Lyapunov–Type Numbers and Torus Breakdown:
Numerical Aspects and a Case Study. [1]

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Abstract. In this work we examine the breakdown mechanism for a 2-torus associated to a system of coupled oscillators. To this end, we rely on computation of the Lyapunov–type numbers, as defined by Fenichel in [6]. We give some general results on these numbers, and some specific constructive results for the case in which there is phase-locking on the torus. This is the situation for the system of coupled oscillator we consider, and it leads to great simplifications in the computation of the Lyapunov–type numbers.

Key Words. Lyapunov–type numbers, invariant tori, coupled oscillator, phase locking.

AMS Subject Classification. 65L

1 Introduction

In this paper we consider dynamical systems depending smoothly on a parameter $a \leq \lambda \leq b$,

$$\frac{dw}{dt} = \Phi(w, \lambda), \quad w(t) \in \mathbb{R}^n \ . \tag{1}$$

[1]This research was supported by NSF grants DMS-9404124, DMS-9306412 and by DOE grant DE-FG03-95ER25235.

The solution of (1) with initial condition $w(0) = x, x \in \mathbb{R}^n$, is denoted by $F^t(x, \lambda), t \in \mathbb{R}$. Suppose for $\lambda = a$ the system has a compact invariant manifold $M(a) \subset \mathbb{R}^n$. We may think of $M(a)$ as a smooth surface in $\mathbb{R}^n$ of dimension $p < n$. (Precise assumptions will be stated in Section 2.) Invariance of $M(a)$ means that, given any $x \in M(a)$, the whole trajectory $F^t(x, a), t \in \mathbb{R}$, lies in $M(a)$.

The study of invariant manifolds plays an important role in the theory of dynamical systems and in applications.

Now suppose the parameter λ varies in an interval $a \leq \lambda \leq a + \epsilon$. Under suitable assumptions on the linearized flow near $M(a)$, it is known that the systems (1) have nearby invariant manifolds $M(\lambda)$ for $a \leq \lambda \leq a + \epsilon$; these manifolds depend smoothly on λ and are all diffeomorphic to $M(a)$. One says that $M(a)$ *persists*, though it actually deforms smoothly. Then, if similar conditions on the linearized flow are satisfied near $M(a + \epsilon)$, the branch $M(\lambda)$ can be continued further, etc.

Perturbation results for invariant manifolds, which can be used to establish such a continuation property, have a long history. In this paper we follow Fenichel [6] where sufficient conditions are formulated in terms of so–called Lyapunov–type numbers. If the conditions are met at an invariant manifold $M(\lambda_0)$, then the branch $M(\lambda)$ can be continued beyond λ_0. If the conditions are violated, then — in general — the branch cannot be uniquely continued, and a large variety of bifurcations is possible. At present, no complete classification of these bifurcations is known, however.

Under rather restrictive assumptions, we will describe how the generalized Lyapunov–type numbers can be computed. Basically, we will assume that $M = M(\lambda_0)$ consists of stationary points, of periodic orbits, and of the unstable manifolds of the stationary points and the periodic orbits in M. Then the Lyapunov–type numbers can be obtained in terms of eigenvalues of linearizations at the stationary points and of Floquet multipliers of the periodic orbits.

These results, though restrictive, are often applicable. We present a case study of a system of two coupled oscillators, depending on two parameters, β and λ. Here λ is the coupling constant. Many properties of the system have been analyzed in the fundamental paper [1], and branches of invariant 2–tori have been computed in [3], [4], [12] with λ as continuation parameter. Breakdown of the tori has been observed in these works, as was predicted in [1]. The exact mechanism of breakdown remained unknown, however. An aim of the present paper is to combine branch following for tori — as described in [4] — with monitoring of Fenichel's Lyapunov–type numbers. For the example of two coupled oscillators, a consistent picture of the breakdown of the invariant tori will emerge.

2 Invariant Manifolds and Lyapunov–Type Numbers

To set notations, we first summarize some simple results on ODEs. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ denote a C^1 vector field. The solution of

$$\frac{dw}{dt} = \Phi(w), \quad w(0) = x \tag{2}$$

is denoted by $w(t) = F^t x$. For simplicity of presentation, we make the technical assumption that $F^t x$ is defined for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. (This can always be achieved by first applying a cut–off process to a given system of ODEs.) The solution operator F^t has the group property

$$F^{s+t}x = F^s F^t x, \quad s, t \in \mathbb{R}, \quad F^0 = id \ . \tag{3}$$

Differentiating the identity

$$\frac{d}{dt} F^t x = \Phi(F^t x) \tag{4}$$

w.r.t. x, we obtain

$$\frac{d}{dt} DF^t(x) = D\Phi(F^t x) DF^t(x) \ , \tag{5}$$

and $DF^0(x) = id$. Here $DF^t(x)$ is the Jacobian of $F^t(\cdot)$ at x, etc. Thus, the linearized solution operator $DF^t(x)$ is the solution operator of the linearized equation

$$\frac{dy}{dt} = D\Phi(F^t x) y \ . \tag{6}$$

Differentiation of (3) w.r.t. x yields

$$DF^{s+t}(x) = DF^s(F^t x) DF^t(x) \ . \tag{7}$$

Manifolds. Let $M \subset \mathbb{R}^n$ denote a compact connected C^1 manifold without boundary of dimension $p < n$. We may think of M as a p dimensional surface in $\mathbb{R}^n$. For $x \in M$ let $T_x(M)$ denote the tangent space of M at x. We may (and will) identify $T_x(M)$ with a p dimensional linear subspace of $\mathbb{R}^n$.

Let

$$\langle u, v \rangle = \sum_{j=1}^n u_j v_j, \quad |u|^2 = \langle u, u \rangle$$

define the Euclidean inner product and norm in $\mathbb{R}^n$. Then

$$N_x(M) = \{u \in \mathbb{R}^n : \langle u, v\rangle = 0 \ \forall \ v \in T_x(M)\}$$

is the normal space of M at x, a subspace of dimension $q = n - p$. For every $x \in M$, each $v \in \mathbb{R}^n$ has a unique orthogonal decomposition

$$v = v^I + v^{II}, \quad v^I \in T_x(M), \quad v^{II} \in N_x(M) \tag{8}$$

and

$$\Pi_x : \mathbb{R}^n \to \mathbb{R}^n, \quad v \to v^{II} \in N_x(M) \subset \mathbb{R}^n \tag{9}$$

denotes the corresponding orthogonal projection operator onto the normal space.

Invariant Manifolds. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ denote a C^1 vector field and let $M \subset \mathbb{R}^n$ denote a compact connected C^1 manifold of dimension p, without boundary. Then M is called invariant under the flow of Φ (for short: invariant for Φ) if for all $x \in M$ the whole trajectory $F^t x, t \in \mathbb{R}$, lies in M. It is not difficult to show that M is invariant for Φ if and only if

$$\Phi(x) \in T_x(M) \quad \text{for all} \quad x \in M \ . \tag{10}$$

Furthermore, if M is invariant for Φ, then for every $x \in M$ and every $t \in \mathbb{R}$ the linearized solution operator $DF^t(x)$, which we view as a map from

$$\mathbb{R}^n = T_x(M) + N_x(M) \quad \text{onto} \quad \mathbb{R}^n = T_{F^t x}(M) + N_{F^t x}(M) \ ,$$

has the property

$$DF^t(x)(T_x(M)) = T_{F^t x}(M) \ . \tag{11}$$

Lyapunov–Type Numbers. Suppose that M is invariant for Φ. Following Fenichel [6], we define for every $x \in M$ and every $t \in \mathbb{R}$ the following two linear operators:

$$A_t(x) : T_x(M) \to T_{F^{-t}x}(M), \qquad v \to DF^{-t}(x)v$$

and

$$B_t(x) : N_{F^{-t}x}(M) \to N_x(M), \qquad v \to \Pi_x DF^t(F^{-t}x)v \ .$$

To give first an intuitive understanding of these operators, we remark the following: Suppose that $t \to \infty$. Then $B_t(x)$ will describe the evolution of normal vectors — under the linearized flow — in forward time. Roughly speaking, if the norm of $B_t(x)$ goes to zero as $t \to \infty$ for every $x \in M$, then the manifold M is locally attracting. This will be quantified by the Lyapunov–type number $\nu(x)$ defined below.

The operators $A_t(x)$ describe the linearized dynamics *within* M in backward time. If the norm of $A_t(x)$ becomes large as $t \to \infty$, then there is attractivity — in forward time — of the flow within M. To obtain a perturbation result for M (i.e., persistence of M under small perturbations of the vector field Φ), it is critical that along any trajectory in M the attractivity *towards* M is stronger than the attractivity *within* M. The ratio of these two attractivity rates is determined by the second Lyapunov–type number $\sigma(x)$, which will also be defined below.

