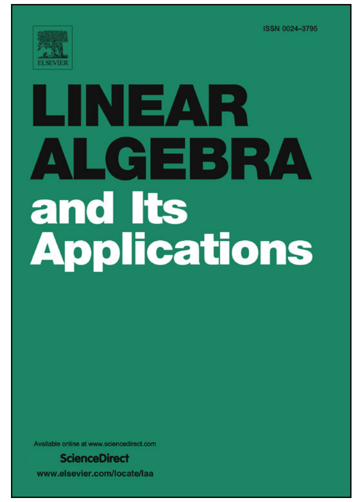
Journal Pre-proof



SVD, joint-MVD, Berry phase, and generic loss of rank for a matrix valued function of 2 parameters

Luca Dieci, Alessandro Pugliese

PII: S0024-3795(24)00311-2
DOI: https://doi.org/10.1016/j.laa.2024.07.021
Reference: LAA 16836

To appear in: *Linear Algebra and its Applications*

Received date: 22 February 2024
Revised date: 23 July 2024
Accepted date: 24 July 2024

Please cite this article as: L. Dieci, A. Pugliese, SVD, joint-MVD, Berry phase, and generic loss of rank for a matrix valued function of 2 parameters, *Linear Algebra Appl.* (2024), doi: https://doi.org/10.1016/j.laa.2024.07.021.

Version of Wednesday 24th July, 2024

SVD, JOINT-MVD, BERRY PHASE, AND GENERIC LOSS OF RANK FOR A MATRIX VALUED FUNCTION OF 2 PARAMETERS

LUCA DIECI AND ALESSANDRO PUGLIESE

ABSTRACT. In this work we consider generic losses of rank for complex valued matrix functions depending on two parameters. We give theoretical results that characterize parameter regions where these losses of rank occur. Our main results consist in showing how following an appropriate smooth SVD along a closed loop it is possible to monitor the Berry phases accrued by the singular vectors to decide if –inside the loop– there are parameter values where a loss of rank takes place. It will be needed to use a new construction of a smooth SVD, which we call the "joint-MVD" (minimum variation decomposition).

Notation. We indicate with Ω an open and simply connected subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. For points in Ω, the symbol ξ will indicate $\xi = (x, y)$ if $\Omega \subset \mathbb{R}^2$ or $\xi = (x, y, z)$ if $\Omega \subset \mathbb{R}^3$. If A is a complex matrix valued function having $k \geq 1$ continuous derivatives on Ω, we write $A \in \mathcal{C}^k(\Omega, \mathbb{C}^{n\times n})$ and call A *smooth*, and (to avoid trivialities), $n \geq 2$ throughout. Unless stated otherwise, we will label singular values of a matrix $A \in \mathbb{C}^{m\times n}$, $m \geq n$, in decreasing order $\sigma_1(A) \geq \ldots \geq \sigma_n(A) \geq 0$, and do the same for the eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n\times n}$: $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. Vectors $v \in \mathbb{R}^n$ are always column vectors. The notation A^* indicates the conjugate transpose of A.

1. Introduction and background

Loss of rank of a matrix $A \in \mathbb{C}^{m\times n}$, $m \geq n$, is an issue of paramount importance in linear algebra, underpinning the concerns of unique solution of a linear system and the equivalent problems of detecting linear independence of a set of vectors and of redundancies in data sets. From a numerical analysis perspective (hence, in finite precision), and ignoring the concerns of computational expense, it is widely accepted that the SVD (singular value decomposition) of A is the most reliable and flexible tool to detect the rank of a matrix, for square as well as rectangular matrices[1]. Our goal in this work is to understand how

1991 *Mathematics Subject Classification.* 15A18, 15A23, 15A99.

Key words and phrases. Matrices depending on parameters, loss of rank, minimum variation decomposition.

We gratefully acknowledge the hospitality of the School of Mathematics of Georgia Tech and the support provided by the University of Bari and by INdAM-GNCS.

[1]E.g., for square A (i.e., $m = n$), criteria based on the smallest singular value of A are much more robust than going through an LU-factorization of A and monitoring the determinant; this is even more true when A is rectangular and one should not form A^*A.

$\mathbb{F}$	codimension rank$(A) = n-1$
$\mathbb{R}$	$m-n+1$
$\mathbb{C}$	$2(m-n)+2$

TABLE 1. Values of the codimension for $A \in \mathbb{F}^{m\times n}$, $m \geq n$, to have rank$(A) = n-1$ in the two cases of $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$.

the SVD can, and should, be used to detect losses of rank for matrix valued functions A smoothly depending on parameters.

Of course, to be interesting and doable, parameter values where a loss of rank occurs should be isolated in parameter space, and moreover we will want to consider problems depending on the minimal possible number of parameters for the phenomenon to occur. A very simple counting of the number of degrees of freedom gives Table 1 for real and complex valued A of size (m,n), $m \geq n$, in order to have rank$(A) = n-1$. The real case tells us that we should expect a loss of rank already when $m = n$ and A depends on one real parameter (after all, this is detected by the scalar relation $\det A = 0$). This case is fairly well understood and already adequately discussed in [2, 3], and see also [4] for numerical methods able to detect and bypass the losses of rank of a smooth function A. The complex case is what we will consider in this work when $m = n$, which has the minimal possible codimension of 2 for a single loss of rank. Again, this setup is easily understood since $\det(A) = 0$ are now two conditions, for the real and imaginary parts of the determinant. For the above reasons, we will consider losses of rank for $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$, where $\Omega \subset \mathbb{R}^2$ is open and simply connected. For a Hermitian function A, however, we will take $\Omega \subset \mathbb{R}^3$, see Section 2.

Finally, we will also require points of loss of rank to be *generic*, a property which we define below. First, recall that a value $v \in \mathbb{R}^n$ is a *regular zero* for a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ if $F(v) = 0$ and the derivative of F at v is invertible.

Definition 1.1. *A point $\xi_0 \in \Omega$ is a* generic point of loss of rank *for $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$ if it is a regular zero for the map*

$$\xi \in \Omega \mapsto \begin{bmatrix} \mathrm{Re}(\det(A(\xi))) \\ \mathrm{Im}(\det(A(\xi))) \end{bmatrix} \in \mathbb{R}^2.$$

Our main contribution in this paper will be to device a topological test for the detection of generic points where a matrix loses rank, a test which also lends itself to a nice algorithmic criterion to detect regions where A loses rank. Our test will be based on an appropriate generalization of the concept of Berry phase[2], by looking at the phase accrued by the singular vectors of a general function A. This will necessitate to find, smoothly, a certain SVD along a closed path, following what we will call the *joint-MVD* along the path. The definition of the joint-MVD is new and to understand it properly we will revisit the Berry phase of a Hermitian eigenproblem and adopt a novel characterization for generic coalescence of eigenvalues of the Hermitian eigenproblem.

[2]ordinarily associated to the eigenvectors of a Hermitian function, see [1] and below

Remark 1.2. *At this stage, we point out that working with the Hermitian eigenproblem for A^*A will not help in finding a useful way to characterize parameter values where a coalescing occurs, regardless of the numerical concerns caused by forming the product. This is already evident in the 1-parameter case for a real square A, whereby through a generic coalescing the function* $\det(A)$ *will change sign, but* $\det(A^TA)$ *will not. Indeed, the net effect of a reformulation like the one above is to turn a generic problem into a non-generic one.*

A plan of the paper is as follows. Section 2 is both a review and a revisitation of the Hermitian eigenproblem and of generic coalescing of eigenvalues and of its relation to the Berry phase accumulated by an eigenvector associated to coalescing eigenvalues. Section 3 is devoted to the joint-MVD and losses of rank, and here we give our main result, Theorem 3.11 and discuss some of its consequences.

