TWO-PARAMETER SVD: COALESCING SINGULAR VALUES AND PERIODICITY

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Abstract. We consider matrix valued functions of two parameters in a simply connected region Ω. We propose a new criterion to detect when such functions have coalescing singular values. For *generic* coalescings, the singular values come together in a "double cone"-like intersection. We relate the existence of any such singularity to the periodic structure of the orthogonal factors in the singular value decomposition of the one-parameter matrix function obtained restricting to closed loops in Ω. Our theoretical result is very amenable to approximate numerically the location of the singularities.

1. Introduction

Matrix valued functions depending on parameters appear pervasively in many applications, notably in the study of dynamical systems. For example, stability and bifurcation studies of fixed points and/or periodic orbits typically reduce to studying spectral properties of parameter dependent matrices (Jacobians or monodromy matrices). It is natural to inquire whether a smooth dependence on the parameters in the entries of these matrices reflect in smooth dependence of the eigenvalues and eigenvectors as well. This is a classical problem, extensively studied; e.g., see the description in [13], and [4, 5, 6, 8, 12, 14, 15] for a sample of relevant references, chiefly for functions depending on one parameter.

In this work, we will consider this problem for matrices depending on two parameters, and will specifically study the symmetric eigenproblem and the singular value decomposition (SVD) of matrix valued functions in two variables.

Notation. With $\Omega \subset \mathbb{R}^2$ we indicate an open simply connected planar region, and $x = (x_1, x_2)$ will be coordinates in Ω. At times we will need to restrict to closed rectangular regions in $\mathbb{R}^2$, which will be indicated by R: $R = \{(x_1, x_2) \in \mathbb{R}^2 \ a \leq x_1 \leq b \ , c \leq x_2 \leq d \ \}$ (and always $a < b, \ c < d$). We write $A \in \mathcal{C}^e(\Omega, \ \mathbb{R}^{n\times n})$ to indicate a $\mathcal{C}^e$ matrix valued function; typically $e \geq 1$ and finite, but also the analytic case of $A \in \mathcal{C}^\omega$ (A is an analytic function) is of interest. We write $A \in \mathcal{C}^e_\tau(\mathbb{R}, \mathbb{R}^{n\times n})$ to indicate a $\mathcal{C}^e$ and τ-periodic matrix valued function of the real variable t; τ is always assumed to be the minimal positive period.

As previously remarked, several existing works on smooth decomposition of matrix valued functions are concerned with the case of $A(t)$, $t \in [a, b] \subseteq \mathbb{R}$. This case is well understood, and results exist both for the analytic and smooth case. For example, it has been known for a long time (see [13]) that symmetric and analytic matrix valued functions admit analytic Schur decompositions;

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similarly, analytic matrix valued functions (of one real variable) admit analytic singular value decompositions, SVD (see [4]). Results on smoothness of the factors in the case of $\mathcal{C}^e$, $e \geq 0$, functions of one real variable are given in [6].

Remark 1.1. In the standard linear algebra setting (see [9]), the SVD of a matrix $A \in \mathbb{R}^{n\times n}$ is the decomposition $A = U\Sigma V^T$, where $U, V \in \mathbb{R}^{n\times n}$ are orthogonal and Σ is a diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$, with the singular values σ_i, $i = 1, \dots, n$, such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. However, in the previously cited works concerned with smoothness of the factors, in order to retain smooth (or analytic) factors, the singular values must be allowed to change ordering when they coalesce and to change sign when they become 0. More properly, one obtains a *signed smooth SVD*. In the present work, unless otherwise stated, when we talk about the SVD we will always mean the one with ordered and non-negative singular values.