In our setting, where the manifold M is embedded in the Euclidean space $\mathbb{R}^n$, the linear operator $A_t(x)$ maps a p dimensional subspace of $\mathbb{R}^n$ onto another such subspace. The vector norm in these subspaces is always taken to be the Euclidean norm of $\mathbb{R}^n$, and $|A_t(x)|$ denotes the corresponding operator norm. The operator norm of $B_t(x)$ is defined similarly. As in [6], we define

$$\nu(x) = \mathrm{limsup}_{t\to\infty} |B_t(x)|^{1/t} \ ,$$

and if $\nu(x) < 1$, we set

$$\sigma(x) = \mathrm{limsup}_{t\to\infty} \frac{\log |A_t(x)|}{-\log |B_t(x)|} \ .$$

For illustration, suppose that

$$|B_t(x)| = c_1 e^{-\beta t}, \quad \beta > 0 \ ,$$

and

$$|A_t(x)| = c_2 e^{\alpha t}, \quad \alpha \in \mathbb{R} \ .$$

Then we have

$$\nu(x) = e^{-\beta} < 1 \quad \text{and} \quad \sigma(x) = \frac{\alpha}{\beta} \ .$$

The following characterization of $\sigma(x)$ is often useful.

Lemma 2.1 *Let $x \in M$ with $\nu(x) < 1$. For $c > \sigma(x)$ we have*

$$|A_t(x)||B_t(x)|^c \to 0 \quad \textit{as} \quad t \to \infty \ . \tag{12}$$

Conversely, if (12) holds, then $c \geq \sigma(x)$. Consequently,

$$\sigma(x) = \inf\{c \in \mathbb{R} : (12) \ \textit{holds}\} \ .$$

Proof: We abbreviate $a_t = |A_t(x)|, b_t = |B_t(x)|$ and note that $b_t \to 0$ as $t \to \infty$ since $\nu(x) < 1$. Therefore, $-\log b_t > 0$ for $t \geq t_0$.

First let $c > \sigma(x)$ be arbitrary. We choose $c > \gamma > \sigma(x)$ and have

$$\frac{\log a_t}{-\log b_t} < \gamma \quad \text{for} \quad t \geq t_1 \geq t_0 \ .$$

Therefore, $\log a_t < \log(b_t^{-\gamma})$, i.e. $a_t b_t^{\gamma} < 1$ for $t \geq t_1$. It follows that

$$a_t b_t^c = a_t b_t^{\gamma} b_t^{c-\gamma} < b_t^{c-\gamma} \to 0 \quad \text{as} \quad t \to \infty \ .$$

Thus we have shown that (12) holds for $c > \sigma$.

Conversely, assume (12) for some $c \in \mathbb{R}$. Clearly, $a_t b_t^c < 1$ for $t \geq t_1 \geq t_0$, thus

$$\log a_t + c \log b_t < 0 \ ,$$

and therefore

$$\frac{\log a_t}{-\log b_t} < c \quad \text{for} \quad t \geq t_1 \geq t_0 \ .$$

This implies $c \geq \sigma(x)$, and the lemma is proved. $\diamond$

Suppose now that the manifold M is an invariant manifold for (1) at a certain parameter value λ_0. Define

$$\nu(M) := \sup_{x \in M} \nu(x) \, ,$$

and — if $\nu(x) < 1$ for all $x \in M$ — also

$$\sigma(M) := \sup_{x \in M} \sigma(x) \, .$$

Then, the perturbation theorem of [6] guarantees that if $\nu(M) < 1$ and $\sigma(M) < 1/k$, there exists a unique C^k invariant manifold diffeomorphic to M for $\lambda = \lambda_0 + \Delta\lambda$ if $\Delta\lambda$ is sufficiently small. In other words, if we could accurately compute the values of $\nu(M)$ and $\sigma(M)$, then we would know whether a certain manifold can be continued. In general, computation of these quantities is non-trivial, however. Still, there are interesting cases in which their computation is more or less trivial. These are examined below. It is clear that Fenichel was aware of the simplifications occurring in these cases, see [6]. However, their precise derivation and quantification was not carried out in [6] nor — to our knowledge — anywhere else. For this reason, we are going to examine these cases in detail.

Fixed Points in M. Let M be invariant for Φ, and let $x \in M$ be a fixed point of the dynamical system (2), i.e. $\Phi(x) = 0$. We have $F^t x \equiv x$, and if we set $L = D\Phi(x)$, then

$$DF^t(x) = e^{Lt} \ .$$

The tangent space $T_x(M)$ is invariant under e^{Lt} for all t, and therefore $L(T_x(M)) \subset T_x(M)$. Let $\lambda_1, \ldots, \lambda_p$ denote the eigenvalues of L corresponding to the invariant subspace

$T_x(M)$, and let $\lambda_{p+1},\ldots,\lambda_n$ denote the remaining eigenvalues of L. (If λ has algebraic multiplicity m, then λ is listed m times.)

We may assume the orderings

$$\operatorname{Re}\lambda_1 \ge \ldots \ge \operatorname{Re}\lambda_p =: -\alpha$$

and

$$-\beta := \operatorname{Re}\lambda_{p+1} \ge \ldots \ge \operatorname{Re}\lambda_n \ .$$

The λ_j for $j = 1,\ldots,p$ correspond to the linearized flow within M, and the λ_j for $j = p+1,\ldots,n$ correspond to the linearized flow towards M. Therefore, we expect that

$$|A_t(x)| \sim e^{\alpha t} \quad \text{as} \quad t \to \infty$$

and

$$|B_t(x)| \sim e^{-\beta t} \quad \text{as} \quad t \to \infty \ .$$

Thus the previous illustration suggests $\nu(x) = e^{-\beta}$ and, if $\beta > 0$, then $\sigma(x) = \alpha/\beta$. This will indeed be shown below.

Assuming that $\beta > 0$, thus $\nu(x) < 1$, the crucial question to obtain a perturbation theorem is if

$$\sigma(x) = \frac{\alpha}{\beta} < 1 \ .$$

Clearly, this is equivalent to $-\alpha > -\beta$, i.e. the crucial estimate $\sigma(x) < 1$ is equivalent to the existence of a gap between the eigenvalues of $L = D\Phi(x)$ that correspond to the invariant subspace $T_x(M)$ and the other eigenvalues of L:

$$\min_{1\le j\le p} \operatorname{Re}\lambda_j =: -\alpha > -\beta := \max_{p+1\le j\le n} \operatorname{Re}\lambda_j \ .$$

Theorem 2.1 *Let M be invariant for Φ, and let $x \in M$ be a fixed point of the dynamical system $dw/dt = \Phi(w)$. Let $\lambda_1,\ldots,\lambda_p$ denote the eigenvalues of $L = D\Phi(x)$ corresponding to the invariant subspace $T_x(M)$ of L and let $\lambda_{p+1},\ldots,\lambda_n$ denote the other eigenvalues of L. Define*

$$\alpha = -\min_{1\le j\le p} \operatorname{Re}\lambda_j, \quad \beta = -\max_{p+1\le j\le n} \operatorname{Re}\lambda_j \ .$$

Then we have $\nu(x) = e^{-\beta}$, and if $\beta > 0$ then $\sigma(x) = \alpha/\beta$.

Proof: Let $v^1, \ldots, v^p$ denote a basis of $T_x(M)$ and let $v^{p+1}, \ldots, v^n$ denote a basis of $N_x(M)$. We will represent all operators by matrices w.r.t. these bases. For example, the $n \times n$ matrix $\hat{L} = (l_{ij})$ determined by $Lv^j = \sum_i l_{ij} v^i$ represents L. Note that $\hat{L}$ has block form,

$$\hat{L} = \begin{pmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{pmatrix}, \quad L_{11} \text{ is } p \times p \ . \tag{13}$$

The solution operator e^{Lt} is represented by

$$e^{\hat{L}t} = \begin{pmatrix} e^{L_{11}t} & R_{12} \\ 0 & e^{L_{22}t} \end{pmatrix} \ . \tag{14}$$

with

$$R_{12} = \int_0^t e^{L_{11}(t-s)} L_{12} e^{L_{22}s} ds \ .$$

(The form of R_{12} is not important for our current argument.) Then $A_t(x)$ and $B_t(x)$ are represented by $e^{-L_{11}t}$ and $e^{L_{22}t}$, respectively. It follows that there are positive constants c_1 and c_2 independent of $t \geq 1$ with

$$c_1 e^{\alpha t} \leq |A_t(x)| \leq c_2 t^{p-1} e^{\alpha t}$$

and

$$c_1 e^{-\beta t} \leq |B_t(x)| \leq c_2 t^{q-1} e^{-\beta t} \ .$$

Here $q = n - p$. This implies

$$\lim_{t \to \infty} |B_t(x)|^{1/t} = e^{-\beta} \ .$$

Also, if $\beta > 0$, then we have

$$\lim_{t \to \infty} \frac{\log |A_t(x)|}{-\log |B_t(x)|} = \frac{\alpha}{\beta}$$

and the theorem is proved. $\diamond$

General Remarks on Lyapunov–Type Numbers. Let $x, y \in M$ lie on the same orbit. We claim that $\nu(x) = \nu(y)$, and if $\nu(x) < 1$ then also $\sigma(x) = \sigma(y)$. This result, already stated in [6], is well–known. The proof — given here for completeness — is based on useful product formulae for $A_t(x)$ and $B_t(x)$.