2. Hermitian problems: generic coalescing of eigenvalues and the Berry phase

In this section we consider $A \in \mathcal{C}^k(\Omega, \mathbb{C}^{n\times n})$, $k \geq 0$; typically A will be Hermitian and $\Omega \subset \mathbb{R}^3$.

In general, it is well known that a continuous matrix function A taking values in $\mathbb{C}^{n\times n}$ has continuous eigenvalues. Likewise, it is also well known that if A is a smooth ($k \geq 1$) Hermitian matrix valued function on Ω, and $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ are its eigenvalues, then A admits a Schur decomposition $A = U\Lambda U^*$ with smooth factors as long as its eigenvalues are distinct everywhere on Ω; further, in this case, when the eigenvalues appear in decreasing order along the diagonal of Λ, the unitary factor U is unique up to post-multiplication by a diagonal unitary matrix $\Phi = \mathrm{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_n})$, where each α_j is a smooth real valued function defined on Ω. A similar result holds for a block decomposition of A. That is, if A has two (or more) blocks of eigenvalues Λ_1 and Λ_2, of size n_1, n_2, that stay disjoint everywhere, then there is a smooth factorization $A = U \left[\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right] U^*$ where B and C are Hermitian $n_1 \times n_1$ and $n_2 \times n_2$, respectively, and have eigenvalues given by those in Λ_1 and Λ_2 (e.g., see [8, 9]).

As it is well understood, the situation is very different when A has a pair of eigenvalues that coalesce and one can end up with no smoothness at all for the eigendecomposition of A (e.g., see [10]). This is why it is important to be able to locate parameter values where eigenvalues coalesce, and it is further mandatory to focus only on those parameter values where coalescing of eigenvalues occur in a generic way. As we noted in [7], generic coalescing of eigenvalues of a Hermitian function is a co-dimension 3 phenomenon, which we characterize next.

Definition 2.1 (Generic coalescence)**.** *Let $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$ be Hermitian, and Ω be an open subset of $\mathbb{R}^3$. Let $\lambda_j(\xi) = \lambda_j(A(\xi))$ be the continuous eigenvalues of A, with $\lambda_1(\xi) \geq \cdots \geq \lambda_n(\xi)$. Suppose*

$$\lambda_j(\xi) = \lambda_{j+1}(\xi) \text{ if and only if } j = h \text{ and } \xi = \xi_0 \in \Omega.$$

Then, ξ_0 is said to be a generic point of coalescence *for the eigenvalues of A according to the following.*

i) If $n = 2$, write $A(\xi) = \begin{bmatrix} a(\xi) & b(\xi) + ic(\xi) \\ b(\xi) - ic(\xi) & d(\xi) \end{bmatrix}$, where a, b, c, d are real valued functions, and let

$$F(\xi) = \begin{bmatrix} a(\xi) - d(\xi) \\ b(\xi) \\ c(\xi) \end{bmatrix}. \tag{1}$$

Then ξ_0 is a generic point of coalescence for the eigenvalues of A if it is a regular zero for $F(\xi)$.

ii) If $n > 2$, let $R \subset \Omega$ be a pluri-rectangular domain containing ξ_0 in its interior, and let

$$A(\xi) = U(\xi) \begin{bmatrix} P(\xi) & 0 \\ 0 & \Lambda(\xi) \end{bmatrix} U^*(\xi) \tag{2}$$

be a $\mathcal{C}^k$ block Schur decomposition of $A(\xi)$ on R, where

$\Lambda(\xi) = \mathrm{diag}(\lambda_1(\xi), \ldots, \lambda_{h-1}(\xi), \lambda_{h+2}(\xi), \ldots, \lambda_n(\xi)) \in \mathbb{R}^{(n-2)\times(n-2)}$,

$P(\xi) \in \mathbb{C}^{2\times 2}$ *has eigenvalues* $\{\lambda_h(\xi), \lambda_{h+1}(\xi)\}$ *for all* $\xi \in R$.

Then, ξ_0 is a generic point of coalescence for the eigenvalues of A if it is a generic point of coalescence for the eigenvalues of P according to point i) above.

Next, we give an alternative condition to characterize genericity of coalescing eigenvalues of an Hermitian function, in a way that will be conducive to characterize generic losses of rank in Section 3, see Theorem 3.1. The stepping stone is the next result, characterizing a regular zero of a $\mathcal{C}^1$ function.

Lemma 2.2. *Let $F : \mathbb{R}^p \to \mathbb{R}^n$, $p \geq 1$, be $\mathcal{C}^1$. Then $x \in \mathbb{R}^p$ is a regular zero for F if and only if $F(x) = 0$ and*

$$\lim_{t\to 0} \frac{\|F(x + tv)\|_2^2}{t^2} > 0 \text{ for any non-zero } v \in \mathbb{R}^p.$$

Proof. Note that

$$\frac{d}{dt} \|F(x + vt)\|_2^2 = 2F(x + vt)^T DF(x + vt)v,$$

where DF is the derivative of F. Since $F(x) = 0$, we can write

$$\lim_{t\to 0} \frac{2F(x + vt)^T DF(x + vt)v}{2t} = \lim_{t\to 0} \frac{F(x + vt)^T - F(x)^T}{t} DF(x + tv)v$$
$$= \frac{d}{dt} F(x + vt)^T DF(x)v = v^T DF(x)^T DF(x)v = \|DF(x)v\|_2^2 \ .$$

Therefore, upon using the L'Hospital's rule we get:

$$\lim_{t\to 0} \frac{\|F(x + tv)\|_2^2}{t^2} = \lim_{t\to 0} \frac{\frac{d}{dt}\|F(x + vt)\|_2^2}{2t} = \|DF(x)v\|_2^2 \, ,$$

from which the statement of the Lemma follows. □

In Theorem 2.5, we will use Lemma 2.2 applied to the discriminant of a Hermitian function, by relating genericity of coalescence of eigenvalues to local properties of the discriminant.

Definition 2.3 (Discriminant). *Let $A \in \mathbb{C}^{n\times n}$ have eigenvalues $\lambda_1, \dots, \lambda_n$. Then the* discriminant *of A is defined as*

$$\operatorname{discr}(A) = \prod_{\ell<j} (\lambda_j - \lambda_\ell)^2.$$

Remark 2.4. *If A is Hermitian, then* $\operatorname{discr}(A)$ *is real valued and non-negative. Further,* $\operatorname{discr}(A) = 0$ *if and only if A has a pair of repeated eigenvalues. Also,* $\operatorname{discr}(A)$ *is a homogeneous polynomial in the entries of A. Therefore, if A is a smooth function of $\xi \in \mathbb{R}^p$, then so is* $\operatorname{discr}(A)$.