Another problem which has received some attention, for 1-parameter functions, is concerned with functions which are periodic in the parameter; see [15] for early work, and [5] for more recent work related to the symmetric eigendecomposition and SVD. For example, in [5] it was proved that "If the singular values of a function $A \in \mathcal{C}^e_1(\mathbb{R}, \mathbb{R}^{n\times n})$, $e \geq 0$, remain distinct over one period, then A admits a $\mathcal{C}^e$ SVD, where the (signed) singular values are 1-periodic, while the orthogonal functions U and V are either 1-periodic or 2-periodic". There was no indication in [5], and see also [15], of when these functions effectively had period 1 or 2. Perhaps surprisingly, we will realize in the present work that this fact can be understood by studying functions of 2 parameters: More precisely, by thinking of the 1-parameter periodic function $A \in \mathcal{C}^e_1(\mathbb{R}, \mathbb{R}^{n\times n})$ (with distinct singular values) as a function on a closed loop in two-parameter space, one will have 1-periodic factors U, V, if the singular values of the underlying two-parameter function do not coalesce inside the loop. [In fact, more is true: Depending on whether and how singular values coalesce inside the loop, the factors will have period 1 or 2; see below].

Results for functions depending on several parameters are much less encouraging than in the 1-parameter case. For example, analytic symmetric functions in 2-parameters do not even admit differentiable eigenvalues.

Example 1.2. [13] Let $(x_1, x_2) \in \mathbb{R}^2$, and consider the function

$$A(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix} .$$

The eigenvalues are $\pm\sqrt{x_1^2 + x_2^2}$ which are not differentiable at $(0, 0)$. Of course, the problem is the lack of global differentiability at the origin, where both eigenvalues are 0. We notice that viewing $A(x_1, x_2)$ as a function of one parameter (holding the other frozen), renders analytic eigenvalues.

Example 1.2 notwithstanding, an important and useful tool in our investigation of matrix valued functions in two-parameters is the "block-diagonalization" result of Hsieh and Sibuya, and Gingold, [12] and [8]. This result allows to focus locally, in the neighborhood of coalescing eigenvalues or singular values.

Theorem 1.3 (Block-Diagonalization)**.** *Let R be a closed rectangular region in $\mathbb{R}^2$. Suppose that the eigenvalues of $A \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$, $e \geq 0$, can be labeled so that they belong to two disjoint sets for all $x \in R$: $\lambda_1(x), \dots, \lambda_p(x)$ in $\Lambda_1(x)$ and $\lambda_{p+1}(x), \dots, \lambda_n(x)$ in $\Lambda_2(x)$, $\Lambda_1(x) \cap \Lambda_2(x) = \emptyset$, $\forall x \in R$. Further, assume that complex conjugate eigenvalues are put in the same group. Then, there exists $M \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$, invertible, such that*

$$M^{-1}(x)A(x)M(x) =: S = \begin{bmatrix} S_1(x) & 0 \\ 0 & S_2(x) \end{bmatrix} , \ \forall x \in R ,$$

where $S_1 \in \mathcal{C}^e(R, \mathbb{R}^{p\times p})$*,* $S_2 \in \mathcal{C}^e(R, \mathbb{R}^{(n-p)\times(n-p)})$*, and the eigenvalues of* $S_i(x)$ *are those in* $\Lambda_i(x)$*, for all* $x \in R$ *and* $i = 1, 2$*.*

We notice that the function M is not unique, in general.

A useful consequence of Theorem 1.3 is the following result, which we state as a Corollary.

Corollary 1.4. *Let* $M \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ *be the function of which in Theorem 1.3. Let* Γ *be a simple closed curve in* R*, parametrized as a* $\mathcal{C}^p$ $(p \geq 0)$ *function* γ *in the variable* t*, so that the function* $\gamma:\ t \in \mathbb{R} \to R$ *is* $\mathcal{C}^p$ *and* 1*-periodic. Let* $m = \min(e, p)$*, and let* M_γ *be the* $\mathcal{C}^m$ *function* $M(\gamma(t))$*,* $t \in \mathbb{R}$*. Then, we have* $M_\gamma \in \mathcal{C}^m_1(\mathbb{R}, \mathbb{R}^{n\times n})$*.*

Proof. The result is immediate upon considering the composite function M_γ and using the stated smoothness and periodicity results. □

Remarks 1.5.