Lemma 2.2 *For all* $x \in M$ *and all* $s, t \in \mathbb{R}$ *we have*

$$B_{s+t}(x) = B_s(x) B_t(F^{-s}x) \tag{15}$$

and

$$A_{s+t}(x) = A_t(F^{-s}x) A_s(x) \ . \tag{16}$$

Proof: In (7) we replace x by $F^{-s-t}x$ to obtain

$$DF^{s+t}(F^{-s-t}x) = DF^s(F^{-s}x) DF^t(F^{-s-t}x) \ . \tag{17}$$

Now let $v \in N_{F^{-s-t}x}(M)$ be arbitrary. Then we have

$$B_{s+t}(x)v = \Pi_x DF^{s+t}(F^{-s-t}x)v \tag{18}$$

and

$$B_t(F^{-s}x)v = \Pi_{F^{-s}x} DF^t(F^{-s-t}x)v \ .$$

Define the intermediate vector

$$w := DF^t(F^{-s-t}x)v$$

and decompose

$$w = w^I + w^{II}, \quad w^{II} = \Pi_{F^{-s}x} w \ .$$

Using (18) and (17) we have

$$B_{s+t}(x)v = \Pi_x DF^s(F^{-s}x)w \ .$$

On the other hand,

$$B_s(x)B_t(F^{-s}x)v = B_s(x)w^{II} = \Pi_x DF^s(F^{-s}x)w^{II} \ .$$

The vector w^I is a tangent vector and is mapped into another tangent vector by $DF^s(F^{-s}x)$. Consequently,

$$\Pi_x DF^s(F^{-s}x)w^I = 0 \ .$$

This shows that

$$B_{s+t}(x)v = B_s(x)B_t(F^{-s}x)v \ ,$$

and formula (15) is proved.

To show (16) we start again with (7) and interchange s and t,

$$DF^{s+t}(x) = DF^t(F^s x) DF^s(x) \ .$$

Replacing t by $-t$ and s by $-s$ we find

$$DF^{-s-t}(x) = DF^{-t}(F^{-s}x) DF^{-s}(x) \ .$$

Restriction of this identity to the corresponding tangent spaces yields

$$A_{s+t}(x) = A_t(F^{-s}x) A_s(x) \ .$$

This finishes the proof of lemma. $\diamond$

Next we show that the Lyapunov–type numbers are constant on orbits.

Lemma 2.3 *Let $x, y \in M$ lie on the same orbit. Then we have $\nu(x) = \nu(y)$, and if $\nu(x) < 1$ then $\sigma(x) = \sigma(y)$.*

Proof: Let $y = F^{-s}x, c = |B_s(x)|$. Then (15) yields

$$|B_{s+t}(x)| \le c|B_t(y)|, \quad t \in \mathbb{R} \ .$$

This implies $\nu(x) \le \nu(y)$, and by a symmetry argument equality, $\nu(x) = \nu(y)$, follows. If $\nu(x) < 1$ then we use (16) to show $\sigma(x) = \sigma(y)$ in a similar way. $\diamond$

Periodic Orbits. Let $x \in M$ and let $F^t x$ have period $\tau > 0$, i.e., $F^\tau x = x$ and $F^t x \ne x$ for $0 < t < \tau$. It follows that $L := DF^\tau(x)$ maps $T_x(M)$ onto itself. (Note that $DF^t(x)$ is always nonsingular since it is a fundamental solution operator of a linear system.) We choose a basis $v^1, \dots, v^p$ of $T_x(M)$ and a basis $v^{p+1}, \dots, v^n$ of $N_x(M)$, and henceforth will identify linear operators with their matrix representations in these bases. By Floquet's theorem, the fundamental matrix $DF^t(x)$ can be written as

$$DF^t(x) = P(t) e^{Rt}$$

where $P(t)$ is a continuous matrix function of period τ. The matrix R is not unique, but the monodromy matrix $L = e^{R\tau} = DF^\tau(x)$ is. Our choice of basis $v^1, \dots, v^n$ implies the block form

$$L = \begin{pmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{pmatrix}, \quad L_{11} \text{ is } p \times p \ . \tag{19}$$

The eigenvalues μ_j of L are — by definition — the Floquet multipliers of the periodic solution $F^t x$. We order the μ_j so that $\mu_1, \dots, \mu_p$ are the eigenvalues of L_{11} and $\mu_{p+1}, \dots, \mu_n$ are the eigenvalues of L_{22}. In other words, $\mu_1, \dots, \mu_p$ are the eigenvalues of L corresponding to the invariant subspace $T_x(M)$ and $\mu_{p+1}, \dots, \mu_n$ are the other eigenvalues of L. Of course, one of the eigenvalues of L_{11} equals 1, $\mu_1 = 1$, say. We write $\mu_j = e^{\lambda_j \tau}$ for $j = 1, \ldots, n$ with $\lambda_j = \alpha_j + i\beta_j$ where α_j, β_j are real and $-\pi \le \beta_j \tau < \pi$. By definition, the λ_j are the Floquet exponents of the periodic solution $F^t x$. Clearly, $\lambda_1 = 0$ since $\mu_1 = 1$. In the following theorem we determine the Lyapunov–type numbers of x in terms of the Floquet exponents.

Theorem 2.2 *Let $F^t x$ denote an orbit of period τ in M with Floquet exponents $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$, where $\lambda_1, \ldots, \lambda_p$ correspond to the invariant subspace $T_x(M)$ of $L = DF^\tau(x)$. Define α, β by*

$$\alpha = -\min_{1 \le j \le p} \operatorname{Re} \lambda_j, \quad \beta = -\max_{p+1 \le j \le n} \operatorname{Re} \lambda_j \; .$$

Then we have $\nu(x) = e^{-\beta}$, and if $\beta > 0$ then $\sigma(x) = \alpha/\beta$.

Proof: First consider $B_t(x)$ for $t = m\tau, m = 1, 2, 3, \ldots$ The operator $B_{m\tau}(x)$ has the eigenvalues $e^{\lambda_j m\tau}, j = p+1, \ldots, n$. As in the proof of Theorem 2.1 we conclude

$$\lim_{m \to \infty} |B_{m\tau}(x)|^{1/(m\tau)} = e^{-\beta} \; . \tag{20}$$

If t is arbitrary, we write $t = m\tau + s$ with integer m and $0 \le s < \tau$. By (15) we have

$$B_t(x) = B_{m\tau}(x) B_s(F^{-m\tau} x) = B_{m\tau}(x) B_s(x) \; .$$

Therefore,

$$B_{m\tau}(x) = B_t(x)(B_s(x))^{-1} \; .$$

Defining

$$c_1 = \max_{0 \le s \le \tau} |B_s(x)|, \quad c_2 = \max_{0 \le s \le \tau} |(B_s(x))^{-1}| \; ,$$

we obtain

$$\frac{1}{c_2} |B_{m\tau}(x)| \le |B_t(x)| \le c_1 |B_{m\tau}(x)| \; .$$

Together with (20) it is then easy to conclude that

$$\nu(x) = \lim_{t \to \infty} |B_t(x)|^{1/t} = e^{-\beta} \; .$$

Now suppose that $\beta > 0$, i.e. $\nu(x) < 1$. Then we obtain $\sigma(x) = \alpha/\beta$ by similar arguments: First we consider $t = m\tau$ for integers $m \to \infty$ and note that $A_{m\tau}(x)$ has the eigenvalues $e^{-\lambda_j m\tau}, j = 1, \ldots, p$. An argument as in the proof of Theorem 2.1 shows that

$$\lim_{m\to\infty} \frac{\log|A_{m\tau}(x)|}{-\log|B_{m\tau}(x)|} = \frac{\alpha}{\beta} \ .$$

Then the limit for general $t \to \infty$ is treated by applying (15) and (16). $\diamond$

Backward Limit Sets. As above, let $M \subset \mathbb{R}^n$ denote a compact manifold of dimension $p < n$ which is invariant under the flow of the dynamical system $dw/dt = \Phi(w)$. We will show two results. In the first we assume that $F^{-t}x \to x_0$ as $t \to \infty$ for some $x \in M$. It follows that $x_0 \in M$ is a fixed point of the evolution and we can compute $\nu(x_0)$ and $\sigma(x_0)$ using Theorem 2.1. We will show that $\nu(x) \le \nu(x_0)$ and, if $\nu(x_0) < 1$ then also $\sigma(x) \le \sigma(x_0)$. In the second result we will prove corresponding estimates if $F^{-t}x$ approaches a periodic orbit in M as $t \to \infty$.

