequence of the fact that the matrix valued function $\big[\nabla h_1(x) \quad \nabla h_2(x)\big]$ is smooth and full rank for all x.

Further, since $\dot{y}_1 = \nabla h_1(x)^T \dot{x}_1$ and $\dot{y}_2 = \nabla h_2(x)^T \dot{x}_2$, the differential inclusion (7) rewrites as

$$\begin{aligned}\dot{y} \in & \frac{1-\sigma(y_1)}{2}\frac{1-\sigma(y_2)}{2}w_1(y) + \frac{1-\sigma(y_1)}{2}\frac{1+\sigma(y_2)}{2}w_2(y) \\ & + \frac{1+\sigma(y_1)}{2}\frac{1-\sigma(y_2)}{2}w_3(y) + \frac{1+\sigma(y_1)}{2}\frac{1+\sigma(y_2)}{2}w_4(y) \ ,\end{aligned}$$

and equation (21) rewrites as

$$\begin{aligned}\dot{y} = & \frac{1-g(y_1)}{2}\frac{1-g(y_2)}{2}w_1(y) + \frac{1-g(y_1)}{2}\frac{1+g(y_2)}{2}w_2(y) \\ & + \frac{1+g(y_1)}{2}\frac{1-g(y_2)}{2}w_3(y) + \frac{1+g(y_1)}{2}\frac{1+g(y_2)}{2}w_4(y) \ , \quad y \in \Sigma_\varepsilon \ ,\end{aligned} \tag{22}$$

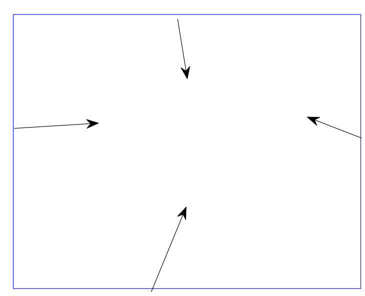
where the first two components of the n-dimensional vector valued functions w_j, $j = 1, \dots, 4$, are the same as w_j^i, $i = 1, 2$, of which in Table 2 (formally, at least, since they may be evaluated at different points).

Remark 3.2. *For later reference, we note that the same change of variable on the system (8) gives*

$$\dot{y} = (1-\alpha)(1-\beta)w_1(y) + (1-\alpha)\beta w_2(y) + \alpha(1-\beta)w_3(y) + \alpha\beta w_4(y) \ , \tag{23}$$

where now it must be understood that this expression is legitimate just for $y \in \Sigma$, that is when $y_1 = y_2 = 0$.

Next, let y be on the boundary of Σ_ε. By virtue of the signs of the w_j^i, $j = 1, \dots, 4$, and $i = 1, 2$, the flow with respect to y_1 and y_2 points inward, and thus the regions Σ_ε are positively invariant sets for the flow of (22) for any $\varepsilon \geq \varepsilon_0 > 0$; naturally, the regions Σ_ε do depend on ε. See the figure on the right. The situation is analogous to having an asymptotically stable node in a planar system, hence the name of *nodal attractivity.*



3.1. **Piecewise constant vector fields.** Next, we consider the simpler –but practically important– case in which the vectors w_j^i, $i = 1, 2$, $j = 1, \dots, 4$, are constant.

Example 3.3. A typical situation when the case under scrutiny presents itself is when in the original problem (5) the vector fields f_i, $i = 1, \dots, 4$, are constant, and the surfaces Σ_1 and Σ_2 are (hyper-)planes. In this case, the gradients $\nabla h_{1,2}$ are constant, and as a consequence so are the vectors w_j. This seemingly simple situation occurs in many problems of practical relevance; e.g., see Example 1.1, as well as [4, 5, 11]. □

In this case, we now show that not only Σ_ε is invariant, but in fact the flow of (22) goes to an equilibrium point relatively to (y_1, y_2), for any positive ε. Notation is the usual one.

Theorem 3.4. *Let $\varepsilon \geq \varepsilon_0 > 0$, and let w_j^i, $i = 1, 2$, $j = 1, \dots, 4$, in (22) be constant in Σ_ε, and satisfy the sign constraints of Table 2. Then, the first two components of the flow of (22) approach an equilibrium point.*

Proof. Under the stated assumptions, the differential system (22) decouples in two systems, one satisfied by y_1, y_2, the other satisfied by $y_3, \dots, y_n$. That is, we have

$$\begin{aligned}\begin{bmatrix}\dot y_1\\ \dot y_2\end{bmatrix} =& \frac{1-g(y_1)}{2}\frac{1-g(y_2)}{2}\begin{bmatrix}w_1^1\\ w_1^2\end{bmatrix} + \frac{1-g(y_1)}{2}\frac{1+g(y_2)}{2}\begin{bmatrix}w_2^1\\ w_2^2\end{bmatrix}\\ &+\frac{1+g(y_1)}{2}\frac{1-g(y_2)}{2}\begin{bmatrix}w_3^1\\ w_3^2\end{bmatrix} + \frac{1+g(y_1)}{2}\frac{1+g(y_2)}{2}\begin{bmatrix}w_4^1\\ w_4^2\end{bmatrix}\ ,\end{aligned} \tag{24}$$

and

$$\begin{aligned}\begin{bmatrix}\dot y_3\\ \vdots\\ \dot y_n\end{bmatrix} =& \frac{1-g(y_1))}{2}\frac{1-g(y_2)}{2}\begin{bmatrix}w_1^3\\ \vdots\\ w_1^n\end{bmatrix} + \frac{1-g(y_1))}{2}\frac{1+g(y_2)}{2}\begin{bmatrix}w_2^3\\ \vdots\\ w_2^n\end{bmatrix}\\ &+\frac{1+g(y_1)}{2}\frac{1-g(y_2)}{2}\begin{bmatrix}w_3^3\\ \vdots\\ w_3^n\end{bmatrix} + \frac{1+g(y_1)}{2}\frac{1+g(y_2)}{2}\begin{bmatrix}w_4^3\\ \vdots\\ w_4^n\end{bmatrix}\ .\end{aligned} \tag{25}$$

In the case under examination here, we notice that (unlike w_j^i, $i = 1, 2$, $j = 1, \dots, 4$) the components w_j^i, $i = 3, \dots, n$, $j = 1, \dots, 4$, are not necessarily constant.

Now, the flow of (24) points inside the region Σ_ε and thus there must be at least one equilibrium point in Σ_ε. To find the equilibria, we let $\alpha_\varepsilon = (1 + g(y_1))/2$, $\beta_\varepsilon = (1 + g(y_2))/2$, and need to solve the system

$$0 = (1-\alpha_\varepsilon)[(1-\beta_\varepsilon)\begin{bmatrix}w_1^1\\ w_1^2\end{bmatrix} + \beta_\varepsilon\begin{bmatrix}w_2^1\\ w_2^2\end{bmatrix}] + \alpha_\varepsilon[(1-\beta_\varepsilon)\begin{bmatrix}w_3^1\\ w_3^2\end{bmatrix} + \beta_\varepsilon\begin{bmatrix}w_4^1\\ w_4^2\end{bmatrix}]\ . \tag{26}$$

But, we know (see (10)-(11)) that this algebraic system has only one solution $\alpha_\varepsilon, \beta_\varepsilon \in [0, 1]^2$. So, there is only one equilibrium of (24) in Σ_ε, which is uniquely found from the relations $y_1 = g^{-1}(2\alpha_\varepsilon - 1)$, $y_2 = g^{-1}(2\beta_\varepsilon - 1)$, since g is monotone.