Theorem 2.5. *Let $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$ be Hermitian, and Ω be an open subset of $\mathbb{R}^3$. Let $\lambda_j(\xi) = \lambda_j(A(\xi))$ be the continuous eigenvalues of A, with $\lambda_1(\xi) \geq \cdots \geq \lambda_n(\xi)$. Suppose*

$$\lambda_j(\xi) = \lambda_{j+1}(\xi) \text{ if and only if } j = h \text{ and } \xi = \xi_0 \in \Omega.$$

Then, ξ_0 is a generic point of coalescence for the eigenvalues of A if and only if

$$\lim_{t\to 0} \frac{\operatorname{discr}(A(\xi_0 + tv))}{t^2} > 0, \text{ for any non-zero } v \in \mathbb{R}^3.$$

Proof. Let

$$A(\xi) = U(\xi) \begin{bmatrix} P(\xi) & 0 \\ 0 & \Lambda(\xi) \end{bmatrix} U^*(\xi)$$

for all ξ inside a pluri-rectangle R whose interior contains ξ_0, as in (2). Then, we can write

$$\operatorname{discr}(A(\xi)) = \operatorname{discr}(P(\xi)) \prod_{\substack{j<\ell \\ (j,\ell)\neq(h,h+1)}} (\lambda_j(\xi) - \lambda_\ell(\xi))^2 = \operatorname{discr}(P(\xi)) g(\xi),$$

where g is defined by the above equation. Note that g is a smooth and strictly positive function of ξ. Then, we can write

$$\lim_{t\to 0} \frac{\operatorname{discr}(A(\xi_0 + tv))}{t^2} = g(\xi_0) \lim_{t\to 0} \frac{\operatorname{discr}(P(\xi_0 + tv))}{t^2}, \text{ for any non-zero } v \in \mathbb{R}^3.$$

The result follows from Lemma 2.2 with F as in (1) since, by letting $P = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$, we have $\operatorname{discr} P = (a-d)^2 + b^2 + c^2$. □

In Section 3, we will need the following elementary result, which will be useful to relate a generic coalescing of eigenvalues to a generic loss of rank.

Lemma 2.6. *Let $A \in \mathbb{C}^{n\times n}$ and $\varepsilon \in \mathbb{R}$, and consider the Hermitian matrix function*

$$M = \begin{bmatrix} \varepsilon I & A \\ A^* & -\varepsilon I \end{bmatrix}. \tag{3}$$

Then M has eigenvalues $\pm\sqrt{\sigma_j^2+\varepsilon^2}$, and

$$\operatorname{discr}(M) = 4^n \prod_{j<\ell} (\sigma_j^2 - \sigma_\ell^2)^4 \prod_j (\sigma_j^2 + \varepsilon^2). \tag{4}$$

Proof. A direct computation gives

$$\det(tI - M) = \det(t^2 I - (A^*A + \varepsilon^2 I)),$$

from which the two statements follow. □

We conclude this section with some known results (mostly from [3, 7]) that allow us to lay the groundwork for detecting losses of rank in Section 3.

2.1. **Hermitian 1 parameter, Berry phase, and covering of a sphere.** First, consider the case of Hermitian A smoothly depending on one parameter: $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n\times n})$. In this case, there is a standard way of resolving the degree of non-uniqueness of its Schur decomposition, which in the end leads to the concept of *Berry phase*. We summarize this as follows, see [7].

Theorem 2.7. *Let $A \in \mathcal{C}^k([0,1], \mathbb{C}^{n\times n})$, $k \geq 1$, be Hermitian with distinct eigenvalues $\lambda_1(t) > \ldots > \lambda_n(t)$ for all $t \in [0,1]$. Then, given a Schur decomposition of A at $t=0$, $A(0) = U_0\Lambda_0U_0^*$, there exists a uniquely defined so called* Minimum Variation Decomposition (MVD) *$A(t) = U(t)\Lambda(t)U^*(t)$, $t \in [0,1]$, satisfying $U(0) = U_0$, $\Lambda(0) = \Lambda_0$, where U minimizes the total variation*

$$\operatorname{Vrn}(U) = \int_0^1 \left\| \dot U(t) \right\|_{\mathrm{F}} dt \tag{5}$$

among all possible smooth unitary Schur factors of A over the interval $[0,1]$.

In addition, suppose that A is 1-periodic and of minimal period 1. Then, we have:

i) *U satisfies*

$$U(0)^*U(1) = \operatorname{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_n}), \tag{6}$$

where each $\alpha_j \in (-\pi, \pi]$, $j = 1, \ldots, n$, is the so called Berry phase *associated to λ_j;*

ii) *if $Q \in \mathcal{C}^k([0,1], \mathbb{C}^{n\times n})$ is a 1-periodic unitary Schur factor for A over $[0,1]$, partitioned by columns $Q = [q_1, \cdots, q_n]$, then*

$$\alpha_j = i \int_0^1 q_j^*(t)\dot q_j(t) dt \mod 2\pi, \text{ for all } j = 1, \ldots, n. \tag{7}$$

For our purposes, the relevance of the Berry phase is because of its relation to detection of coalescing eigenvalues in a region of $\mathbb{R}^3$, homotopic to a sphere, as we recall next.

Let $\mathbb{S}_r = \left\{\xi \in \mathbb{R}^3 : \|\xi\|_2 = r\right\}$ be the sphere of radius $r > 0$ centered at the origin in $\mathbb{R}^3$, and consider for $\mathbb{S}_r$ the following parametrization:

$$\begin{cases} x(s,t) = r\sin(\pi s)\cos(2\pi t) \\ y(s,t) = r\sin(\pi s)\sin(2\pi t) \\ z(s,t) = r\cos(\pi s) \end{cases}, \tag{8}$$

with $(s,t) \in [0,1] \times [0,1]$. The sphere $\mathbb{S}_r$ can be thought of as covered by the family of loops $\{X_s\}_{s\in[0,1]}$,

$$X_s(\cdot) = (x(s,\cdot), y(s,\cdot), z(s,\cdot)),$$

as s increases from $s = 0$ to $s = 1$.

Let $A : \xi \in \mathbb{R}^3 \mapsto A(\xi) \in \mathbb{C}^{n\times n}$ be a $\mathcal{C}^k$ Hermitian matrix valued function, and suppose that all eigenvalues of A are distinct on some sphere $\mathbb{S}_r$, $r > 0$. Then, the restriction of A to each loop in $\{X_s\}_{s\in[0,1]}$ is a 1-periodic function and therefore, according to Theorem 2.7, each eigenvector of A continued along X_s accrues a Berry phase $\alpha_j(s)$, $j = 1, \dots, n$. All $\alpha_j(s)$'s can be defined to be continuous functions of s, again see [7]. Moreover, since X_0 and X_1 are just points, the corresponding MVD (see Theorem 2.7) of A must have constant factors, and therefore

$$\begin{aligned} \alpha_j(0) &= 0 \mod 2\pi, \\ \alpha_j(1) &= 0 \mod 2\pi. \end{aligned}$$

Let $\mathbb{B}_r$ be the solid ball $\mathbb{B}_r = \left\{\xi \in \mathbb{R}^3 : \|\xi\|_2 \le r\right\}$, so that $\mathbb{S}_r$ is the boundary of $\mathbb{B}_r$.