Theorem 2.3 *Let $x, x_0 \in M$ and assume $F^{-t}x \to x_0$ as $t \to \infty$. Then $\nu(x) \le \nu(x_0)$ and, if $\nu(x_0) < 1$ then also $\sigma(x) \le \sigma(x_0)$.*

Proof: a) Let $a > \nu(x_0)$ be arbitrary. There exists $\tau > 0$ such that

$$|B_\tau(x_0)|^{1/\tau} < a \ .$$

By continuity, there is a compact neighborhood U of x_0 in M with

$$|B_\tau(y)| < a^\tau \quad \text{for all } y \in U \subset M \ . \tag{21}$$

Since $\nu(\cdot)$ is constant on the orbit $F^t x$ and since $F^{-t}x \to x_0$ as $t \to \infty$ we may assume that $F^{-t}x \in U$ for all $t \ge 0$. Recall the product formula (15), which can inductively be generalized to

$$B_{t_1+t_2+\ldots+t_j}(x) = B_{t_1}(x) B_{t_2}(F^{-t_1}x) \cdots B_{t_j}(F^{-(t_1+t_2+\cdots+t_{j-1})}x)$$

for arbitrary $t_i \in \mathbb{R}$. If $t > 0$ is given, we write $t = (j-1)\tau + s$ with integer j and $0 \le s < \tau$, and obtain

$$B_t(x) = B_\tau(x) B_\tau(F^{-\tau}x) \cdots B_\tau(F^{(-j+2)\tau}x) B_s(F^{(-j+1)\tau}x) \ . \tag{22}$$

Using (21) we find

$$|B_t(x)| \le a^{(j-1)\tau} |B_s(F^{(-j+1)\tau}x| \ .$$

Setting

$$c_1 = \max\{a^{-s}|B_s(y)| : \ y \in U, \ 0 \le s \le \tau\}$$

we obtain

$$|B_t(x)| \le c_1 a^{(j-1)\tau} a^s = c_1 a^t$$

where $t = (j-1)\tau + s > 0$ was arbitrary. This implies

$$\nu(x) = \limsup_{t\to\infty} |B_t(x)|^{1/t} \le a \ ,$$

and since $a > \nu(x_0)$ was arbitrary, the estimate $\nu(x) \le \nu(x_0)$ is proved.

b) Let $\nu(x_0) < 1$. To show $\sigma(x) \le \sigma(x_0)$, we use Lemma 2.1 and let $c > \sigma(x_0)$ be arbitrary. There exists $\tau > 0$ and a compact neighborhood U of x_0 in M with

$$|A_\tau(y)||B_\tau(y)|^c \le \frac{1}{2} \quad \text{for all } y \in U \subset M \ . \tag{23}$$

We may assume $F^{-t}x \in U$ for $t \ge 0$, and write any given $t > 0$ as $t = (j-1)\tau + s$ with $0 \le s < \tau$. The product formula

$$A_t(x) = A_s(F^{(-j+1)\tau}x)A_\tau(F^{(-j+2)\tau}x)\cdots A_\tau(F^{-\tau}x)A_\tau(x)$$

follows inductively from (16). Together with (22) and (23) we obtain

$$|A_t(x)||B_t(x)|^c \le (\frac{1}{2})^{j-1}c_2$$

where

$$c_2 = \max\{|A_s(y)||B_s(y)|^c : \ y \in U, \ 0 \le s \le \tau\} \ .$$

Thus $|A_t(y)||B_t(y)|^c \to 0$ as $t \to \infty$, and Lemma 2.1 implies $\sigma(x) \le c$. Since $c > \sigma(x_0)$ was arbitrary, the estimate $\sigma(x) \le \sigma(x_0)$ is proved. $\diamond$

In the next result we assume $F^t x_0$ to be periodic for some $x_0 \in M$ and let $\gamma = \{F^t x_0 : t \in \mathbb{R}\}$ denote the corresponding periodic orbit. We consider a point $x \in M$ for which $F^{-t}x$ approaches γ as $t \to \infty$. The notation

$$\mathrm{dist}(z,\gamma) = \min\{|z-y| : y \in \gamma\}$$

is used.

Theorem 2.4 *Let $x \in M$ and let $\gamma \subset M$ be a periodic orbit. We assume that*

$$\mathit{dist}(F^{-t}x,\gamma) \to 0 \quad \mathit{as}\ t \to \infty \ .$$

Then $\nu(x) \le \nu(x_0)$ and, if $\nu(x_0) < 1$ then also $\sigma(x) \le \sigma(x_0)$. Here $x_0 \in \gamma$ is arbitrary.

Proof: Let $\tau > 0$ denote the period of γ and let $x_0 \in \gamma$ denote a chosen point which is kept fixed in the following. For any $z \in \gamma$ there is $0 \le s < \tau$ with $z = F^{-s}x_0$. By (15) we have

$$B_{s+t}(x_0) = B_s(x_0)B_t(z)$$

and if we set

$$c_1 = \max_{0 \le s \le \tau} |(B_s(x_0))^{-1}|$$

then the estimate

$$|B_t(z)| \le c_1 |B_{s+t}(x_0)|$$

follows. Applying (15) again (with s and t interchanged), we also have

$$B_{s+t}(x_0) = B_t(x_0)B_s(F^{-t}x_0) \ .$$

Setting

$$c_2 = \max\{|B_s(y)| : \ 0 \le s \le \tau, \ y \in \gamma\}$$

we conclude with the previous estimate

$$|B_t(z)| \le c_1 c_2 |B_t(x_0)| \ .$$

Now let $a > \nu(x_0)$ be chosen arbitrarily. There exists $T > 0$ with

$$|B_T(z)|^{1/T} \le (c_1 c_2)^{1/T} |B_T(x_0)|^{1/T} < a \ .$$

It is important to note that T is independent of $z \in \gamma$. For $\epsilon > 0$ we define the compact neighborhoods

$$U_M(\gamma, \epsilon) = \{y \in M : \mathrm{dist}(y, \gamma) \le \epsilon\}$$

of γ in M. Compactness of γ yields existence of $\epsilon > 0$ with

$$|B_T(y)|^{1/T} < a \quad \text{for all } y \in U_M(\gamma, \epsilon) \ .$$

The remaining part of the proof that $\nu(x) \le \nu(x_0)$ can now be given in the same way as in the case where x_0 is a fixed point. See part a) of the proof of Theorem 2.3. Also, if $\nu(x_0) < 1$ then $\sigma(x) \le \sigma(x_0)$ follows by similar arguments. $\diamond$

Remark 2.1. Under the assumptions of Theorems 2.3 and 2.4 we conjecture that equalities $\nu(x) = \nu(x_0)$ and $\sigma(x) = \sigma(x_0)$ hold. However, since we do not need this here, we have not investigated this question carefully.

In the remaining part of this section, we indicate some generalizations and open questions. For any $x \in \mathbb{R}^n$ the backward limit set $\alpha(x)$ consists, by definition, of all $z \in \mathbb{R}^n$ for which there is a sequence $t_j \to \infty$ with $F^{-t_j}x \to z$. Equivalently,

$$\alpha(x) = \bigcap_{\tau \geq 0} cl\{F^{-t}x : t \geq \tau\}$$

where clA denotes the closure of the set $A \subset \mathbb{R}^n$. Now let $x \in M$. Then $\alpha(x) \subset M$ is nonempty and compact. (This follows since the sets $cl\{F^{-t}x : t \geq \tau\}, \tau \geq 0$, are compact and nested.) Furthermore, $\alpha(x)$ is invariant under backward evolution, i.e., $F^{-t}(\alpha(x)) = \alpha(x)$ for all $t \geq 0$. In such a situation, where $K \subset M$ is a nonempty compact set invariant under backward evolution, an argument as in Fenichel's uniformity lemma (see [6]) can be used to show that

$$\sup_{z \in K} \nu(z) =: \nu(K)$$

and — if $\nu(z) < 1$ for all $z \in K$ — then also

$$\sup_{z \in K} \sigma(z) =: \sigma(K)$$

are *attained*, i.e., there exist $x_0, x_1 \in K$ with

$$\nu(K) = \nu(x_0) \quad \text{and} \quad \sigma(K) = \sigma(x_1) \ .$$

Furthermore, if $x \in M$ is a point with $\text{dist}(F^{-t}x, K) \to 0$ as $t \to \infty$ then arguments as given in the proofs of Theorems 2.3 and 2.4 show that

$$\nu(x) \leq \nu(K) \quad \text{and} \quad \sigma(x) \leq \sigma(K) \ .$$

We have only treated the two simplest cases where either $K = \alpha(x) = \{x_0\}$ consists of a fixed point or $K = \alpha(x) = \gamma$ is a periodic orbit. These seem to be the only cases where a computation of $\nu(K)$ and $\sigma(K)$ is straightforward.