Now, since the system (24) is 2-dimensional and autonomous, if we rule out the existence of a periodic orbit in Σ_ε, it will follow that the equilibrium is attracting, and the flow will approach that of the reduced system. We use Green's theorem to rule out periodic orbits (this is a standard argument; e.g., see [15]). For notational simplicity, rewrite the system (24) as

$$\begin{bmatrix}\dot y_1\\ \dot y_2\end{bmatrix} = \begin{bmatrix}F_1(y_1, y_2)\\ F_2(y_1, y_2)\end{bmatrix} \equiv F(y_1, y_2)\ .$$

Suppose that there is a periodic orbit in Σ_ε; hence, there is a closed loop in (y_1, y_2), call it Γ (which we can take it to be parametrized by t), enclosing a region R in Σ_ε. Green's theorem states that

$$\iint_R \mathrm{div} F dy_1 dy_2 \;=\; \oint_\Gamma (F \cdot n) dt$$

where n is the normal to Γ. But of course if Γ is a periodic orbit, then n is orthogonal to F (since the vector field is tangent to the trajectory). So, $\oint_\Gamma (F \cdot n) dt = 0$. On the other hand, explicit computation of $\mathrm{div} F$ gives

$$\begin{aligned} \mathrm{div} F = &- \frac{g'(y_1)}{2} \Big[\frac{1-g(y_2)}{2} w_1^1 + \frac{1+g(y_2)}{2} w_2^1 - \frac{1-g(y_2)}{2} w_3^1 - \frac{1+g(y_2)}{2} w_4^1 \Big] + \\ &- \frac{g'(y_2)}{2} \Big[\frac{1-g(y_1)}{2} w_1^2 - \frac{1-g(y_1)}{2} w_2^2 + \frac{1+g(y_1)}{2} w_3^2 - \frac{1+g(y_1)}{2} w_4^2 \Big] \ . \end{aligned}$$

Now, recalling that g is monotone increasing, that it takes values in $[-1, 1]$, and that w_j^i, $i = 1, 2$, $j = 1, \dots, 4$, satisfy the signs of which in Table 2, we immediately get that $\mathrm{div} F < 0$, and hence there cannot be a periodic orbit in Σ_ε. □

Next, under the assumptions of Theorem 3.4, we want to compare the reduced flow of (22) to that of (23). First, rewrite (25) with notation from Theorem 3.4 as

$$(27) \quad \begin{bmatrix} \dot y_3 \\ \vdots \\ \dot y_n \end{bmatrix} = (1-\alpha_\varepsilon)\Big[(1-\beta_\varepsilon) \begin{bmatrix} w_1^3 \\ \vdots \\ w_1^n \end{bmatrix} + \beta_\varepsilon \begin{bmatrix} w_2^3 \\ \vdots \\ w_2^n \end{bmatrix}\Big] + \alpha_\varepsilon \Big[(1-\beta_\varepsilon) \begin{bmatrix} w_3^3 \\ \vdots \\ w_3^n \end{bmatrix} + \beta_\varepsilon \begin{bmatrix} w_4^3 \\ \vdots \\ w_4^n \end{bmatrix}\Big] \ .$$

Next, recall that the system (23) is defined on Σ, and therefore $y_1 = y_2 = 0$ (for all $t \geq 0$). This means that also (23) rewrites in a manner similar to (26)-(27), namely:

$$(28) \qquad 0 = (1-\alpha)\Big[(1-\beta) \begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix} + \beta \begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix}\Big] + \alpha \Big[(1-\beta) \begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix} + \beta \begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix}\Big] \ ,$$

and

$$(29) \qquad \begin{bmatrix} \dot y_3 \\ \vdots \\ \dot y_n \end{bmatrix} = (1-\alpha)\Big[(1-\beta) \begin{bmatrix} w_1^3 \\ \vdots \\ w_1^n \end{bmatrix} + \beta \begin{bmatrix} w_2^3 \\ \vdots \\ w_2^n \end{bmatrix}\Big] + \alpha \Big[(1-\beta) \begin{bmatrix} w_3^3 \\ \vdots \\ w_3^n \end{bmatrix} + \beta \begin{bmatrix} w_4^3 \\ \vdots \\ w_4^n \end{bmatrix}\Big] \ ,$$

where $y \in \Sigma$. From (28) and (26), since w_j^i, $i = 1, 2$, $j = 1, \dots, 4$, are constant, we immediately have that $\alpha_\varepsilon = \alpha$ and $\beta_\varepsilon = \beta$. However, although formally (29) and (27) are the same vector fields, the functions w_j^i, $i = 3, \dots, n$, $j = 1, \dots, 4$, depend on y, and the arguments y to be used in (29) and (27) are not the same, since for (29) we must have $y_1 = y_2 = 0$, while for (27) we have y_1 and y_2 from solving $g(y_1) = 2\alpha - 1$, $g(y_2) = 2\beta - 1$. As a consequence, the trajectories described by (29) and (27) will not be the same, in general. Nevertheless, there is an important situation when they are the same.

Theorem 3.5. *Let $\varepsilon \geq \varepsilon_0 > 0$, and let w^i_j, $i = 1, 2, \dots, n$, $j = 1, \dots, 4$, in (22) be constant in Σ_ε, with the usual sign constraints of Table 2 satisfied. Let y_1^*, y_2^*, be the equilibrium point of (24), and consider the reduced systems (27) and (29). Then, their respective flows are identical.*

Proof. Under the stated assumptions, the differential systems (27) and (29) are the same, hence their solution trajectories are as well, for same initial conditions. □

Remark 3.6. *As a consequence of Theorem 3.5, and under its assumptions, let $v \in \mathbb{R}^{n-2}$ be a fixed vector, and suppose we take ICs for (24)-(25) given by $y(0) = \begin{bmatrix} y_1^* \\ y_2^* \\ v \end{bmatrix}$, while we take ICs for (23) given by $y(0) = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}$. Then, $y_3(t), \dots, y_n(t)$, are identical $\forall t \geq 0$. In other words, the reduced dynamics of (22) follows the same vector field as the dynamics of (23). Except for the first two components of the solution, which are usually at a different equilibrium, the remaining solution components are the same. To fix ideas, suppose that we have chosen the function g: $g(y) = y/\varepsilon$, $y \in [-\varepsilon, \varepsilon]$, and that we have solved the algebraic system (28) and obtained the unique pair $\alpha_\varepsilon, \beta_\varepsilon \in [0, 1]^2$. This will give us $y_1^* = \varepsilon(2\alpha_\varepsilon - 1) = \varepsilon(2\alpha - 1)$, $y_2^* = \varepsilon(2\beta_\varepsilon - 1) = \varepsilon(2\beta - 1)$.*