Theorem 2.8. (Adapted from [7, Theorems 4.6, 4.8 and 4.10]) *Let $A \in \mathcal{C}^1(\mathbb{B}_r, \mathbb{C}^{n\times n})$ be Hermitian, and let $\lambda_1(\xi), \lambda_2(\xi), \dots, \lambda_n(\xi)$ be its continuous eigenvalues, not necessarily labelled in descending order, and let $\alpha_j(s)$, $s \in [0,1]$, be the continuous Berry phase functions associated to λ_j over $\mathbb{S}_r$, for all $j = 1, \dots, n$.*

(i) *If $\lambda_1(\xi), \lambda_2(\xi), \dots, \lambda_n(\xi)$ are distinct for all $\xi \in \mathbb{S}_r$, then*

$$\sum_{j=1}^{n} \alpha_j(s) = \sum_{j=1}^{n} \alpha_j(0), \text{ for all } s \in [0,1],$$

(ii) *If $\lambda_1(\xi), \lambda_2(\xi), \dots, \lambda_n(\xi)$ are distinct for all $\xi \in \mathbb{B}_r$, then*

$$\alpha_j(1) = \alpha_j(0) \text{ for all } j = 1, \dots n.$$

(iii) *Finally, suppose that $\lambda_j(\xi) = \lambda_k(\xi)$ if and only if $(j,k) = (h_1, h_2)$ and $\xi = 0$, and that $\xi = 0$ is a generic point of coalescence for the eigenvalues of A. Then:*

$$\begin{cases} \alpha_j(1) = \alpha_j(0), & \text{for all } j \neq h_1, h_2 \\ \alpha_{h_1}(1) = \alpha_{h_1}(0) \pm 2\pi, \\ \alpha_{h_2}(1) = \alpha_{h_2}(0) \mp 2\pi. \end{cases}$$

3. SVD, joint-MVD, and generic losses of rank

We are ready to characterize generic losses of rank for a smooth general matrix function of two parameters, $A = A(x, y)$ (see Definition 1.1). In particular, in Theorem 3.1 we will relate a generic loss of rank to the local behavior of the smallest singular value of A.

Theorem 3.1. *A point $\xi_0 \in \Omega$ is a* generic point of loss of rank *for $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$ if and only if*

$$\lim_{t\to 0^+} \frac{\sigma_n(A(\xi_0 + tv))}{t} > 0 \text{ for any non-zero } v \in \mathbb{R}^2.$$

Proof. Consider the following $2n \times 2n$ Hermitian matrix

$$B(\xi) = \begin{bmatrix} 0 & A(\xi) \\ A^*(\xi) & 0 \end{bmatrix}.$$

Its eigenvalues are $\pm\sigma_1(\xi), \dots, \pm\sigma_n(\xi)$, and therefore we can write

$$\det(B(\xi)) = (-1)^n \left|\det(A(\xi))\right|^2 = (-1)^n \prod_j \sigma_j(\xi)^2.$$

On the other hand, we have

$$\det(B(\xi)) = (-1)^n \left(\mathrm{Re}(\det(A(\xi)))^2 + \mathrm{Im}(\det(A(\xi)))^2\right).$$

Let

$$F(\xi) = \begin{bmatrix} \mathrm{Re}(\det(A(\xi))) \\ \mathrm{Im}(\det(A(\xi))) \end{bmatrix} \in \mathbb{R}^2.$$

Then, by virtue of Definition 1.1 and Lemma 2.2, ξ_0 is a generic point of loss of rank if and only if

$$\lim_{t\to 0} \frac{\|F(\xi_0 + tv)\|_2^2}{t^2} > 0 \text{ for any non-zero } v \in \mathbb{R}^2,$$

and this is equivalent to

$$\lim_{t\to 0} \frac{(-1)^n \det(B(\xi_0 + tv))}{t^2} > 0 \text{ for any non-zero } v \in \mathbb{R}^2.$$

Since the ordered singular values are continuous, and we have $\sigma_1 \geq \ldots \geq \sigma_{n-1} > 0$ in a neighborhood of ξ_0, we can write

$$\lim_{t\to 0} \frac{\sigma_n^2(A(\xi_0 + tv))}{t^2} > 0 \text{ for any non-zero } v \in \mathbb{R}^2.$$

Then, the sought statement follows from taking the square root of the limit. □

Next, in Theorem 3.2, we relate a generic loss of rank to the coalescing of the eigenvalues of a Hermitian function of 3 parameters.

Theorem 3.2. *Let $A = A(\xi) \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$, $\Omega \subset \mathbb{R}^2$, have distinct singular values for all $\xi \in \Omega$. Consider*

$$M(\eta) = \begin{bmatrix} \varepsilon I & A(\xi) \\ A^*(\xi) & -\varepsilon I \end{bmatrix},$$

where $\eta = (\xi, \varepsilon) \in \Omega \times \mathbb{R}$. Let ξ_0 be the only point in Ω where A loses rank. Then, $\xi_0 \in \Omega$ is a generic point of loss of rank for A if and only if $\eta_0 = (\xi_0, 0)$ is a generic point of coalescence for the eigenvalues of M.

Proof. We will show that Theorems 3.1 and 2.5 are equivalent through Lemma 2.6. First, note that η_0 is the only point in $\Omega \times \mathbb{R}$ where two eigenvalues of M coalesce. Obviously, the coalescing pair is $\pm\sqrt{\sigma_n^2 + \varepsilon^2}$.

Because of Theorem 3.1, ξ_0 is a generic point of loss of rank for A if and only if

$$\lim_{t\to 0} \frac{\sigma_n^2(A(\xi_0 + tv))}{t^2} > 0 \text{ for any non-zero } v \in \mathbb{R}^2 \tag{9}$$

and this is equivalent to

$$\lim_{t\to 0} \frac{\sigma_n^2(A(\xi_0 + tv)) + \gamma^2 t^2}{t^2} > 0 \text{ for any non-zero } (v, \gamma) \in \mathbb{R}^2 \times \mathbb{R}.$$

Now, equation (4) can be rewritten as

$$\operatorname{discr}(M) = \left(4^n \prod_{j<\ell} (\sigma_j^2 - \sigma_\ell^2)^4 \prod_{j=1}^{n-1} (\sigma_j^2 + \varepsilon^2)\right) (\sigma_n^2 + \varepsilon^2),$$

and the first of the two factors of this product is strictly positive in $\Omega \times \mathbb{R}$. Then, through Lemma 2.6, Equation (9) is equivalent to

$$\lim_{t\to 0} \frac{\operatorname{discr}(M(\eta_0 + tv))}{t^2} > 0, \text{ for any non-zero } v \in \mathbb{R}^3, \tag{10}$$

which, by Theorem 2.5, expresses the fact that η_0 is a generic point of coalescence for the eigenvalues of M. □

We will leverage the relation between losses of rank of A and coalescing of eigenvalues of M of Theorem 3.2, but of course without forming M but working directly with an appropriate SVD of A. The stepping stone will be Theorem 3.4, for whose proof the next Lemma will be handy.

Lemma 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A of multiplicity 1 such that λ^2 is an eigenvalue of A^2 of multiplicity 2. Let $u, v \in \mathbb{C}^n$ span the eigenspace of A^2 associated to the eigenvalue λ^2. Then, the eigenspace of A associated to λ is spanned by a linear combination of u and v.*

Proof. If A had an eigenvector not in $\operatorname{span}\{u, v\}$, then A^2 would have a 3-dimensional invariant subspace associated to λ^2, contradicting the hypothesis on the multiplicity of λ^2. □

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ and $A = U\Sigma V^*$ be a SVD of A, where $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$. Suppose that all singular values of A are distinct and non-zero, and let $\varepsilon \in \mathbb{R}$. Let M be given by (3). Then, M admits the following eigendecomposition*

$$W^* M W = \begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix},$$

where $S = \operatorname{diag}\left(\sqrt{\sigma_1^2+\varepsilon^2}, \dots, \sqrt{\sigma_n^2+\varepsilon^2}\right)$,

$$W = \begin{bmatrix} UC & -UD \\ VD & VC \end{bmatrix},$$

$C = \operatorname{diag}(c_1, \dots, c_n)$, $D = \operatorname{diag}(d_1, \dots, d_n)$, *and the diagonal entries of* C *and* D *are given by*

$$\begin{cases} c_j = \dfrac{1}{\sqrt{2}} \dfrac{\sigma_j}{\sqrt{\sigma_j^2+\varepsilon^2 - \varepsilon\sqrt{\sigma_j^2+\varepsilon^2}}} \\ d_j = \dfrac{1}{\sqrt{2}} \dfrac{\sqrt{\sigma_j^2+\varepsilon^2}-\varepsilon}{\sqrt{\sigma_j^2+\varepsilon^2 - \varepsilon\sqrt{\sigma_j^2+\varepsilon^2}}} \end{cases}, \quad j = 1, \dots, n. \tag{11}$$

Proof. Recall that the eigenvalues of M are $\pm\sqrt{\sigma_j^2+\varepsilon^2}$ and they are all distinct as long as the singular values of A are all distinct and non-zero.