(i) Naturally, Theorem 1.3 can be refined to any number of disjoint groups of eigenvalues. In the limiting case, if all eigenvalues are distinct, Theorem 1.3 says that we can find a $\mathcal{C}^e$ basis of real eigenvectors, and S is diagonal, with (2×2) bumps along the diagonal corresponding to complex conjugate eigenvalues.

(ii) In case A is also symmetric, which will be the case of interest for us, then M can be taken orthogonal and S stays symmetric. In this case, if the eigenvalues are distinct in R, then the orthogonal function M has diagonalized A: One has a $\mathcal{C}^e$ Schur decomposition. Naturally, in this case, M is essentially unique: the degree of non-uniqueness is solely determined by the ordering of the eigenvalues and the signs of the columns of M. Further, Corollary 1.4 will give M_γ as a 1-periodic function.

The following result is another useful consequence of Theorem 1.3, and clarifies the degree of non-uniqueness in the block-diagonalization result for symmetric functions. We state it as a Theorem (see [5, Lemma 2.2 and Theorem 2.3]).

Theorem 1.6. *Let* $A \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$*,* $e \geq 0$*, be symmetric. Suppose that the eigenvalues of* A *can be labeled so that they belong to two disjoint sets for all* $x \in R$*:* $\lambda_1(x), \dots, \lambda_p(x)$ *in* $\Lambda_1(x)$ *and* $\lambda_{p+1}(x), \dots, \lambda_n(x)$ *in* $\Lambda_2(x)$*,* $\Lambda_1(x) \cap \Lambda_2(x) = \emptyset$ *,* $\forall x \in R$*.*

Consider $Q \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$*, orthogonal, guaranteed to exist by Theorem 1.3, such that*

$$Q^T(x)A(x)Q(x) =: S = \begin{bmatrix} S_1(x) & 0 \\ 0 & S_2(x) \end{bmatrix} \ , \ \forall x \in R \ ,$$

where $S_1(x) = S_1^T(x) \in \mathbb{R}^{p\times p}$*,* $S_2(x) = S_2^T(x) \in \mathbb{R}^{(n-p)\times(n-p)}$*, for all* $x \in R$*, and the eigenvalues of* $S_i(x)$ *are those in* $\Lambda_i(x)$*, for all* $x \in R$ *and* $i = 1, 2$*.*

Then, any other $\mathcal{C}^e$ *orthogonal function* U *achieving a block diagonalization of* A *in two groups corresponding to the eigenvalues in* Λ_1, Λ_2*, must have the form:*

$$U(x) \ = \ Q(x) \begin{bmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{bmatrix} \ ,$$

where the $\mathcal{C}^e$ *functions* V_1 *and* V_2 *are orthogonal, taking values in* $\mathbb{R}^{p\times p}$ *and* $\mathbb{R}^{(n-p)\times(n-p)}$ *respectively.*

Proof. The proof is a consequence of the fact that orthonormal bases of invariant subspaces associated to disjoint group of eigenvalues of symmetric functions are mutually orthogonal. In other words, if U is an orthogonal function having achieved the block reduction of which in the theorem, writing $Q = [Q_1, Q_2]$ and $U = [U_1, U_2]$ with the partitioning inherited by the dimensions of the eigenvalues' groups, we must have that U_i and Q_i, $i = 1, 2$, span the same subspace and are thus related as stated. □

Again, Theorem 1.6 can be refined to any number of disjoint blocks of eigenvalues as well.

To properly characterize periodicity of decompositions, we will also need the Lemma below, which is a generalization of an example given by Sibuya in [15]. Its main point is that, for a given continuous 1-periodic matrix function A, continuous 1-periodic and 2-periodic decompositions cannot coexist. We stress that this is true since we take the factors to be real-valued. In a somewhat more general fashion than the previous cases, we give the result for a general eigendecomposition. (A similar result holds for the SVD as well; the extension is straightforward and, therefore, omitted.)