Example 3.7. Consider this simple example. Let the surfaces $\Sigma_{1,2} = \{x : \; h_{1,2}(x) = 0\}$ be given by $h_1(x) = x_1$ and $h_2(x) = x_2$. Further, take the following constant vector fields in the regions R_i, $i = 1, \dots, 4$:

$$f_1 = \begin{bmatrix} a \\ a \\ 1 \end{bmatrix} \;,\; f_2 = \begin{bmatrix} a \\ -b \\ -1 \end{bmatrix} \;,\; f_3 = \begin{bmatrix} -b \\ a \\ -1 \end{bmatrix} \;,\; f_4 = \begin{bmatrix} -b \\ -b \\ 1 \end{bmatrix} \;,\; a, b > 0 \;.$$

As regularizing function, we choose $g(y) = y/\varepsilon$. So, we obtain $\alpha = \alpha_\varepsilon = \beta = \beta_\varepsilon = \frac{a}{a+b}$, and from this $x_1^* = x_2^* = \varepsilon\frac{a-b}{a+b}$, while x_3 satisfies the differential equation $\dot{x}_3 = \left(\frac{b-a}{b+a}\right)^2$. Except when $a = b$, the equilibrium point for the regularized problem is not at the origin (that is, not on Σ). □

To conclude this section, we observe that the equilibrium for the reduced problem (24) is asymptotically stable (and reached exponentially fast). [This is a consequence of the fact that the function $V(y_1, y_2) = (y_1 - y_1^*)^2 + (y_2 - y_2^*)^2$ is a strict Lyapunov function.]

3.2. **General case.** Let us now consider the general case, when in the system (24) the vector valued functions w_j depend on the state variables y.

Let us still formally rewrite the system as in (24)-(25), where of course all the variables w^i_j's now depend on y. We note again that the dynamics of the system are restricted to live in Σ_ε, in particular $y_{1,2} \in [-\varepsilon, \varepsilon]$. Now, let $v \in \mathbb{R}^{n-2}$ be a vector of admissible values for the variables $y_3, \dots, y_n$, and consider the functions w^i_j

$(j = 1, \dots, 4; i = 1, 2)$ in (24) as functions of y_1 and y_2 only, for fixed v: $w_j^i(y_1, y_2, v)$. As a consequence of the restricted dynamical behavior of $y_{1,2}$, then there must be a steady state relatively to y_1 and y_2, inside Σ_ε.

Theorem 3.8. *Let the function $g(y)$ of (19) be of the form (20), $g(y) = y/\varepsilon$, $y \in [-\varepsilon, \varepsilon]$. Also, let the first partial derivatives of w_j^i $(j = 1, \dots, 4; i = 1, 2)$ with respect to y_1 and y_2 be uniformly bounded in $[-\varepsilon, \varepsilon]^2$.*

Then, the algebraic system

$$
\begin{aligned}
0 =& \frac{1-g(y_1)}{2}\Big[\frac{1-g(y_2)}{2} w_1^1(y_1, y_2, v) + \frac{1+g(y_2)}{2} w_2^1(y_1, y_2, v)\Big] \\
&+ \frac{1+g(y_1)}{2}\Big[\frac{1-g(y_2)}{2} w_3^1(y_1, y_2, v) + \frac{1+g(y_2)}{2} w_4^1(y_1, y_2, v)\Big] , \\
0 =& \frac{1-g(y_2)}{2}\Big[\frac{1-g(y_1)}{2} w_1^2(y_1, y_2, v) + \frac{1+g(y_1)}{2} w_3^2(y_1, y_2, v)\Big] \\
&+ \frac{1+g(y_2)}{2}\Big[\frac{1-g(y_1)}{2} w_2^2(y_1, y_2, v) + \frac{1+g(y_1)}{2} w_4^2(y_1, y_2, v)\Big]
\end{aligned}
\tag{30}
$$

has a unique solution (y_1, y_2) in $[-\varepsilon, \varepsilon]^2$, for ε sufficiently small. The solution of course depends on v.

Furthermore, the evolution of y_1 and y_2 according to (24) will reach this equilibrium configuration.

After verifying Theorem 3.8, by freeing v, then we will have a co-dimension 2 surface of equilibria, $S = \{y_1(v), y_2(v)\}$, parametrized by v. If we now let the vector v describe the evolution of the variables $y_3, \dots, y_n$, according to (25), we have the following very interesting conclusion (we use [15, Theorem 0.3.2]):

Corollary 3.9. *There is a surface S of equilibria for (24), parametrizable by $y_3, \dots, y_n$. As a consequence, the system is constrained to move on the co-dimension 2 surface $S_\varepsilon := \{(y_1, y_2) \in S\}$, rather than on Σ. The surface S_ε is $\mathcal{O}(\varepsilon)$ close to Σ.* □

Next, let us proceed to prove Theorem 3.8.

Proof of Theorem 3.8. Rewrite (30) by setting $\alpha_\varepsilon = (1 + g(y_1))/2$ and $\beta_\varepsilon = (1 + g(y_2))/2$, for all $(y_1, y_2) \in [-\varepsilon, \varepsilon]^2$. This way, we can formally rewrite (30) as

$$
\begin{aligned}
0 =& (1-\alpha_\varepsilon)\Big[(1-\beta_\varepsilon)\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix}(y_1, y_2, v) + \beta_\varepsilon \begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix}(y_1, y_2, v)\Big] \\
&+ \alpha_\varepsilon\Big[(1-\beta_\varepsilon)\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix}(y_1, y_2, v) + \beta_\varepsilon \begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix}(y_1, y_2, v)\Big] ,
\end{aligned}
\tag{31}
$$

and, on this system, we consider the iteration for $k=0,1,\dots$:

$$
\begin{aligned}
0 =&(1-\alpha_\varepsilon^{(k+1)})\big[(1-\beta_\varepsilon^{(k+1)})\begin{bmatrix} w_1^1 \\ w_1^2\end{bmatrix}(y_1^{(k)},y_2^{(k)},v)+\beta_\varepsilon^{(k+1)}\begin{bmatrix} w_2^1 \\ w_2^2\end{bmatrix}(y_1^{(k)},y_2^{(k)},v)\big] \\
&+\alpha_\varepsilon^{(k+1)}\big[(1-\beta_\varepsilon^{(k+1)})\begin{bmatrix} w_3^1 \\ w_3^2\end{bmatrix}(y_1^{(k)},y_2^{(k)},v)+\beta_\varepsilon^{(k+1)}\begin{bmatrix} w_4^1 \\ w_4^2\end{bmatrix}(y_1^{(k)},y_2^{(k)},v)\big]\ ,
\end{aligned}
\tag{32}
$$

where (y_1^0,y_2^0) is a given value in $[-\varepsilon,\varepsilon]^2$.