Let $A = U\Sigma V^*$ be a singular value decomposition of A, with $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ and $U = [u_1, \dots, u_n]$, $V = [v_1, \dots, v_n]$, be partitioned by columns. Then, we have that

$$M^2 = \begin{bmatrix} \varepsilon^2 I + AA^* & 0 \\ 0 & \varepsilon^2 I + A^*A \end{bmatrix}, M^2 \begin{bmatrix} u_j \\ 0 \end{bmatrix} = (\sigma_j^2+\varepsilon^2) \begin{bmatrix} u_j \\ 0 \end{bmatrix}, \; M^2 \begin{bmatrix} 0 \\ v_j \end{bmatrix} = (\sigma_j^2+\varepsilon^2) \begin{bmatrix} 0 \\ v_j \end{bmatrix},$$

for all $j = 1, \dots, n$. If $\sigma_1 > \dots > \sigma_n > 0$, it follows from Lemma 3.3 that, for all $j = 1, \dots, n$, there exist $c_j, d_j \in \mathbb{C}$ not both zero such that

$$M \begin{bmatrix} c_j u_j \\ d_j v_j \end{bmatrix} = \sqrt{\sigma_j^2+\varepsilon^2} \begin{bmatrix} c_j u_j \\ d_j v_j \end{bmatrix}. \tag{12}$$

These equations can be rewritten as

$$\begin{bmatrix} (\varepsilon c_j + \sigma_j d_j) u_j \\ (\sigma_j c_j - \varepsilon d_j) v_j \end{bmatrix} = \begin{bmatrix} c_j \sqrt{\sigma_j^2+\varepsilon^2} u_j \\ d_j \sqrt{\sigma_j^2+\varepsilon^2} v_j \end{bmatrix}, \; j = 1, \dots, n, \tag{13}$$

and thus

$$\begin{cases} \left(\varepsilon - \sqrt{\sigma_j^2+\varepsilon^2}\right) c_j + \sigma_j d_j = 0 \\ \sigma_j c_j - \left(\varepsilon + \sqrt{\sigma_j^2+\varepsilon^2}\right) d_j = 0 \end{cases}, \; j = 1, \dots, n. \tag{14}$$

Now, for any $j = 1, \dots, n$, (14) has infinitely many non trivial solutions (c_j, d_j), where c_j and d_j are real valued and cannot have opposite sign. Imposing the normalization conditions $c_j^2 + d_j^2 = 1$, and settling (without loss of generality) on the positive solutions, we get the expressions for c_j and d_j given in (11). Finally, let $C = \operatorname{diag}(c_1, \dots, c_n)$, $D = \operatorname{diag}(d_1, \dots, d_n)$, $S = \operatorname{diag}\left(\sqrt{\sigma_1^2+\varepsilon^2}, \dots, \sqrt{\sigma_n^2+\varepsilon^2}\right)$. Then equations (12) read

$M\begin{bmatrix}UC\\VD\end{bmatrix} = \begin{bmatrix}UC\\VD\end{bmatrix}S$, and by letting $W = \begin{bmatrix}UC & -UD\\VD & VC\end{bmatrix}$, one easily sees that W is unitary and that

$$W^*MW = \begin{bmatrix}S & 0\\0 & -S\end{bmatrix},$$

which is the desired result. □

Remark 3.5. *Each c_j and d_j in* (11) *depends smoothly on ε and on σ_j, and moreover $C(-\varepsilon) = D(\varepsilon)$ and $D(-\varepsilon) = C(\varepsilon)$, while $S(\varepsilon) = S(-\varepsilon)$ because of how S is defined. Now, consider:*

$$M(\varepsilon) = \begin{bmatrix}UC(\varepsilon) & -UD(\varepsilon)\\VD(\varepsilon) & VC(\varepsilon)\end{bmatrix}\begin{bmatrix}S(\varepsilon) & 0\\0 & -S(\varepsilon)\end{bmatrix}\begin{bmatrix}C(\varepsilon)U^* & D(\varepsilon)V^*\\-D(\varepsilon)U^* & C(\varepsilon)V^*\end{bmatrix},$$

and observe that $C(\varepsilon), D(\varepsilon), S(\varepsilon)$ depend smoothly on ε (since the singular values of A are distinct and non zero). Then one has

$$\begin{aligned}M(-\varepsilon) &= \begin{bmatrix}UC(-\varepsilon) & -UD(-\varepsilon)\\VD(-\varepsilon) & VC(-\varepsilon)\end{bmatrix}\begin{bmatrix}S(-\varepsilon) & 0\\0 & -S(-\varepsilon)\end{bmatrix}\begin{bmatrix}C(-\varepsilon)U^* & D(-\varepsilon)V^*\\-D(-\varepsilon)U^* & C(-\varepsilon)V^*\end{bmatrix}\\ &= \begin{bmatrix}UD(\varepsilon) & -UC(\varepsilon)\\VC(\varepsilon) & VD(\varepsilon)\end{bmatrix}\begin{bmatrix}S(\varepsilon) & 0\\0 & -S(\varepsilon)\end{bmatrix}\begin{bmatrix}D(\varepsilon)U^* & C(\varepsilon)V^*\\-C(\varepsilon)U^* & D(\varepsilon)V^*\end{bmatrix}.\end{aligned}$$

The main result of this paper will show that a loss of rank is detected by the phases accumulated by the singular vectors for an appropriate smooth decomposition of $A(x, y)$ along a closed loop. To properly define/understand our result, it is necessary to clarify what "appropriate" means, and this requires looking at how to define/compute a smooth SVD along a closed loop.

3.1. Smooth SVD: 1 parameter.

We will follow the approach of [3]. We have a smooth function A, depending on a real parameter $t \in [0, 1]$ and with distinct singular values for all t. Then, the SVD factors of A are smooth and satisfy the system of differential equations given in (15).