Other cases are also of interest. For example, assume that M is a 2-torus with parallel flow and that every orbit $F^t x, x \in M$, is dense in M. If $\nu(x) < 1$ then $\sigma(x) = 0$ since there is no attractivity of the flow within M. In this case the backward limit set is $\alpha(x) = M$ for every $x \in M$. Using ergodic theory results (e.g., see [9]), in this case one can show that $\nu(\cdot)$ is constant almost everywhere on M.

3 Invariant 2-Tori

Let us restrict now to the case in which the invariant manifold M for (1) is a 2-torus. That is, there is a diffeomorphism

$$\omega : \; T^2 \to M \tag{24}$$

such that M is invariant for Φ, and $T^2 = (\mathbb{R} \bmod 2\pi)^2$ is the standard 2-torus. Let us denote by θ_1 and θ_2 the angular coordinates describing T^2; we can then visualize T^2 as the periodic square $[0, 2\pi] \times [0, 2\pi]$ in the (θ_1, θ_2)-plane.

Clearly, the solution operator F^t of $dw/dt = \Phi(w)$ determines a flow on M, whereas

$$\phi^t := \omega^{-1} F^t \omega \tag{25}$$

is a flow on T^2. The scenario of possibilities for the behavior of the flow ϕ^t on T^2 (equivalently, of F^t on M) is greatly simplified if ϕ^t induces a diffeomorphic *circle map* on a suitable cross section. In this case, let

$$\mathcal{C} : T^1 \to T^1 \tag{26}$$

be such a circle map. (E.g., we can think of $\mathcal{C}$ as obtained by taking a cross section on T^2 with $\theta_2 = 0$ or some other constant, and $\mathcal{C}$ is the associated return map.) Let ρ denote the *rotation number* of $\mathcal{C}$. Then, it is well understood (e.g., see [2, p.27]) that $\mathcal{C}$ has periodic points if and only if ρ is rational. Rephrased in terms of the flow on M, if ρ is rational, then every orbit on M is either periodic or transient, converging to a periodic orbit as $t \to \pm\infty$. We refer to the case of rational ρ as the *phase locking* situation. Assuming in addition that there are only finitely many periodic orbits, we can use our previous results on type numbers to compute $\nu(M)$ and $\sigma(M)$, which is what we will do in the next section.

Remark 3.1. The case in which ρ is irrational does not appear in our case study, at least for the parameter values considered. However, this case is clearly very interesting. If $\mathcal{C}$ is twice continuously differentiable (see Denjoy's theorem, [2, p.28]), then $\mathcal{C}$ is conjugate to a rigid rotation, and every orbit in M is dense in M.

To help our exposition in the next section, consider the case of phase-locking on a 2–torus T^2_λ of the parameter dependent system $dw/dt = \Phi(w, \lambda)$. Suppose that the system is n-dimensional ($n \geq 3$) and that the 2–torus T^2_λ consists of finitely many periodic orbits (stable and unstable) $\omega_j(\lambda)$, $j = 1, \dots, J$, and of the unstable manifolds of the unstable orbits.

With $\lambda_{ij} = \lambda_{ij}(\lambda)$ we denote the Floquet exponents of $\omega_j(\lambda)$ for $i = 1, \dots, n$, $j = 1, \dots, J$. Here we always adopt the following ordering: we let $\lambda_{1j} = 0$ and we let λ_{2j}

measure the rate of attractivity to (or repulsion from) $\omega_j(\lambda)$ *tangential to* T^2_λ. We note that the Floquet exponent $\lambda_{2j}(\lambda)$ is always real. In other words, $\lambda_{1j} = 0$ and $\lambda_{2j} \in \mathbb{R}$ are expression of the motion on the torus, while $\lambda_{3j}, \ldots, \lambda_{nj}$ are expression of the motion towards the torus. In addition, we assume that T^2_λ and the periodic orbits $\omega_j(\lambda)$ depend smoothly on λ, and thus we may assume continuous dependence of $\lambda_{ij}(\lambda)$ on the parameter λ.

Motivated by our case study presented in the next section, we assume that the branch of invariant tori T^2_λ exists for some parameter range $a \le \lambda < b$ with the above properties, and that the branches of periodic orbits $\omega_j(\lambda)$ exist for $a \le \lambda \le c$ with $c > b$. We further assume that

$$\nu(T^2_\lambda) < 1 \quad \text{and} \quad \sigma(T^2_\lambda) < 1 \quad \text{for} \quad a \le \lambda < b \; . \tag{27}$$

As described in the previous section, we can express the functions $\nu(T^2_\lambda)$ and $\sigma(T^2_\lambda)$ in terms of the Floquet exponents of the $\omega_j(\lambda)$. First note that

$$\nu(T^2_\lambda) = \max_j \nu(\omega_j(\lambda)), \quad \sigma(T^2_\lambda) = \max_j \sigma(\omega_j(\lambda)) \; ,$$

and if we set

$$\alpha_j(\lambda) = -\min\{0, \lambda_{2j}(\lambda)\} \; ,$$

$$\beta_j(\lambda) = -\max\{\mathrm{Re}\, \lambda_{ij}(\lambda), 3 \le i \le n\} \; ,$$

then

$$\nu(\omega_j(\lambda)) = e^{-\beta_j(\lambda)}, \qquad \sigma(\omega_j(\lambda)) = \frac{\alpha_j(\lambda)}{\beta_j(\lambda)}$$

for $a \le \lambda < b$. The above formulae show continuous dependence of $\nu(T^2_\lambda)$ and $\sigma(T^2_\lambda)$ on $a \le \lambda < b$. Also, the estimates (27) are equivalent to the strict inequalities

$$\mathrm{Re}\, \lambda_{ij}(\lambda) < 0 \quad \text{for} \quad 3 \le i \le n,\ 1 \le j \le J,\ a \le \lambda < b \tag{28}$$

and

$$\mathrm{Re}\, \lambda_{ij}(\lambda) < \lambda_{2j}(\lambda) \quad \text{for} \quad 3 \le i \le n,\ 1 \le j \le J,\ a \le \lambda < b \; . \tag{29}$$

Therefore, concerning the limiting behavior as λ approaches b, there are three possible cases:

Case 1: The strict inequalities (28), (29) remain valid for $\lambda = b$.
Case 2: There exists $1 \le j \le J$ and $3 \le i \le n$ with $\mathrm{Re}\, \lambda_{ij}(b) = 0$.
Case 3: All strict inequalities (28) extend to $\lambda = b$, but (29) becomes violated; that is, there exists $1 \le j \le J$ and $3 \le i \le n$ with $\mathrm{Re}\, \lambda_{ij}(b) = \lambda_{2j}(b)$.

Under additional assumptions, one can show that in case 1 the branch T^2_λ can be extended to $a \le \lambda < b + \Delta\lambda$ for some $\Delta\lambda > 0$. (To prove this, it suffices to give a lower bound on the *size* of the C^1–neighborhood $\mathcal{N}_\lambda$ of the vector field $w \to \Phi(w, \lambda)$, so that T^2_λ persists for every vector field in $\mathcal{N}_\lambda$, see [6]. It is crucial to estimate the size of this neighborhood from below independently of $a \le \lambda < b$. To this end, it appears necessary to assume that the coordinate system determined by tangents and normals to $M(\lambda)$ can be smoothly extended into a neighborhood of $M(\lambda)$ which is independent of $a \le \lambda < b$.)

In case 2 we have

$$\lim_{\lambda \to b} \nu(T^2_\lambda) = 1 \ .$$

This case often occurs if one of the Floquet exponents $\lambda_{ij}, i \ge 3$, of an orbit $\omega_j(\lambda)$ crosses 0 as λ crosses b, leading (typically) to a pitchfork bifurcation of periodic orbits from the branch $\omega_j(\lambda)$ at $\lambda = b$. In general, one cannot guarantee that the branch T^2_λ can be continued beyond $\lambda = b$. However, we believe that the concept of pseudo–hyperbolicity (see, e.g., [11] for an invariant manifold theorem of a pseudo–hyperbolic fixed point of a map) will be applicable under suitable additional assumptions. We expect, then, that the branch T^2_λ *can* be continued smoothly beyond $\lambda = b$, where T^2_λ is non–attracting for $\lambda > b$, but pseudo–hyperbolic. In addition, we expect other attracting tori to bifurcate from the branch T^2_λ at $\lambda = b$. This is in correspondence to the pitchfork bifurcation of periodic orbits from $\omega_j(\lambda)$ at $\lambda = b$. So far, however, we were not able to confirm this picture by numerical computation.

In case 3, the function $\nu(T^2_\lambda)$ remains bounded away from 1 in $a \le \lambda < b$, but

$$\lim_{\lambda \to b} \sigma(T^2_\lambda) = 1 \ .$$

Typically, the real (for $a \le \lambda \le b$) Floquet exponent $\lambda_{2j}(\lambda)$ meets another real Floquet exponent $\lambda_{ij}(\lambda)$ for $\lambda = b$. If these exponents remain real for $\lambda > b$ by passing each other, then we expect that T^2_λ becomes replaced by a nonsmooth surface for $\lambda > b$, with a cusp in each cross section, the tip of the cusp traveling along $\omega_j(\lambda)$. (Compare the example in [8, p.244].) If $\lambda_{2j}(\lambda)$ and $\lambda_{ij}(\lambda)$ form a complex conjugate pair for $\lambda > b$, then we expect a spiral instead of a cusp (see [6]).