Now, each iterate of (32) gives a unique well defined value $(\alpha_\varepsilon^{(k+1)},\beta_\varepsilon^{(k+1)})$ in $[0,1]^2$, which in general will depend on $y^{(k)}$ through the dependence of the w_j^i's on $y^{(k)}$. In fact, since the $w_j^i(y_1^{(k)},y_2^{(k)},v),\ i=1,2,\ j=1,\dots,4$, satisfy the nodal attractivity conditions of which in Table 2, then –arguing as in (10-11)– we can infer uniqueness of $\alpha_\varepsilon^{(k+1)}\in[0,1]$ and $\beta_\varepsilon^{(k+1)}\in[0,1]$ for each given $(y_1^{(k)},y_2^{(k)})$. Further, we notice that from Lemma 1.5 the Jacobian of the system (32) at this root $(\alpha_\varepsilon^{(k+1)},\beta_\varepsilon^{(k+1)})$ is nonsingular.

Correspondingly, we define the unique value (next iterate) $(y_1^{(k+1)},y_2^{(k+1)})$ in $[-\varepsilon,\varepsilon]^2$ by exploiting the monotonicity of the function g:

$$
\begin{bmatrix} y_1^{(k+1)} \\ y_2^{(k+1)}\end{bmatrix}=\begin{bmatrix} g^{-1}(\alpha_\varepsilon^{(k+1)}(y^{(k)})) \\ g^{-1}(\beta_\varepsilon^{(k+1)}(y^{(k)}))\end{bmatrix}\ ,
\tag{33}
$$

where we have highlighted that the values of $\alpha_\varepsilon^{(k+1)}$ and $\beta_\varepsilon^{(k+1)}$ depend on $y^{(k)}$ (since the w_j^i's do). In short, we have the iteration process

$$
\begin{bmatrix} y_1^{(k)} \\ y_2^{(k)}\end{bmatrix}\to\begin{bmatrix} y_1^{(k+1)} \\ y_2^{(k+1)}\end{bmatrix}\ ,\quad k=0,1,2,\dots,
$$

defined by the combination: Solve (32) for $(\alpha_\varepsilon^{(k+1)},\beta_\varepsilon^{(k+1)})$, and use (33) to define $(y_1^{(k+1)},y_2^{(k+1)})$. Since this fixed point iteration maps $[-\varepsilon,\varepsilon]^2$ into itself, then it has a fixed point.

We want to show that the fixed point is unique and that the iteration converges to it for any given (fixed) v, and any initial condition (y_1^0,y_2^0). A sufficient condition for convergence, and for uniqueness of the fixed point, is of course contractivity of the iteration map. In particular, if the derivative of the map is less than 1 in norm, contractivity will follow.

In the specific case of g we are considering, we have

$$
\begin{bmatrix} y_1^{(k+1)} \\ y_2^{(k+1)}\end{bmatrix}=\varepsilon\begin{bmatrix} 2\alpha_\varepsilon^{(k+1)}(y^{(k)})-1 \\ 2\beta_\varepsilon^{(k+1)}(y^{(k)})-1\end{bmatrix}\ .
$$

As a consequence, contractivity of the iteration map will follow (for ε sufficiently small) as long as the first partial derivatives of $\alpha_\varepsilon^{(k+1)}(y^{(k)})$ and $\beta_\varepsilon^{(k+1)}(y^{(k)})$ (viewed as functions of $y^{(k)}$) are bounded (uniformly in ε). In what follows, for simplicity, we will avoid writing the superscript $(k+1)$.

Below, we show boundedness for $\partial_{y_1}\alpha_\varepsilon$ and $\partial_{y_1}\beta_\varepsilon$; the case of $\partial_{y_2}\alpha_\varepsilon$ and $\partial_{y_2}\beta_\varepsilon$ is much the same.

To obtain expressions for $\partial_{y_1}\alpha_\varepsilon$ and $\partial_{y_1}\beta_\varepsilon$, we differentiate with respect to y_1 the system satisfied by $\alpha_\varepsilon, \beta_\varepsilon$ (see (32)):

$$(1-\alpha_\varepsilon)\big[(1-\beta_\varepsilon)\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix} + \beta_\varepsilon \begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix}\big] + \alpha_\varepsilon\big[(1-\beta_\varepsilon)\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix} + \beta_\varepsilon \begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix}\big] = 0\,. \tag{34}$$

This gives

$$\begin{aligned}
&-\frac{\partial \alpha_\varepsilon}{\partial y_1}\big[(1-\beta_\varepsilon)w_1 + \beta_\varepsilon w_2\big] + \frac{\partial \alpha_\varepsilon}{\partial y_1}\big[(1-\beta_\varepsilon)w_3 + \beta_\varepsilon w_4\big] \\
&-\frac{\partial \beta_\varepsilon}{\partial y_1}\big[(1-\alpha_\varepsilon)w_1 + \alpha_\varepsilon w_3\big] + \frac{\partial \beta_\varepsilon}{\partial y_1}\big[(1-\alpha_\varepsilon)w_2 + \beta_\varepsilon w_4\big] \\
&+(1-\alpha_\varepsilon)\big[(1-\beta_\varepsilon)\frac{\partial w_1}{\partial y_1} + \beta_\varepsilon \frac{\partial w_2}{\partial y_1}\big] + \alpha_\varepsilon\big[(1-\beta_\varepsilon)\frac{\partial w_3}{\partial y_1} + \beta_\varepsilon \frac{\partial w_4}{\partial y_1}\big] = 0\,,
\end{aligned}$$

and this can be rewritten as the following linear system for $\begin{bmatrix} \partial_{y_1}\alpha_\varepsilon \\ \partial_{y_1}\beta_\varepsilon \end{bmatrix}$:

$$M \begin{bmatrix} \partial_{y_1}\alpha_\varepsilon \\ \partial_{y_1}\beta_\varepsilon \end{bmatrix} = b\,,$$

where in b we have put all the terms containing the partial derivatives of the w_i's.

The key observation now is to notice that M is exactly the Jacobian of the system (34) (that is, of (32)) at its unique root in $[0,1]^2$. But then, by virtue of Lemma 1.5, the matrix M is invertible and thus $\begin{bmatrix} \partial_{y_1}\alpha_\varepsilon \\ \partial_{y_1}\beta_\varepsilon \end{bmatrix}$ exists unique and it is bounded (uniformly in ε), and thus the iteration map is contracting uniformly in ε.