Theorem 3.6 (Adapted from [3])**.** *Let $A \in \mathcal{C}^k([0, 1], \mathbb{C}^{n\times n})$, $k \geq 1$, have distinct singular values $\sigma_1(t) > \ldots > \sigma_n(t) > 0$ for all $t \in [0, 1]$. Then, given any initial singular value decomposition $A(0) = U_0\Sigma_0V_0^*$, there exists a $\mathcal{C}^k$ singular value decomposition $A(t) = U(t)\Sigma(t)V^*(t)$, $t \in [0, 1]$, defined as solution of the following differential-algebraic initial value problem:*

$$\begin{cases}\dot\Sigma = U^*\dot AV - H\Sigma + \Sigma K,\\ \dot U = UH,\\ \dot V = VK,\\ U(0) = U_0, \Sigma(0) = \Sigma_0, V(0) = V_0.\end{cases} \tag{15}$$

The matrix functions H and K are skew-Hermitian on $[0,1]$, with entries given by

$$(16)\qquad \begin{aligned} H_{j\ell} &= \frac{\sigma_\ell (U^*\dot{A}V)_{j\ell} + \sigma_j (U^*\dot{A}V)_{\ell j}}{\sigma_\ell^2 - \sigma_j^2} \\ K_{j\ell} &= \frac{\sigma_\ell (U^*\dot{A}V)_{\ell j} + \sigma_j (U^*\dot{A}V)_{j\ell}}{\sigma_\ell^2 - \sigma_j^2} \end{aligned}$$

for all $j \neq \ell$. The diagonal entries of H and K are real valued and satisfy

$$(17)\qquad H_{jj} - K_{jj} = \frac{\mathrm{Im}\left((U^*\dot{A}V)_{jj}\right)}{\sigma_j}, \quad \text{for all } j = 1, \dots, n.$$

Remark 3.7. *Obviously, the requirement* (17) *does not fully determine the diagonal entries of H and K and we are left with n conditions to impose. To uniquely determine a smooth SVD path, one possibility was suggested in* [3], *simply set $H_{jj} = 0$ (or $K_{jj} = 0$) for all j, and this was shown in* [7] *to be equivalent to selecting the SVD path that minimizes the total variation of U (or V) on $[0,1]$ given in* (5) *and defined originally in* [2]*; we call these the U-MVD or V-MVD, respectively.*

None of the options of Remark 3.7 to select a smooth SVD path would be useful for our purposes of detecting when A loses rank. The correct smooth SVD path for us is identified in the next Theorem.

Theorem 3.8. *Let $A \in \mathcal{C}^k([0,1], \mathbb{C}^{n\times n})$, $k \geq 1$, have distinct singular values $\sigma_1(t) > \dots > \sigma_n(t)$ for all $t \in [0,1]$. Then, given any initial singular value decomposition $A(0) = U_0 \Sigma_0 V_0^*$, there exists a uniquely defined $\mathcal{C}^k$ singular value decomposition $A(t) = U(t)\Sigma(t)V^*(t)$, $t \in [0,1]$, satisfying $U(0) = U_0, \Lambda(0) = \Lambda_0, V(0) = V_0$ and such that the pair (U, V) minimizes the quantity*

$$(18)\qquad \int_0^1 \sqrt{\left\|\dot{U}(t)\right\|_{\mathrm{F}}^2 + \left\|\dot{V}(t)\right\|_{\mathrm{F}}^2}\, dt$$

among all possible smooth unitary SVD factors of A over the interval $[0,1]$.

Proof. Since the Frobenius norm is unitarily invariant, to minimize the quantity in (18) is the same as to minimize

$$\int_0^1 \sqrt{\left\|U^*(t)\dot{U}(t)\right\|_{\mathrm{F}}^2 + \left\|V^*(t)\dot{V}(t)\right\|_{\mathrm{F}}^2}\, dt.$$

We now show that all the singular value decompositions satisfying eqs. (15) to (17) share the same value for

$$\sum_{j\neq\ell} \left|\left(U^*(t)\dot{U}(t)\right)_{j\ell}\right|^2 + \left|\left(V^*(t)\dot{V}(t)\right)_{j\ell}\right|^2, \text{ for all } t \in [0,1].$$

In fact recall that, being the singular values all distinct, each unitary factor (U or V) is unique up to post-multiplication by a smooth diagonal unitary matrix function $\Phi(t) =$

$\operatorname{diag}\left(e^{i\phi_1(t)}, \ldots, e^{i\phi_n(t)}\right)$. Let $U(t)$ and $Q(t) = U(t)\Phi(t)$ be two matrices of left singular vectors satisfying eqs. (15) to (17). A simple computation shows that

$$\left|\left(U^*(t)\dot{U}(t)\right)_{j\ell}\right| = \left|\left(Q^*(t)\dot{Q}(t)\right)_{j\ell}\right|, \text{ for all } j \neq \ell \text{ and } t \in [0,1].$$

Of course, analogus considerations hold for $V(t)$. Therefore, minimizing (18) is equivalent to minimizing

$$\int_0^1 \sqrt{\left\|\operatorname{diag}\left(U^*(t)\dot{U}(t)\right)\right\|_{\mathrm{F}}^2 + \left\|\operatorname{diag}\left(V^*(t)\dot{V}(t)\right)\right\|_{\mathrm{F}}^2}\, dt,$$

that is

$$\int_0^1 \sqrt{\sum_{j=1}^n \left(|H_{jj}(t)|^2 + |K_{jj}(t)|^2\right)}\, dt$$
$$= \int_0^1 \sqrt{\frac{1}{2}\sum_{j=1}^n \left(|H_{jj}(t) - K_{jj}(t)|^2 + |H_{jj}(t) + K_{jj}(t)|^2\right)}\, dt.$$

Since the difference $H_{jj} - K_{jj}$ is prescribed by (17), the minimizing choice is given by

$$H_{jj}(t) + K_{jj}(t) = 0, \text{ for all } t \in [0,1] \text{ and all } j = 1, \ldots, n.$$

Using this, along with eqs. (15) to (17), yields (uniquely) the desired unitary factors U and V. □

Definition 3.9. *Any smooth singular value decomposition of* $A \in \mathcal{C}^k([0,1], \mathbb{C}^{n\times n})$ *satisfying* (18) *will be called a* joint minimum variation decomposition*, or simply "joint-MVD".*

Remarks 3.10.

(i) *If A is Hermitian, then U and V are equal, and unique up to changes of sign of their columns. In this case, the joint-MVD of A is effectively the MVD of Theorem 2.7.*

(ii) *If A is periodic, then –using the joint-MVD– each singular vector acquires a phase factor over one period, and the corresponding left and right singular vectors acquire the same phase. This can be thought of as a generalization of the Berry phase to non-Hermitian matrix functions and in fact it is the same value as the Berry phase accrued by the eigenvectors of* $\begin{bmatrix} 0 & A \\ A* & 0 \end{bmatrix}$.

3.2. **Main Result.** We can finally formulate and prove the main result of this paper, showing that a loss of rank inside a closed loop is detected by the phases accumulated by the singular vectors.