4 A Case Study: Two Coupled Oscillators

We consider the following system of two linearly coupled oscillators

$$\begin{aligned}
\dot{x}_1 &= x_1 + \beta y_1 - (x_1^2 + y_1^2)x_1 - \lambda(x_1 + y_1 - x_2 - y_2) \\
\dot{y}_1 &= -\beta x_1 + y_1 - (x_1^2 + y_1^2)y_1 - \lambda(x_1 + y_1 - x_2 - y_2) \\
\dot{x}_2 &= x_2 + \beta y_2 - (x_2^2 + y_2^2)x_2 + \lambda(x_1 + y_1 - x_2 - y_2) \\
\dot{y}_2 &= -\beta x_2 + y_2 - (x_2^2 + y_2^2)y_2 + \lambda(x_1 + y_1 - x_2 - y_2),
\end{aligned} \tag{30}$$

where $\beta > 0$ and $\lambda \geq 0$. This system has been studied in much detail by Aronson et al. in [1], to which we refer for the derivation and motivation of (30). The authors provide analytical and numerical studies of the bifurcations of fixed points and periodic orbits of the system. We will use many of their results, and will adopt their notation as much as possible, in order to facilitate cross-reading. From now on, we will restrict β: $0 < \beta < 1$, and consider the range of λ: $0 \leq \lambda \leq \beta/2$. This turns out to be sufficient for our purpose.

For $\lambda = 0$, the two oscillators both have the periodic attracting limit cycle $\Omega_0 = \{\cos \beta t, -\sin \beta t\}$, and therefore the full system has an attracting invariant torus $T_0^2 = \Omega_0 \times \Omega_0$. For fixed β sufficiently large ($1/2 < \beta < 1$), numerical studies have been performed attempting to follow (in λ) the branch of invariant 2-tori T_λ^2 emanating from T_0^2. Typical approaches are in [3], [4], [5], [7], [10], [12]. Regardless of the relative merits of each approach, all of them eventually are unable to continue the branch. Loosely speaking, the torus seems to "break down", but a convincing argument of why the branch could not be continued was not provided in any of the above works. It should be said that the inability to continue the branch of tori is not likely to be due to numerical artifacts. In particular, the approach in [4] is a discrete analogue of the Hadamard graph transform technique of [6]. A convergence analysis for this approach is given in [4], assuring that the method will perform continuation of the branch of tori as long as the analytical conditions of [6] are met, and one has sufficient numerical resolution.

Then, somehow, the conditions in [6] become violated at or near the value of λ where we fail to continue the branch of tori. Thus, to understand failure to continue the branch of tori, we have to understand the motion on the torus and towards it. To this end, [1] provides many of the needed tools. Relying on the results in [1], and on our computations, we will present a picture of the torus breakdown, which is consistent with the view presented in the previous section.

Let us begin by noticing that T_0^2 is covered by a 2–parameter family of periodic solutions, each member of which is specified by a pair of initial phase angles. However, since the origin of t is arbitrary, there is only a 1–parameter family of orbits; the initial phase difference can be taken as this parameter. Since all periodic orbits on T_0^2 are degenerate with two Floquet multipliers equal to 1, one cannot expect the whole 1-parameter family to persist for weak coupling. In fact, see [1], only two periodic orbits persist for small coupling λ: ω_0 and ω_π. (The orbit ω_π depends on λ, but we suppress this in our notation.) The orbit ω_0 represents in–phase solutions (synchronized at $\lambda = 0$), and ω_π represents solutions which are out of phase by π at $\lambda = 0$. Finally, for $\lambda = 0$, an easy computation reveals that $\Omega_0(t)$ has Floquet exponents 0 and -2. Therefore, at $\lambda = 0$, we have parallel flow on the torus T_0^2, and a rate of approach of $\mathcal{O}(e^{-2t})$ towards it; as a consequence, we have $\nu = e^{-2} < 1$ and $\sigma = 0$ for T_0^2. Therefore, for small coupling λ, T_0^2 can be continued, and — as proved in [1] — there is phase–locking on the torus: T_λ^2 is made up of ω_0, ω_π, and the unstable manifold of ω_π . As long as phase–locking on the torus persists as λ increases, we can use our previous setup to decide upon the possibility to continue T^2_λ. We have performed several numerical experiments to verify whether or not this was the case and will report on two of them, which have revealed both non-trivial and typical. They correspond to the cases

$$(a) \quad \beta = 0.5165\,, \qquad \text{and} \qquad (b) \quad \beta = 0.55\,.$$

By using the approach in [4], in case (a) we were able to continue the branch of invariant tori up to $\lambda \approx 0.2497$, and in case (b) up to $\lambda \approx 0.26$ (somewhat higher values of λ had been obtained with the approach in [3]). In both cases, our numerical computations showed that indeed phase-locking persisted on the torus up to the indicated values of λ. The corresponding circle map always has zero rotation number, since ω_0 always leads to a fixed point of the circle map. In case (b), the circle map has precisely two fixed points, corresponding to ω_0 and ω_π. In case (a), further fixed points develop in the window of stability of ω_π; see below.

In Figure 1 we show the circle maps for case (b) for the values of $\lambda = 0.01$, 0.04, 0.1, 0.15 and in Figure 2 for $\lambda = 0.2525$; for larger values of λ the picture just got flatter.

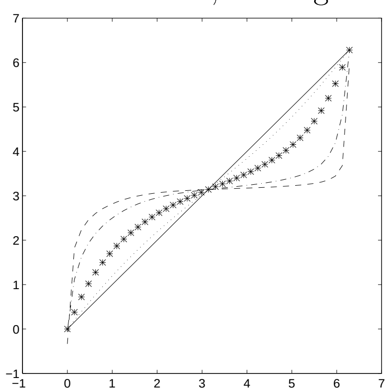

Figure 1.

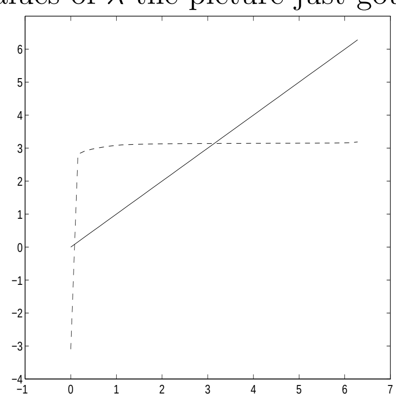

Figure 2.

These figures show the circle maps $\theta_1 \to \theta_1' = \mathcal{C}(\theta_1)$ corresponding to the flow on the torus, see (25), obtained by restricting to the cross section $\theta_2 = \pi$; qualitatively identical results were obtained for different cross sections. The straight line in Figures 1 and 2 represents the line $\theta_1 = \theta_1'$, and the intersections of the other curves with this line are the fixed points of the circle maps. (The stable fixed point $\theta_1 = \pi$ corresponds to ω_0 and the unstable fixed point $\theta_1 = 0 = 2\pi$ corresponds to ω_π.) In case (a), we have obtained qualitatively similar pictures. In Figure 3, we show the flow on the torus for case (a) for the value of $\lambda = 0.2497$.

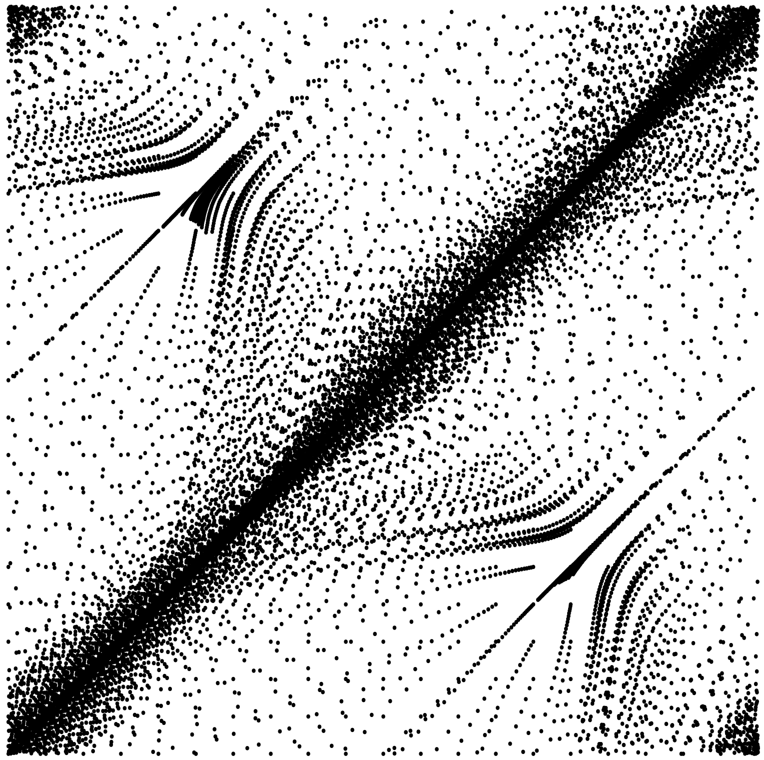

Figure 3.