Next, we need to argue that the evolution of y_1 and y_2 according to (24) reaches the equilibrium configuration. This is guaranteed if the reduced system (24) for fixed v, that is with $w_j^i(y_1, y_2, v)$ (for $i = 1, 2$, $j = 1, \dots, 4$), does not admit periodic orbits. To rule out periodic orbits, we resort again to Green's theorem. The argument is similar to the one we used for the case of constant w_j^i's, except that now the smallness of ε plays a key role. Rewrite (24) in the present case:

$$\begin{aligned}
\begin{bmatrix} \dot y_1 \\ \dot y_2 \end{bmatrix} =& \frac{1-g(y_1)}{2}\Big[\frac{1-g(y_2)}{2}\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix}(y_1,y_2,v) + \frac{1+g(y_2)}{2}\begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix}(y_1,y_2,v)\Big] \\
&+\frac{1+g(y_1)}{2}\Big[\frac{1-g(y_2)}{2}\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix}(y_1,y_2,v) + \frac{1+g(y_2)}{2}\begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix}(y_1,y_2,v)\Big]\,,
\end{aligned}$$

and compute div$F$, F being the right-hand side of this system above. Writing g_i for $g(y_i)$, $i = 1, 2$, and using a similar notation for the derivatives, we get:

$$\begin{aligned}\mathrm{div}F = & -\frac{g_1'}{2}\Big[\frac{1-g_2}{2}w_1^1 + \frac{1+g_2}{2}w_2^1 - \frac{1-g_2}{2}w_3^1 - \frac{1+g_2}{2}w_4^1\Big] + \\ & -\frac{g_2'}{2}\Big[\frac{1-g_1}{2}w_1^2 - \frac{1-g_1}{2}w_2^2 + \frac{1+g_1}{2}w_3^2 - \frac{1+g_1}{2}w_4^2\Big] \\ & + \frac{1-g_1}{2}\Big[\frac{1-g_2}{2}(\partial_{y_1}w_1^1 + \partial_{y_2}w_1^2) + \frac{1+g_2}{2}(\partial_{y_1}w_2^1 + \partial_{y_2}w_2^2)\Big] \\ & + \frac{1+g_1}{2}\Big[\frac{1-g_2}{2}(\partial_{y_1}w_3^1 + \partial_{y_2}w_3^2) + \frac{1+g_2}{2}(\partial_{y_1}w_4^1 + \partial_{y_2}w_4^2)\Big] .\end{aligned}$$

Because of the signs in Table 2, and the fact that $g' > 0$, we know that the terms in the first two rows are negative, but we do not control the signs of the terms in the 3rd and 4th rows. However, since $g_1' = g_2' = 1/\varepsilon$, we immediately get that div$F < 0$ for ε sufficiently small and Theorem 3.8 is verified. □

From contractivity (see above), we also obtain that the equilibrium point (y_1, y_2) is asymptotically stable. Moreover, again from contractivity of the fixed point iteration, we also have that the fixed point (y_1, y_2) depends smoothly on v.

Finally, we would like to remark on the use of a more general function $g(y)$, rather than simply $g(y) = y/\varepsilon$. In our arguments, we used two key properties of the function g, the first of which is a consequence of monotonicity, while the second is not necessarily satisfied for all choices of smooth monotone interpolant g: (a) $g^{-1}(z) \in [-\varepsilon, \varepsilon]$ for any $z \in [-1, 1]$ (used for contractivity), and (b) $g'(y) = \mathcal{O}(1/\varepsilon)$ for all $y \in [-\varepsilon, \varepsilon]$ (used to rule out periodic orbits). Any choice of function g satisfying these properties would work. For example, the cubic function of Remark 3.1, or more generally any function g admitting an expansion of the type $g(z) = z/\varepsilon\big(A + \mathcal{O}(z/\varepsilon)^2\big)$ with A a constant. [We note that the above restriction (b) was not needed in the case of constant vector fields in order to rule out periodic orbits.]

Remark 3.10. *Observe that the argument we used to verify Theorem 3.8 requires unique solvability of the system (32) and invertibility of the Jacobian at the iterates* $\alpha_\varepsilon^{(k+1)}, \beta_\varepsilon^{(k+1)}$. *These facts follow from* (10)-(11) *and Lemma 1.5, and have been inferred regardless of the small value of* ε. *It is only through the contractivity argument (and then also to infer negative divergence) that the need for a sufficiently small* ε *comes into play.*

Finally, we now compare the vector field of the regularized system on S_ε with the vector field of [8] on Σ. That is, we want to compare the vector fields (a) and (b) below.

(a) Vector field from [8]. We have $y_1 = y_2 = 0$, and the differential algebraic system (28)-(29), where the w_i's are functions of y (with $y_1 = y_2 = 0$).

(b) Regularized vector field on S_ε. We have y_1 and y_2 from $y_1^\varepsilon = g^{-1}(\alpha_\varepsilon)$, $y_2^\varepsilon = g^{-1}(\beta_\varepsilon)$ and the differential algebraic system (26)-(27), where the w_i's are functions of y^ε (and now, in general $y_1^\varepsilon \neq 0$, $y_2^\varepsilon \neq 0$).

To recap, let us explicitly rewrite case (a) as

$$\begin{aligned} 0 =& (1-\alpha)\big((1-\beta)\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix}(y) + \beta \begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix}(y)\big) \\ &+\alpha\big((1-\beta)\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix}(y) + \beta \begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix}(y)\big) , \quad y_1 = y_2 = 0 , \end{aligned}$$

$$\begin{bmatrix} \dot y_3 \\ \vdots \\ \dot y_n \end{bmatrix} = (1-\alpha)\big((1-\beta)\begin{bmatrix} w_1^3 \\ \vdots \\ w_1^n \end{bmatrix}(y) + \beta \begin{bmatrix} w_2^3 \\ \vdots \\ w_2^n \end{bmatrix}(y)\big) + \alpha\big((1-\beta)\begin{bmatrix} w_3^3 \\ \vdots \\ w_3^n \end{bmatrix}(y) + \beta \begin{bmatrix} w_4^3 \\ \vdots \\ w_4^n \end{bmatrix}(y)\big) ,$$

and let us rewrite (b) as

$$0 = (1-\alpha_\varepsilon)\big((1-\beta_\varepsilon)\begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix}(y^\varepsilon) + \beta_\varepsilon \begin{bmatrix} w_2^1 \\ w_2^2 \end{bmatrix}(y^\varepsilon)\big) + \alpha_\varepsilon\big((1-\beta_\varepsilon)\begin{bmatrix} w_3^1 \\ w_3^2 \end{bmatrix}(y^\varepsilon) + \beta_\varepsilon \begin{bmatrix} w_4^1 \\ w_4^2 \end{bmatrix}(y^\varepsilon)\big) ,$$

$$\begin{bmatrix} \dot y_3^\varepsilon \\ \vdots \\ \dot y_n^\varepsilon \end{bmatrix} = (1-\alpha_\varepsilon)\big((1-\beta_\varepsilon)\begin{bmatrix} w_1^3 \\ \vdots \\ w_1^n \end{bmatrix}(y^\varepsilon) + \beta_\varepsilon \begin{bmatrix} w_2^3 \\ \vdots \\ w_2^n \end{bmatrix}(y^\varepsilon)\big) + \alpha_\varepsilon\big((1-\beta_\varepsilon)\begin{bmatrix} w_3^3 \\ \vdots \\ w_3^n \end{bmatrix}(y^\varepsilon) + \beta_\varepsilon \begin{bmatrix} w_4^3 \\ \vdots \\ w_4^n \end{bmatrix}(y^\varepsilon)\big) .$$

Comparison of the two vector fields makes sense when they are evaluated at arguments $y_0 = (0, 0, v)$ and $y_0^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, v)$, respectively, where $y_1^\varepsilon = g^{-1}(\alpha_\varepsilon)$, $y_2^\varepsilon = g^{-1}(\beta_\varepsilon)$. In this case, we make the following

Claim. *The two vector fields, on Σ and S_ε respectively, are $\mathcal{O}(\varepsilon)$-close to each other.*

[From this, it will follow that the vector fields are also $\mathcal{O}(\varepsilon)$-close when evaluated at $(0, 0, v)$ and $(y_1^\varepsilon, y_2^\varepsilon, \tilde v)$ with v and $\tilde v$ being $\mathcal{O}(\varepsilon)$-close].