Theorem 3.11. *Let* $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$, $\Omega \subset \mathbb{R}^2$, *have distinct eigenvalues everywhere in* Ω. *Suppose that* $\xi_0 \in \Omega$ *is a generic point of loss of rank for A, and that A has full rank everywhere else in* Ω. *Let* Γ *be a circle centered at* ξ_0 *entirely contained in* Ω, *and let it be parametrized by* $\gamma(t) = \xi_0 + [r\cos(2\pi t), r\sin(2\pi t)]$, $t \in [0,1]$. *Let* $A(\gamma(t)) = U(t)\Sigma(t)V^*(t)$

be the joint-MVD of $A(\gamma(t))$ over the interval $[0,1]$. Let $\beta_j \in (-\pi,\pi]$, $j=1,\dots,n$, be defined through the following equation:

$$U^*(0)U(1) = V^*(0)V(1) = \mathrm{diag}(e^{i\beta_1},\dots,e^{i\beta_n}).$$

Then, we have

$$\sum_{j=1}^{n} \beta_j = \pi \mod 2\pi.$$

Proof. Without loss of generality, we may take $\xi_0 = (0,0)$. Consider the Hermitian matrix function of three parameters

$$M(x,y,z) = \begin{bmatrix} zI & A(x,y) \\ A^*(x,y) & -zI \end{bmatrix}, (x,y) \in \Omega \text{ and } z \in \mathbb{R}.$$

Because of Theorem 3.4, for all (x,y,z), M has the Schur eigendecomposition

$$W(x,y,z)^* M(x,y,z) W(x,y,z) = \begin{bmatrix} S(x,y,z) & 0 \\ 0 & -S(x,y,z) \end{bmatrix},$$

where

$$S(x,y,z) = \mathrm{diag}\left(\sqrt{\sigma_1(x,y)^2+z^2},\dots,\sqrt{\sigma_n^2(x,y)+z^2}\right),$$
$$W(x,y,z) = \begin{bmatrix} U(x,y)C(x,y,z) & -U(x,y)D(x,y,z) \\ V(x,y)D(x,y,z) & V(x,y)C(x,y,z) \end{bmatrix}.$$

Let us label the eigenvalues $\lambda_1,\dots,\lambda_{2n}$ of M in the same order as they appear along the diagonal of $\begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix}$, that is so that $\lambda_j = \sqrt{\sigma_j^2+z^2}$, $\lambda_{n+j} = -\sqrt{\sigma_j^2+z^2}$, for $j=1,\dots,n$.

Consider the sphere $\mathbb{S}_r$ parametrized by $(s,t) \in [0,1]\times[0,1]$ as in (8). It follows from Theorem 3.2 that M and its eigenvalues satisfy the hypotheses of Theorem 2.8-(i,ii) on $\mathbb{B}_r$, with the pair of eigenvalues that undergoes coalescence being (λ_n,λ_{2n}). For each $j=1,\dots,2n$, let $\alpha_j(s)$, $s\in[0,1]$, be the continuous Berry phase function associated to λ_j over $\mathbb{S}_r$, where we choose $\alpha_j(0)=0$ for all $j=1,\dots,2n-1$, and $\alpha_{2n}(0)=2\pi$. Then, Theorem 2.8 gives

$$\sum_{j=1}^{2n} \alpha_j(s) = 2\pi, \text{ for all } s, \quad \text{and} \quad \begin{cases} \alpha_j(1)=0, & \text{for all } j\neq n,2n, \\ \alpha_n(1)=2\pi, \\ \alpha_{2n}(1)=0. \end{cases} \tag{19}$$

Now, let

$$\varphi(s) := \sum_{j=1}^{n} \alpha_j(s),\ s\in[0,1].$$

From the conclusion of Remark 3.5, and through (7) and (8), we have that $\alpha_{j+n}(s) = \alpha_j(1-s)$, for all $j=1,\dots,n$ and all $s\in[0,1]$. Therefore, using (19), we have $\varphi(s) + \varphi(1-s) = 2\pi$ for all $s\in[0,1]$, and in particular $\varphi\left(\frac{1}{2}\right) = \pi$. Finally, note that, taking

$s = \widehat{s} := \frac{1}{2}$, we have $z(\widehat{s}, t) = 0$, and therefore

$$M(x(\widehat{s}, t), y(\widehat{s}, t), z(\widehat{s}, t)) = \begin{bmatrix} 0 & A(\gamma(t)) \\ A^*(\gamma(t)) & 0 \end{bmatrix}, \quad \text{for all } t \in [0, 1].$$

From the previous identity, and from the definition of joint-MVD, it follows that, for each $j = 1, \dots, n$, the phase β_j accrued by the j-th singular vectors of the joint-MVD of $A(\gamma(\cdot))$ along Γ coincides with the Berry phase $\alpha_j(\widehat{s})$ accrued by the eigenvector of M associated to the eigenvalue λ_j of M along the circle $(x(\widehat{s}, t), y(\widehat{s}, t), 0)$. This concludes the proof. □

Theorem 3.12. *With the same notation and hypotheses of Theorem 3.11 above, except for A being full rank everywhere in Ω, we have:*

$$\sum_{j=1}^{n} \beta_j = 0 \mod 2\pi.$$

Proof. The proof follows the same line as that of Theorem 3.11, using Theorem 2.8-(i,iii). □

Finally, we have the following result that follows at once from Theorems 3.11 and 3.12.

Corollary 3.13. *Let $A \in \mathcal{C}^1(\Omega, \mathbb{C}^{n\times n})$, $\Omega \subset \mathbb{R}^2$, have distinct singular values everywhere on Ω. Let Γ be a circle entirely contained in Ω, and $\beta_1, \dots, \beta_n$ be the phases accrued by the singular vectors of the joint-MVD of A along Γ. Suppose*

$$\sum_{j=1}^{n} \beta_j = \pi \mod 2\pi.$$

Then, there exists a point of loss of rank for A inside the region enclosed by Γ.

Remark 3.14. *Corollary 3.13 was formulated relative to a circle Γ. However, this is not necessary. Using the same homotopy argument we adopted in* [6]*, it is enough to have Γ be a simple closed curve.*

4. Examples

The first example illustrates how Corollary 3.13 is used to infer the presence of a point of loss of rank inside the region bounded by a closed curve.

Example 4.1. *For this example we have explicitly constructed the 4×4 matrix function in* (20)*, where the entries of the matrices M_0 to M_2 are pseudorandom numbers uniformly distributed in* $[-1, 1]$*, rounded up to the nearest hundredth:*

$$A(x, y) = M_0 + xM_1 + yM_2\,,\ (x, y) \in \mathbb{R}^2, \tag{20}$$

	Γ_1	Γ_2
β_1	-0.0206	$+0.7928$
β_2	-2.5572	-0.7905
β_3	$+2.6831$	$+0.0004$
β_4	$+3.0363$	-0.0027
$\sum_{j=1}^{4} \beta_j$	$+3.1416$	$+0.0000$

TABLE 2. Numerically computed phases for Example 4.1.

with

$$M_0 = \begin{bmatrix} 0.03+0.23i & -0.71+0.16i & -0.43-0.53i & 0.90-0.40i \\ 0.11-0.96i & -0.84-0.16i & 0.26+0.72i & -0.20-0.62i \\ 0.40-0.98i & 0.96+0.12i & 0.19-0.25i & 0.33+0.26i \\ -0.76-0.46i & 0.37-0.22i & -0.50-0.28i & 0.41-0.76i \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -0.02-0.79i & 0.48-0.76i & -0.63+0.45i & 0.42-0.03i \\ -0.22+0.19i & 0.99+0.05i & -0.80+0.77i & -0.48-0.73i \\ 0.34+0.83i & 0.94+0.48i & -0.77-0.74i & -0.39-0.26i \\ 0.10-0.56i & -0.63-0.14i & 0.94+0.00i & 0.09-0.64i \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.13+0.40i & 0.43-0.83i & -0.53+0.15i & 0.59+0.59i \\ 0.07+0.29i & 0.85-0.07i & -0.08-0.86i & 0.75-0.16i \\ -0.53+0.26i & 0.17+0.61i & 0.53-0.16i & 0.86-0.83i \\ 0.95+0.56i & 0.23+0.69i & -0.29-0.99i & -0.46+0.19i \end{bmatrix}.$$

A visual inspection of the surface $\sigma_4(A(x,y))$ suggests the presence of a point of loss of rank for A, see Figure 1. So, we have numerically computed the joint-MVD of A along two circles, a larger one Γ_1 enclosing the supposed point of loss of rank, and a smaller one Γ_2 not enclosing the point, see again Figure 1. Thus, we have computed $\beta_1, \dots, \beta_4$, i.e. the phases accrued by the four columns of the unitary factors of the joint-MVD of A along the two circles. Table 2 shows the computed phases, rounded up to the fourth decimal place. The outcome of the computation clearly confirms the expectation of Theorems 3.11 and 3.12: there is a loss of rank inside Γ_1, but not inside Γ_2. All the computations have been performed using the `MATLAB` *function* `complexSvdCont` *available at* https://www.mathworks.com/matlabcentral/ fileexchange/160876-smooth-singular-value-decomp-of-complex-matrix-function. *The* `MATLAB` *code follows closely the algorithm proposed in* [5] *for the computation of the MVD of Hermitian matrix functions. In a nutshell, it performs a variable-stepsize continuation of the smooth SVD of a matrix function of one parameter where, at each step, a suitable Procrustes problem is solved to minimize the quantity in* (18).