Having established that the flow on the torus leads to a circle map with zero rotation number, it remains to compute the Floquet exponents of all periodic orbits as functions of λ. We have done this only for ω_0 and ω_π, which turns out to be sufficient for case (b). For case (a), see below. In general, accurate computation of Floquet exponents is difficult, but in the present case great simplifications are provided once again by the work [1].

First of all, it is shown in [1] that ω_0 lives in the "symmetric" plane

$$\mathcal{S} := \{(x_1, y_1, x_2, y_2) : x_1 = x_2,\ y_1 = y_2\}$$

and is stable with respect to perturbations of initial conditions in $\mathcal{S}$. Correspondingly, ω_π lives in the "antisymmetric" plane

$$\mathcal{A} := \{(x_1, y_1, x_2, y_2) : x_1 = -x_2,\ y_1 = -y_2\}$$

and is stable with respect to perturbations of initial conditions in $\mathcal{A}$. Useful analytical expressions for the orbits ω_0 and ω_π are also in [1].

A further important observation in [1] is the following. Linearization of (30) about any solution in $\mathcal{S}$ or $\mathcal{A}$ leads to a four dimensional linear system that decouples into two systems of dimension two. One system determines λ_1 and λ_3, the other determines λ_2 and λ_4. Further, since $\lambda_1 = 0$, one can compute λ_3 analytically from a trace formula. (This is Liouville's theorem, see [8, p.120].) Thus, numerical computations need to be performed for one 2×2 linear system of ODEs only.

For the in–phase orbit ω_0 the following analytical results can be easily obtained.

(i) For $\lambda = 0$ it holds that

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, -2, -2) \ .$$

(ii) For $\lambda > 0$ it holds that $\lambda_3 = -2, \operatorname{Re}\lambda_2 < 0, \operatorname{Re}\lambda_4 < 0$.

(iii) For $\lambda \geq 0$ it holds that (by Liouville's theorem)

$$\lambda_2 + \lambda_4 = -2(1 + \lambda) \ .$$

Result (ii) implies that $\nu(\omega_0)$ remains bounded away from 1 in any bounded interval of existence of T^2_λ. Thus, torus breakdown cannot be due to $\nu(\omega_0)$ approaching 1.

To obtain $\sigma(\omega_0)$, we computed the Floquet exponents λ_2 and λ_4 of ω_0 as functions of λ numerically, with error tolerances at machine precision, and the analytical result $\lambda_2 + \lambda_4 = -2(1 + \lambda)$ was used as an accuracy check. This relation was always fulfilled by more than 10 digits, leading us to conclude that the computation of the Floquet exponents was sufficiently accurate to draw the conclusions given below.

The resulting functions $\lambda \to \sigma(\omega_0)$ are plotted in Figures 4 and 5 for the cases (a) and (b).

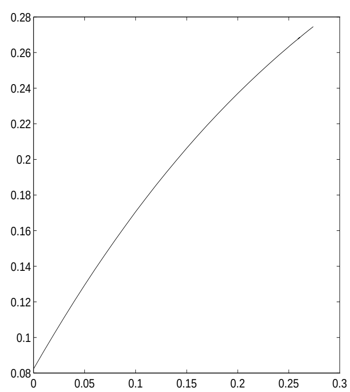

Figure 4.

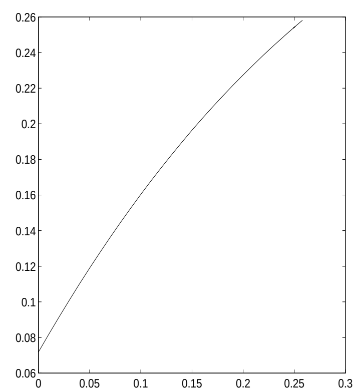

Figure 5.

Quite clearly, in both cases $\sigma(\omega_0)$ stays below 1/4 for the parameter range of interest, and hence we should expect at least a C^4 torus based on the information gathered from ω_0. Thus, we can turn our attention to ω_π.

The following analytical results can be obtained about the orbit ω_π and its Floquet exponents $\lambda_i = \lambda_i(\lambda)$.

(i) For $\lambda = 0$ it holds that

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, -2, -2) \ .$$

(ii) The periodic orbit ω_π exists for $0 \le \lambda < \beta/2$, and in this interval $\lambda_3 = -2$. As $\lambda \to \beta/2$, two fixed points develop, which are saddle–nodes, see [1]. The periodic orbit ω_π approaches the two heteroclinic connections between these two saddle–nodes as $\lambda \to \beta/2$. Thus the period of ω_π tends to infinity as $\lambda \to \beta/2$. Notice that appearance of these fixed points could in principle be responsible for failure to achieve $\sigma < 1$ (as restricted to the orbit ω_π), and hence for failure of the continuation algorithm for the branch of tori. However, our inability to continue the branch of tori occurs before the point $\lambda = \beta/2$, so that some other cause must be responsible for it.

(iii) For $0 \le \lambda < \beta/2$ we have $\lambda_2 + \lambda_4 = 2(4\lambda - 1)$ from Liouville's theorem.

(iv) For small $\lambda > 0$, we have $\lambda_2 > 0 > \lambda_4$. Thus, for $0 < \lambda \le \epsilon(\beta)$, ω_π repels the flow within T^2_λ, but attracts in the directions normal to T^2_λ, see [1].

Again, we computed λ_2 and λ_4 numerically by working with a 2×2 linear system, and the formula $\lambda_2 + \lambda_4 = 2(4\lambda - 1)$ was used as accuracy check. It was fulfilled by more than 10 digits.

Results on ω_π for case (a), $\beta = 0.5165$.

1) For $0 < \lambda < \lambda_s$ we have

$$\lambda_2(\lambda) > 0 > \lambda_4(\lambda) \ ,$$

where $0.24998216527 < \lambda_s < 0.24998216528$, and

$$\lambda_2(\lambda_s) = 0 > \lambda_4(\lambda_s) \ .$$

2) For $\lambda_s < \lambda < 0.25$ (exactly) we have $\lambda_2(\lambda) < 0$ and $\lambda_4(\lambda) < 0$. (In the so–called window of stability, $\lambda_s < \lambda < 0.25$, the orbit ω_π is stable, which motivates the index s.)

3) There is a value λ_b with

$$\lambda_s < \lambda_b < 0.25$$

and

$$\lambda_4(\lambda) < \lambda_2(\lambda) < 0 \quad \text{for} \quad \lambda_s < \lambda < \lambda_b \ ,$$

$$\lambda_4(\lambda) = \lambda_2(\lambda) < 0 \quad \text{for} \quad \lambda = \lambda_b \ .$$

We obtained the bounds $0.24998216530 < \lambda_b < 0.24998216531$. (At $\lambda = \lambda_b$ we expect torus breakdown, which motivates the index b.)

4) For $\lambda_b < \lambda < 0.25$, the Floquet exponents λ_2 and λ_4 form a complex conjugate pair with $\mathrm{Re}\,\lambda_{2,4} < 1$. At $\lambda = 0.25$, we have $\mathrm{Re}\,\lambda_{2,4} = 0$. (This is in agreement with the formula $\lambda_2 + \lambda_4 = 2(4\lambda - 1)$ stated above.)

The Floquet multipliers μ_2 and μ_4 corresponding to the exponents λ_2 and λ_4 are shown in Figures 6 and 7 below.

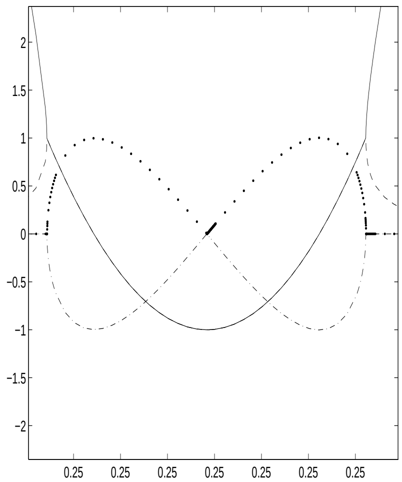

Figure 6.