To verify the claim, let $z_j = \begin{bmatrix} w_j^3 \\ \vdots \\ w_j^n \end{bmatrix}(y)$ and and $z_j^\varepsilon = \begin{bmatrix} w_j^3 \\ \vdots \\ w_j^n \end{bmatrix}(y^\varepsilon)$, $j = 1, \dots, 4$, and write

$$\begin{aligned} \begin{bmatrix} \dot y_3 \\ \vdots \\ \dot y_n \end{bmatrix} - \begin{bmatrix} \dot y_3^\varepsilon \\ \vdots \\ \dot y_n^\varepsilon \end{bmatrix} =& (1-\alpha)\big[(1-\beta)(z_1 - z_1^\varepsilon) + \beta(z_2 - z_2^\varepsilon)\big] + \alpha\big[(1-\beta)(z_3 - z_3^\varepsilon) + \beta(z_4 - z_4^\varepsilon)\big] \\ &+ \big[(1-\alpha)(1-\beta) - (1-\alpha_\varepsilon)(1-\beta_\varepsilon)\big] z_1^\varepsilon + \big[(1-\alpha)\beta - (1-\alpha_\varepsilon)\beta_\varepsilon\big] z_2^\varepsilon \\ &+ \big[\alpha(1-\beta) - \alpha_\varepsilon(1-\beta_\varepsilon)\big] z_3^\varepsilon + \big[\alpha\beta - \alpha_\varepsilon\beta_\varepsilon\big] z_4^\varepsilon . \end{aligned}$$

So, the stated claim will follow if

(i) $\alpha - \alpha_\varepsilon = \mathcal{O}(\varepsilon)$, $\beta - \beta_\varepsilon = \mathcal{O}(\varepsilon)$, and
(ii) $w_j^i(y_0) - w_j^i(y_0^\varepsilon) = \mathcal{O}(\varepsilon)$, $i = 1, \dots, n$, $j = 1, \dots, 4$.

But, (ii) holds because of the way the w_j's are defined, smoothness of the functions $h_1(x)$ and $h_2(x)$ for x in a neighborhood of Σ, and closeness of y_0 and y_0^ε. Also, that (i) holds is a consequence of nonsingularity of the Jacobian of the nonlinear system (9). Therefore, the claim follows.

4. Numerical Experiments

Here we present results of numerical simulation[2] to highlight the behavior of the two proposed regularizations considered in this work. For the regularized problems, one has to choose the value of the regularization parameter, ε; in principle, this should be small enough so to remain close to the original non-regularized problem, but of course the smaller the value of ε the stiffer is the resulting system, which leads to more challenging computations (see [16]). Practically speaking, we found the value $\varepsilon = 10^{-2}$ to be a meaningful value for our purposes.

The first problem we consider is one with a discontinuity surface of co-dimension 2 and nonconstant vector fields, where the solution should slide on the co-dimension 2 manifold for a while before eventually leaving it and coming back to it, periodically. The second problem is actually a study of what can be expected when Σ is of co-dimension 3.

It is easy to formally generalize to the case of a discontinuity surface of co-dimension 3 the time and space regularizations considered in the present work. To witness, we have eight different regions $R_1, \dots, R_8$, separated by three (hyper-)surfaces $\Sigma_{1,2,3}$ each identified with the zero set of a smooth function $h_{1,2,3}$, so that $\Sigma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$. We have eight (typically different) vector fields f_i, $i = 1, \dots, 8$, in these regions,

$$\dot{x} = f_i(x) \ , \ \ x \in R_i \ , \ \ i = 1, \dots, 8 \ ,$$

and we can assume that the regions are labeled as follows:

$$\begin{array}{llllll}
R_1: & f_1 \quad \text{when} & h_1<0\,,\ h_2<0\,,\ h_3<0\,, & R_2: & f_2 \quad \text{when} & h_1<0\,,\ h_2<0\,,\ h_3>0\,, \\
R_3: & f_3 \quad \text{when} & h_1<0\,,\ h_2>0\,,\ h_3<0\,, & R_4: & f_4 \quad \text{when} & h_1<0\,,\ h_2>0\,,\ h_3>0\,, \\
R_5: & f_5 \quad \text{when} & h_1>0\,,\ h_2<0\,,\ h_3<0\,, & R_6: & f_6 \quad \text{when} & h_1>0\,,\ h_2<0\,,\ h_3>0\,, \\
R_7: & f_7 \quad \text{when} & h_1>0\,,\ h_2>0\,,\ h_3<0\,, & R_8: & f_8 \quad \text{when} & h_1>0\,,\ h_2>0\,,\ h_3>0\,.
\end{array}$$

For $x \in \Sigma$, the extension of (8) becomes

(35)
$$\dot{x} = (1-\alpha)(1-\beta)(1-\gamma)f_1(x) + (1-\alpha)(1-\beta)\gamma f_2(x) + (1-\alpha)\beta(1-\gamma)f_3(x) +$$
$$(1-\alpha)\beta\gamma f_4(x) + \alpha(1-\beta)(1-\gamma)f_5(x) + \alpha(1-\beta)\gamma f_6(x) + \alpha\beta(1-\gamma)f_7(x) + \alpha\beta\gamma f_8(x) \ ,$$

[2]The results in this section, as well as on the other problems considered in this paper, have been obtained by using the codes `RADAR5` and `RADAU5` by N. Guglielmi and E. Hairer, and by E. Hairer and G. Wanner, which are freely downloadable from `http://www.unige.ch/math/folks/-hairer/software.html`

where the functions α, β, γ, take values in $[0,1]$ and must be found so that $\nabla h_1^T(x)\dot{x} = \nabla h_2^T(x)\dot{x} = \nabla h_3^T(x)\dot{x} = 0$. In the present context, the condition of nodal attractivity corresponds to the sign restrictions of Table 3 below, where as usual $w_i^j = (\nabla h_j)^T f_i$, $j = 1,2,3$, $i = 1,\dots,8$. The functions α, β, γ, to be used in (35) will need to satisfy

TABLE 3. Nodal Attractivity, co-dimension 3.