The next example shows that, in general, by taking the MVD of just U and/or V will not produce a phase accumulation revealing the presence of a generic point of loss of rank, and that taking the joint MVD is necessary.

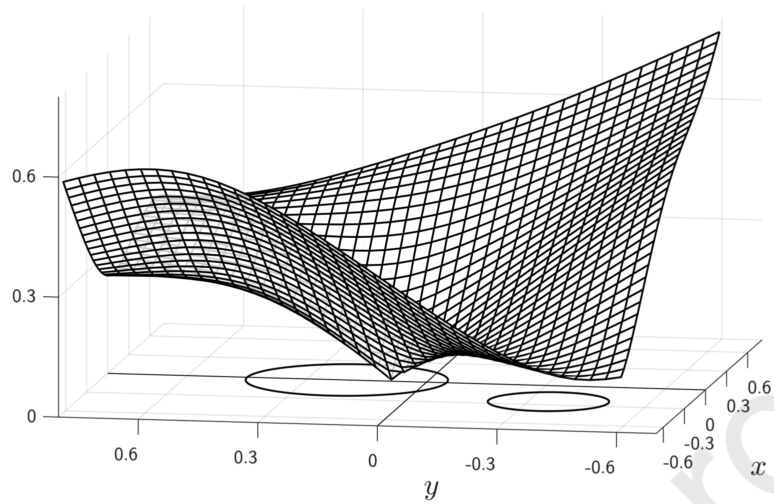


FIGURE 1. Reference figure for Example 1: graph of the smallest singular value $\sigma_4(A(x,y))$ and two circles Γ_1 and Γ_2 in parameters' space, Γ_1 being the largest one on the left.

Example 4.2. *Let*

$$A(x,y) = \begin{bmatrix} 1 & 1 \\ 0 & x - iy \end{bmatrix}, (x,y) \in \mathbb{R}^2,$$

and Γ_r *be the circle parametrized by*

$$\gamma(t) = r[\cos(2\pi t), \sin(2\pi t)], r > 0, t \in [0,1].$$

Notice that A *is full rank everywhere except at the origin* $(0,0)$*, where it has a generic point of loss of rank. By direct computation, it is easy to obtain that:*

i) *letting* β_1, β_2 *be the phases accrued by, respectively, the first and second column of the unitary factors of the joint-MVD of* A *along* Γ_r*, one has*

$$\beta_1(r) = \pi \frac{r^2}{\left(\frac{1}{2}\left(\sqrt{r^4+4} - r^2\right) + 1\right)^2 + r^2}, \quad \beta_2(r) = \pi - \beta_1(r), \quad \textit{for all } r \geq 0,$$

so that $\sum \beta_j = \pi$ *and, in agreement with Corollary 3.13, the point of loss of rank at the origin is properly detected;*

ii) *letting* α_1, α_2 *be the phases accrued by the columns of the unitary factors of the* U*-MVD of* A *along* Γ_r*, one has*

$$\alpha_1(r) = 2\beta_1(r) \mod 2\pi, \quad \alpha_2(r) = -\alpha_1(r), \quad \textit{for all } r \geq 0;$$

iii) *no phase is accrued by the columns of the unitary factors of the* V*-MVD of* A *along* Γ_r*, for any value of* r*.*

In other words, the MVD of just U and/or V does not produce a phase accumulation revealing the presence of a generic point of loss of rank, whereas the joint MVD does. Moreover, to detect the presence of a generic point of loss of rank, one has to consider the phase accrued by all singular vectors, as looking solely at the singular vectors corresponding to the smallest singular value is not sufficient.

5. Conclusions

In this work we considered how to detect generic losses of rank for a complex valued matrix function A smoothly depending on two parameters. We proved that a generic loss of rank is detected by monitoring the (Berry) phases accrued by the singular vectors of an appropriate SVD along closed loops in parameter space containing the value where the loss of rank occurs. To achieve this, we had to introduce a novel smooth path of the SVD, which we called "joint MVD" (joint minimum variation decomposition) for the singular vectors. We complemented our theoretical results with numerical examples both to locate losses of rank, and to show the necessity of considering the joint MVD.

Although we have considered a single loss of rank within a planar region Ω, it should be possible to deal with the case of multiple (generic) losses of rank in a similar way to how we dealt with multiple coalescing eigenvalues in [6, Section 3], and we plan to look at this problem in the future.

References

[1] M.V. Berry, Quantal phase factors accompanying adiabatic changes. *Proc. Roy. Soc. Lond.*, A392:45–57, 1984.

[2] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols. Numerical computation of an analytic singular value decomposition by a matrix valued function. *Numer. Math.*, 60:1–40, 1991.

[3] L. Dieci, T. Eirola. On smooth decompositions of matrices. *SIAM J. Math. Anal.*, 20(3):800–819, 1999.

[4] L. Dieci, M.G. Gasparo, A. Papini. Path following by SVD. *Lecture Notes in Computer Science* 3994, Springer–Verlag, 677–684, 2006.

[5] L. Dieci, A. Papini, A. Pugliese. Approximating Coalescing Points for Eigenvalues of Hermitian Matrices of Three Parameters. *SIAM J. Matrix Anal. Appl.*, 34(2):519–541, 2013.

[6] L. Dieci, A. Pugliese, Two-parameter SVD: Coalescing singular values and periodicity. *SIAM J. Matrix Analysis*, Volume 31-2:375–403, 2009.

[7] L. Dieci, A. Pugliese, Hermitian matrices depending on three parameters: Coalescing eigenvalues. *Linear Algebra Appl.*, Volume 436-11:4120–4142, 2012.

[8] H. Gingold. A method of global blockdiagonalization for matrix-valued functions. *SIAM J. Math. Anal.*, 9-6:1076–1082, 1978.

[9] P.F. Hsieh and Y. Sibuya. A global analysis of matrices of functions of several variables. *J. Math. Anal. Appl.*, 14:332–340, 1966.

[10] Tosio Kato. *A short introduction to perturbation theory for linear operators*. Springer-Verlag, New York, 1982.

[11] J. B. Keller. Multiple eigenvalues. *Linear Algebra Appl.*, 429(8):2209–2220, 2008.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 U.S.A.
Email address: dieci@math.gatech.edu

Dept. of Mathematics, University of Bari "Aldo Moro", Via Orabona 4, 70125 Bari, Italy
Email address: alessandro.pugliese@uniba.it