Lemma 1.7. *Let $A \in \mathcal{C}^0_1(\mathbb{R}, \mathbb{R}^{n\times n})$ be such that*

$$A(t) = S(t)\Lambda(t)S^{-1}(t) \ , \ \forall t \ ,$$

with:

(i) $\Lambda \in \mathcal{C}^0_1(\mathbb{R}, \mathbb{R}^{n\times n})$ *diagonal with distinct diagonal entries,*
(ii) $S \in \mathcal{C}^0_2(\mathbb{R}, \mathbb{R}^{n\times n})$ *invertible, with*

$$S(t+1) = S(t)\ D \ , \ \forall t \in \mathbb{R} \ ,$$

where D is diagonal with $D_{ii} = \pm 1$ for all i, but $D \neq I_n$.

Then, there is no matrix function T such that:

$T \in \mathcal{C}^0_1(\mathbb{R}, \mathbb{R}^{n\times n})$*, invertible, and* $T(t)A(t)T^{-1}(t) = \Lambda(t)$ *for all* $t \in \mathbb{R}$.

Proof. Assume that such a function T exist. Then we have $\Lambda(t)T(t)S(t) = T(t)S(t)\Lambda(t)$, for all $t \in \mathbb{R}$. Since $\Lambda(t)$ has distinct diagonal entries for all $t \in \mathbb{R}$, it follows that $B(t) := T(t)S(t)$ is diagonal for all $t \in \mathbb{R}$. Let us denote its diagonal entries by $b_1(t), \dots, b_n(t)$, for all t. Being $B(t)$ nonsingular for all $t \in \mathbb{R}$, the scalar functions b_i never vanish. On the other hand, we have $B(t+1) = T(t+1)S(t+1) = T(t)S(t)D = B(t)D$, for all $t \in \mathbb{R}$, hence there must exist an index i for which $b_i(t+1) = -b_i(t)$ for all $t \in \mathbb{R}$. This is a contradiction, because the functions b_i are continuous for $t \in \mathbb{R}$. □

In the remainder of this introduction we give some results from differential geometry which are needed to justify our later assumptions. We refer to [11] for background on these concepts. First of all, we recall the *Regular Value Theorem* (see [11, Theorem 3.2, p.22]) in a simplified form sufficient for our purposes.

Theorem 1.8. [Regular Value Theorem] *If $f : \Omega \to \mathbb{R}$ is a $\mathcal{C}^e$ map, $e \geq 1$, and 0 is a regular value of f, then $f^{-1}(0)$ is a $\mathcal{C}^e$ sub-manifold of Ω. In other words, the set $\{x \in \Omega \ : \ f(x) = 0\}$ is a (collection of) $\mathcal{C}^e$ curve(s) in Ω.*

Remark 1.9. The assumption that 0 be a regular value of f translates into the requirement that at values $x \in \Omega$, where $f(x) = 0$, we have $\nabla f(x) \neq 0$. We notice that (in case $e \geq 2$) the Morse-Sard theorem (see [11, Theorem 1.3, p.69]) tells us that we should expect that 0 be a regular value. Also, notice that Theorem 1.8 allows for $f^{-1}(0)$ to be given by the union of several non-intersecting curves, which may be either closed or extend forever; of course, given ξ_0 such that $f(\xi_0) = 0$, there is a unique curve through ξ_0.

We will also need the concept of *transversal intersection* of two smooth curves.

Definition 1.10. When two $\mathcal{C}^e$, $e \geq 1$, curves in $\Omega \subseteq \mathbb{R}^2$ intersect each other at a point ξ_0, we call the intersection transversal if the tangent vectors to the two curves at ξ_0 are not multiple of each other.

In the cases of interest to us, transversal intersection can be characterized as follows.