Component	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
w_i^1, $i=1:8$	>0	>0	>0	>0	<0	<0	<0	<0
w_i^2, $i=1:8$	>0	>0	<0	<0	>0	>0	<0	<0
w_i^3, $i=1:8$	>0	<0	>0	<0	>0	<0	>0	<0

(at each $x \in \Sigma$) the algebraic system

$$\begin{aligned}(1-\alpha)\big[(1-\beta)\big((1-\gamma)w_1+\gamma w_2\big)+\beta\big((1-\gamma)w_3+\gamma w_4\big)\big)\big] \\ +\,\alpha\big[(1-\beta)\big((1-\gamma)w_5+\gamma w_6\big)+\beta\big((1-\gamma)w_7+\gamma w_8\big)\big]=0\ .\end{aligned} \tag{36}$$

Naturally, there are similar extensions for the time and space regularizations (13) and (18). However, the difficulty is that now not only the time regularization cannot be generally expected to select a unique vector field, but also the space regularization presents some lack of uniqueness. This is due to the fact that the proposed space regularization will mimic the behavior of (and converge to) the sliding Filippov solution selected by (35); alas, the latter technique does not guarantee uniqueness in co-dimension 3 (or higher)! This lack of uniqueness was observed by D. Ulmer and A. Leykin ([20]), who produced a family of constant vector fields whose associated algebraic system (36) has 2 or 3 solutions in the unit cube. As a consequence, the arguments of Theorems 3.4 and 3.8 cannot be generalized to co-dimension 3, nor does the Jacobian necessarily remain nonsingular. What we expect, in this case of co-dimension 3 and constant vector fields, is that if the solutions selected by the technique of [8] (that is, the solutions of (36)) are isolated, then the Jacobian will be nonsingular, the map will contract locally, and the space regularization will converge to one of these sliding vector fields. Different initial conditions should select a different limit. In other words, whenever the solution of (36) is not unique in $[0,1]^3$, the following possibilities are to be expected: (1) There are two solutions, one of which is a double root, and at the double root the Jacobian will be singular; (2) There are three distinct roots, with associated nonsingular Jacobian, and each of them with its own basin of attraction. In Example 4.2, we will present a numerical experiment to clarify what happens in practice.

Example 4.1. Consider the following problem, which represents a variation on the standard stick-slip system (e.g., see [7] and references there).

We have the two co-dimension 1 surfaces $\Sigma_1 = \{x \in \mathbb{R}^3 \ : \ x_2 - 0.2 = 0\}$ and $\Sigma_2 = \{x \in \mathbb{R}^3 \ : \ x_3 - 0.4 = 0\}$, $\Sigma = \Sigma_1 \cap \Sigma_2$, and we have the four vector fields:

(37)
$$f_1(x) = \begin{bmatrix} (x_2+x_3)/2 \\ -x_1 + \frac{1}{1.2-x_2} \\ -x_1 + \frac{1}{1.4-x_3} \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} (x_2+x_3)/2 \\ -x_1 + \frac{1}{1.2-x_2} \\ -x_1 - \frac{1}{0.6+x_3} \end{bmatrix},$$
$$f_3(x) = \begin{bmatrix} (x_2+x_3)/2 \\ -x_1 - \frac{1}{0.8+x_2} \\ -x_1 + \frac{1}{1.4-x_3} \end{bmatrix}, \quad f_4(x) = \begin{bmatrix} (x_2+x_3)/2 + x_1(x_2+0.8)(x_3+0.6) \\ -x_1 - \frac{1}{0.8+x_2} \\ -x_1 - \frac{1}{0.6+x_3} \end{bmatrix}.$$

The nodal attractivity conditions of Table 2 are satisfied for $x_1 \in (-1, 1)$. On Σ, the construction of a Filippov sliding vector field according to (2)-(3) renders the one parameter family of solutions $\lambda_1 = x_1 + \lambda_4$, $\lambda_2 = \lambda_3 = (1 - x_1)/2 - \lambda_4$, with λ_4 a value in $[0, (1 - x_1)/2]$ (here, $x_1 \in [-1, 1]$). As a consequence, we expect the time regularization to select one of these possible vector fields and to exhibit oscillatory behavior around the sliding surface. On the other hand, the vector field (8) is well defined, with $\alpha = \beta = (1 - x_1)/2$ (which corresponds to $\lambda_4 = \big((1 - x_1)/2\big)^2$), and we expect that the space regularization technique has a behavior resembling the one associated with this specific sliding motion.

Indeed, below we show some typical plots of the solution computed by the two regularizations considered in this work to confirm the aforementioned expectations.

In Figure 3 we show the solutions corresponding to the ICs $(-0.1, -0.1, 0.2)$ by replacing (37) with its space regularization with linear function g in (19) and parameter $\varepsilon = 10^{-2}$. The solution profile is effectively much the same as the one of sliding motion associated to the vector field (8). Different initial conditions (and/or smaller values of ε) do not change appreciably this picture.

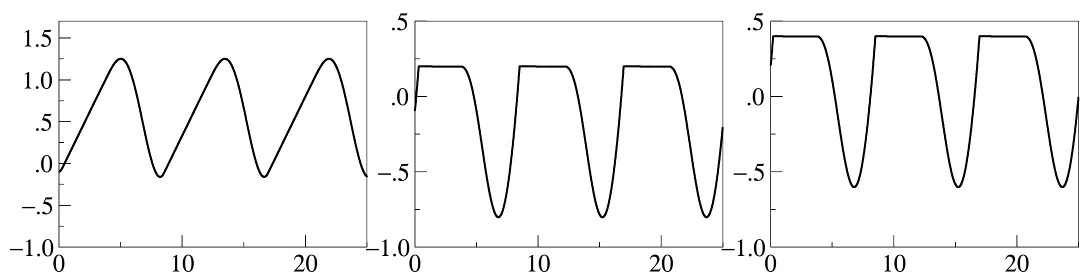


Figure 3. From the left, we show the graphs of $x_1^\varepsilon(t)$, $x_2^\varepsilon(t)$ and $x_3^\varepsilon(t)$, versus time, for initial values $x_1, x_2 = -0.1, x_3 = 0.2$, otained with the regularization in space with $\varepsilon = 10^{-2}$. Note that x_2^ε and x_3^ε periodically approximate sliding modes and classical solutions.

In Figure 4, instead, we show plots obtained by using the time regularization with a relatively large parameter $\varepsilon = 10^{-1}$. The peculiar difference in this case is appreciated by zooming into the flat sections of the solutions: when $x_2 \approx 0.2$ and $x_3 \approx 0.4$ we observe small wrinkles of amplitude and wavelength $\mathcal{O}(\varepsilon)$.

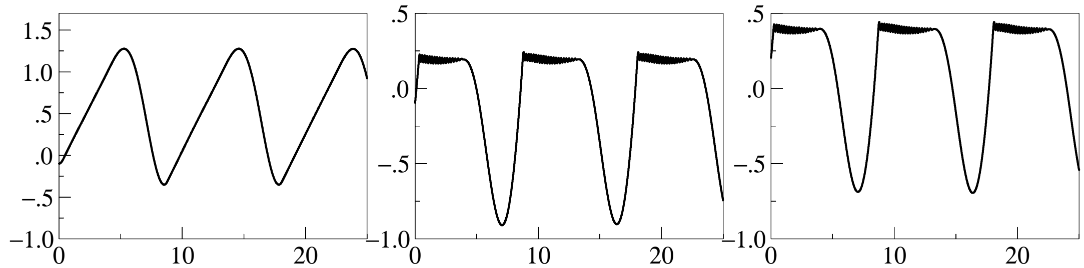


FIGURE 4. From the left, we show the graphs of $x_1^\varepsilon(t)$, $x_2^\varepsilon(t)$ and $x_3^\varepsilon(t)$, versus time, for initial values $x_1, x_2 = -0.1, x_3 = 0.2$, obtained with the regularization in time with $\varepsilon = 10^{-1}$. The appearence of wrinkles is evident in x_2^ε and x_3^ε.