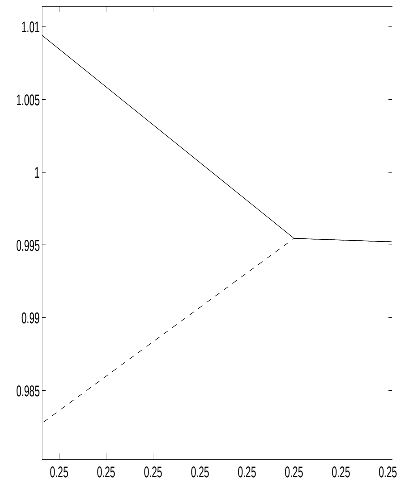

Figure 7.

In Figure 6, we are showing the real and imaginary parts of the multipliers μ_2 and μ_4, as functions of λ, in the region of interest. Figure 7 is an enlargement of Figure 6 showing the multipliers becoming complex conjugate after entering the window of stability; the λ–value at which they meet is our λ_b.

To explain the numerical results, let us tentatively assume that the branch of tori T^2_λ exists for $0 \le \lambda < \lambda_b$. Then the following picture is obtained.

a) For $0 \le \lambda < \lambda_b$ the Floquet exponents $\lambda_{3,4}$ are negative, and bounded away from zero. Consequently we have $\nu(\omega_\pi) \le 1 - \epsilon$ for $0 \le \lambda < \lambda_b$. Thus, $\nu(\omega_\pi)$ cannot be responsible for breakdown.

b) For $0 \le \lambda \le \lambda_s$ we have $\lambda_2 \le 0$, and consequently $\sigma(\omega_\pi) = 0$, because ω_π does not attract the flow within T^2_λ.

c) For $\lambda_s < \lambda < \lambda_b$ we have $\sigma(\omega_\pi) < 1$, but $\sigma(\omega_\pi) \to 1$ as $\lambda \to \lambda_b$ since $\lambda_3 < \lambda_2 = \lambda_4 < 0$ at $\lambda = \lambda_b$. Consequently, we are in case 3 described in the previous section. Since λ_2 and λ_4 move into the complex plane for $\lambda > \lambda_b$, we expect that T^2_λ becomes replaced by a non–smooth surface with a spiral, when $\lambda_b < \lambda < \lambda_b + \epsilon$.

d) At $\lambda = \lambda_s$, where the window of stability for ω_π begins, the Floquet exponent λ_2 passes through zero, as a decreasing function of λ. One expects a pitchfork bifurcation of periodic orbits from the branch ω_π. Also, for $\lambda_s < \lambda < \lambda_b$ the orbits ω_0 and ω_π are both attracting. Consequently, there must exist (at least) two unstable periodic orbits on T^2_λ between ω_0 and ω_π. It is reasonable to believe that these are the orbits generated in the pitchfork bifurcation from ω_π at $\lambda = \lambda_s$. In principle, to confirm our picture, we should compute the Floquet exponents of these orbits, since they could be responsible for torus breakdown before $\lambda = \lambda_b$ is reached. However, since λ_b exceeds λ_s by less than

$4 * 10^{-11}$, this turned out to be too small of a scale for our computing resolution.
Remark: At $\lambda = 0.25$ the pair of complex conjugate Floquet multipliers μ_2, μ_4 leaves the unit circle. One can expect a secondary Hopf bifurcation to occur from the branch ω_π, leading to new invariant tori.

Results on ω_π for case (b), $\beta = 0.55$.

Our computations show that we have

$$\lambda_2(\lambda) > 0 > \lambda_4(\lambda) \quad \text{for} \quad 0 < \lambda < \lambda_b$$

and

$$\lambda_2(\lambda_b) > 0 = \lambda_4(\lambda_b)$$

where $0.260524 < \lambda_b < 0.260525$. Therefore, for $0 \leq \lambda < \lambda_b$ we have $\nu(\omega_\pi) < 1, \sigma(\omega_\pi) = 0$, but $\nu(\omega_\pi) \to 1$ as $\lambda \to \lambda_b$. Thus, case 2 of the previous section prevails.

The approach $\nu(\omega_\pi) \to 1$ as $\lambda \to \lambda_b$ indicates that the torus T^2_λ loses its attractivity as λ approaches λ_b. The Floquet exponent $\lambda_4(\lambda)$ crosses zero at $\lambda = \lambda_b$, as an increasing function of λ. One expects a symmetry breaking pitchfork bifurcation of periodic orbits to occur from the branch ω_π at $\lambda = \lambda_b$.

Let us speculate about the behavior for $\lambda_b < \lambda < \lambda_b + \epsilon$. Taking a suitable cross section $\mathcal{Z}$, the orbit ω_π corresponds to a fixed point Q_π of the Poincaré map P. The Floquet exponent $\lambda_4(\lambda)$ is positive but small, by continuity. The positivity of λ_4 leads to a repulsive direction for Q_π, and this direction is approximately normal to the tori T^2_λ existing for $\lambda < \lambda_b$, in the cross section $\mathcal{Z}$. As long as the repulsion in the direction corresponding to $\lambda_4(\lambda)$ is smaller than the repulsion in the direction corresponding to $\lambda_2(\lambda)$, one can apply an invariant manifold theorem for pseudo–hyperbolic fixed points (see [11]). This theorem establishes the existence of a local one–dimensional invariant manifold (a line) of the Poincaré map in the cross section $\mathcal{Z}$. The line passes through the fixed point Q_π and is tangent to the λ_2–eigenspace at Q_π. If one makes the basic assumption that no major global changes of the flow occur, it is then plausible that this line (the local invariant manifold) can be extended to a global invariant manifold, which connects with the other fixed point Q_0 of P corresponding to ω_0. This all plays in the cross section $\mathcal{Z}$ and invariance means invariance under the Poincaré map P. This invariant curve in $\mathcal{Z}$ then corresponds to an invariant torus in the whole space for the given dynamical system. The torus is non–hyperbolic, since along ω_0 the normal bundle has two attracting dimensions, but along ω_π the normal bundle has one attracting and one repelling dimension.

Further, consider the flow of the Poincaré map P near the fixed point Q_π and ignore the attractive λ_3–direction since it plays no role for our argument. Then P corresponds to a planar map with fixed point Q_π, and there are two eigendirections, corresponding to

λ_2 and to λ_4. Since $0 < \lambda_4 < \lambda_2$, the flow lines of P are tangent to the λ_4–eigenspace. For global reasons, one can expect that these flow lines also can be extended to Q_0, leading to a continuum of invariant sets with a cusp at Q_π. In our numerical computations, for λ near λ_b, we indeed observe approximations to T^2_λ which show such a cusp.

When λ crosses λ_b, the Floquet exponent $\lambda_4(\lambda)$ of the periodic orbit ω_π crosses zero, and one can expect a symmetry breaking pitchfork bifurcation of periodic orbits from the branch ω_π. These orbits lead to two fixed points of P near Q_π for each $\lambda_b < \lambda < \lambda_b + \epsilon$. For global reasons, it is plausible that these fixed points are again connected by invariant curves with Q_0. If this is so, then each of the bifurcating periodic orbits leads to an invariant torus containing ω_0. Thus, if our picture is correct, then for some interval $\lambda_b < \lambda < \lambda_b + \epsilon$ there are three invariant tori, all tangent to each other along ω_0. The torus containing Q_π is non–hyperbolic whereas the other two tori are expected to be hyperbolic and attracting. The torus containing Q_π fulfills symmetry conditions, whereas the other tori break this symmetry. To summarize, at $\lambda = \lambda_b$ we expect a symmetry breaking pitchfork bifurcation of invariant tori.

5 Conclusions

In this paper we have analyzed the simplifications that occur in the Fenichel theory [6] if we have an invariant torus with phase-locking. We have given formulae to obtain the important Lyapunov–type numbers on which this theory rests, and then analyzed their use in understanding the bifurcation behavior for tori associated with a system of coupled oscillators.

In this case, we have brought our understanding *up to* the bifurcation point. There, either attractivity is lost ($\nu \to 1$) or the attractivity within the torus becomes as strong as the attractivity towards the torus ($\sigma \to 1$). In both cases, the theory of [6] no longer applies at and beyond the bifurcation point, and the branch following routine of [4] is also expected to fail. Still, the branch of tori might exist beyond the bifurcation point if the Lyapunov–type number ν becomes 1, the tori becoming pseudo–hyperbolic. At the same time, in a pitchfork bifurcation, new tori might emerge, and one could conceivable follow these numerically.

In fact, the next challenge will be to understand what happens after the bifurcation point. There are many possibilities for torus bifurcations and inherent theoretical difficulties in analyzing them. In general, the lack of a finite dimensional procedure analogous to the Lyapunov-Schmidt reduction, renders a full classification and understanding of all bifurcations of tori extremely difficult. (There is an infinity of possibilities for resonances.)

Consequently, it is hard to imagine ever being able to obtain *general* numerical procedures allowing branch switching for tori. (We grew accustomed to such procedures in the case of periodic orbits.) Nonetheless, there is room for improvement if phase-locking on the tori persists up to the bifurcation point. In this case — via Floquet theory — the bifurcation phenomena become finite dimensional again. This is the case for our model. We plan to consider the resulting numerical aspects in future work.

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