A projection of the solution into the section (x_1, x_3) of the phase space is plotted in Figure 5 for both considered regularizations. □

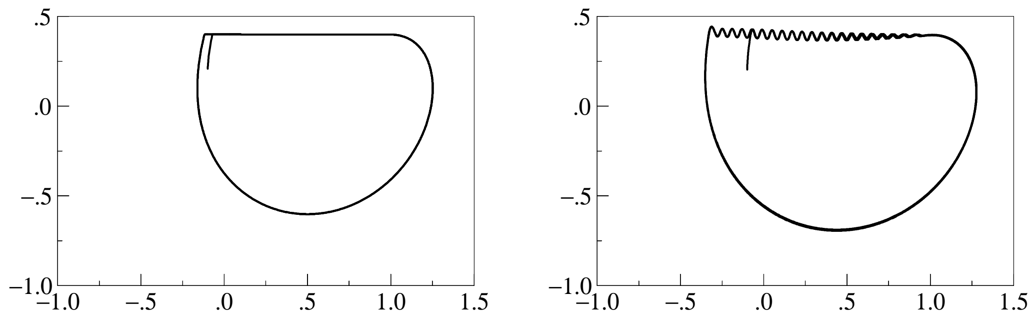


FIGURE 5. Plots of the solution in the phase space $(x_1^\varepsilon, x_3^\varepsilon)$. The spatial regularization is on the left, the time regularization is on the right. In both cases, the solution components $(x_1^\varepsilon, x_3^\varepsilon)$ remain close to the limit cycle of (37).

Example 4.2. We consider the following problem having a codimension-3 sliding manifold. Let $h_i(x) = x_i$, $i = 1, 2, 3$, so that $\Sigma = \{x \in \mathbb{R}^n \ : \ x_1 = x_2 = x_3 = 0\}$ $(n \geq 3)$, and consider the vector fields ([20])

$$f_1(x) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} 3 \\ 1 \\ -9 \\ \vdots \end{pmatrix}, \quad f_3(x) = \begin{pmatrix} 1 \\ -11 \\ 5 \\ \vdots \end{pmatrix}, \quad f_4(x) = \begin{pmatrix} 11 \\ -3 \\ -1 \\ \vdots \end{pmatrix}$$

and

$$f_5(x) = \begin{pmatrix} -7 \\ 3 \\ 1 \\ \vdots \end{pmatrix}, \quad f_6(x) = \begin{pmatrix} -1 \\ 11 \\ -5 \\ \vdots \end{pmatrix}, \quad f_7(x) = \begin{pmatrix} -3 \\ -1 \\ 9 \\ \vdots \end{pmatrix}, \quad f_8(x) = \begin{pmatrix} -5 \\ -1 \\ -1 \\ \vdots \end{pmatrix}$$

(components past the 3rd one are not relevant for our purposes). There are two stationary points (for the first three components),

$$S_1 = (0, 0, 0, \ldots) \qquad \text{and} \qquad S_2 \approx \varepsilon(-0.3368, -0.4175, -0.3839, \ldots)\ ,$$

where S_1 is a double root (with associated singular Jacobian), whereas S_2 is a stable node. In Figure 6, we show different set of initial conditions on the boundary of the cube $(x_1, x_2, x_3) \in [-\varepsilon, \varepsilon]^3$, from where the solutions of the generalization of the regularized system (18) applied to (35) reach one or the other stationary point. As anticipated, we converge to one of the equilibria depending on the starting values. Inside the cube, there is a surface (not shown) separating the basins of attraction of the two equilibria. In this problem, there appear to be no attracting periodic orbit. □

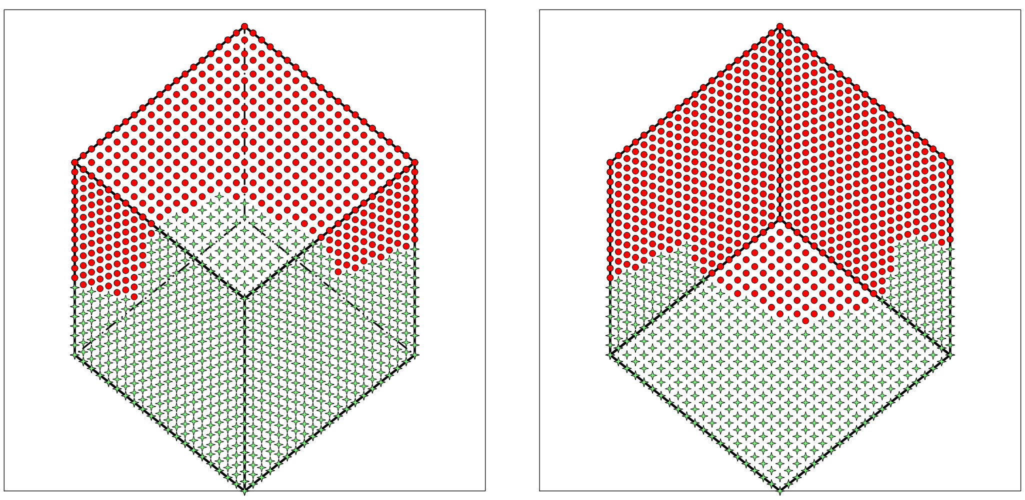


FIGURE 6. The cube $[-\varepsilon, \varepsilon]^3$ and initial conditions from where solutions of the (spatially) regularized system converge to either S_1 or S_2. Red circles indicate initial conditions on the boundary of the cube from where we converge to the stationary point S_1; green stars indicate initial conditions from where we converge to the stationary point S_2. In the left picture, we show the faces $(x_1, x_2, \varepsilon), (-\varepsilon, x_2, x_3), (x_1, -\varepsilon, x_3)$; in the right picture, we show the faces $(x_1, x_2, -\varepsilon), (\varepsilon, x_2, x_3), (x_1, \varepsilon, x_3)$.

5. Conclusions and future work

In this paper we have discussed regularization of piecewise smooth differential equations in the case in which the discontinuities occur on a smooth co-dimension 2 surface Σ. In the particular case of nodally attractive Σ, we have presented and compared a time averaging regularization and a space regularization.

Our results show that the time averaging process in general does retain the same ambiguity in selecting a Filippov sliding vector field as the original Filippov convexification method. On the other hand, the proposed space regularization process selects a well defined trajectory which remains close to the surface Σ. Furthermore, the vector field corresponding to this space regularization is close (uniformly in the regularization parameter) to the Filippov vector field proposed in [8].

In future work, we plan to examine the spatial regularization technique under the exhaustive attractivity conditions for Σ recently used in [6] to justify the Filippov vector field in (8).

Finally, the case where the discontinuity surface has co-dimension 3 (or higher) is more elusive, in the sense that even the space regularization considered in this work fails to select a unique limiting sliding vector field. We hope to come back to this case in future work.

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