We have two $\mathcal{C}^e$, $e \geq 1$, functions from Ω to $\mathbb{R}$, f_1 and f_2, which vanish at a point $\xi_0 \in \Omega$: $f_1(\xi_0) = f_2(\xi_0) = 0$, and $\nabla f_1(\xi_0) \neq 0$, $\nabla f_2(\xi_0) \neq 0$. Assume that 0 is a regular value for f_1 and f_2. Thus, by Theorem 1.8, there are two well defined smooth curves through ξ_0: $\{x \in \Omega \ : \ f_1(x) = 0\}$ and $\{x \in \Omega \ : \ f_2(x) = 0\}$. Transversal intersection of these two curves at ξ_0 means that the Jacobian $\begin{bmatrix} \nabla f_1(x) \\ \nabla f_2(x) \end{bmatrix}$ is invertible at ξ_0: $\begin{bmatrix} \nabla f_1(\xi_0) \\ \nabla f_2(\xi_0) \end{bmatrix}$ invertible. As it is well known, transversal intersection of two smooth curves is a *generic* property. Thus, if two curves intersect non-transversally, there is an arbitrarily small perturbation for which the two curves will intersect transversally (or not at all).

Genericity (again, see [11] for background on this concept) is an important situation to consider when dealing with matrix valued functions depending on parameters. Indeed, consideration of generic cases allows us to get rid of pathological –and somewhat artificial– behavior of a specific function A, and to focus on the behavior of the entire class of $\mathcal{C}^e$ functions depending on parameters. In our results in Sections 2 and 3, we will make assumptions which reflect precisely generic properties of $\mathcal{C}^e$ matrix valued functions A.

At the same time, what makes a certain property generic depends very much on the dimension of the embedding space (i.e., on the number of parameters). For example, having a coalescing pair of eigenvalues of a symmetric matrix valued function, or a coalescing pair of singular values for a general matrix valued function, is a *co-dimension* 2 phenomenon (e.g., see [6]). This means that for a smooth matrix valued function depending on one parameter, having a pair of coalescing singular values is not a generic property (and hence it should not be observed in 1-parameter functions). But, it is a generic property for smooth functions depending on two parameters, in which case the generic property is that singular values will coalesce at isolated values of the parameters; this fact had already been observed by von Neumann and E. Wigner almost 80 years ago, [16]. From this, since singular values describe surfaces, we can justify the name of *conical* intersections which is often given to this situation, with the consequent lack of differentiability at the intersection point. We elucidate this last point in the Example 1.11 below for a symmetric function (similar considerations apply for coalescing singular values of a general function).

Example 1.11. Consider the symmetric matrix valued function below

$$P(x) \ = \ \begin{bmatrix} a(x) & b(x) \\ b(x) & d(x) \end{bmatrix} \ , \quad x = (x_1, x_2) \in \Omega \, ,$$

and let ξ_0 be a point where the eigenvalues of P coalesce. In other words, since the (continuous) eigenvalues have the form

$$\lambda_{\pm}(x) \ = \ \frac{a(x) + d(x)}{2} \pm \frac{1}{2}\sqrt{(a(x) - d(x))^2 + 4b(x)^2} \, ,$$

at ξ_0 we have $(a-d)(\xi_0) = 0$ and $b(\xi_0) = 0$. As we previously remarked, assume that $\nabla(a-d)(\xi_0) \neq 0$ and $\nabla b(\xi_0) \neq 0$ and that the Jacobian $\begin{bmatrix} \nabla(a-d) \\ \nabla b \end{bmatrix}_{\xi_0}$ is nonsingular: Transversal intersection at ξ_0.

We write the eigenvalues in the form $\lambda_{\pm} = \frac{(a+d)(x)}{2} \pm \frac{1}{2}\sqrt{h(x)}$ and expand the function $h(x)$ at ξ_0. We get

$$h(x) = h(\xi_0) + \nabla h(\xi_0)(x - \xi_0) + \frac{1}{2}(x - \xi_0)^T H(\xi_0)(x - \xi_0) + \dots \, ,$$

and a simple computation gives $h(\xi_0) = 0$, $\nabla h(\xi_0) = 0$, and

$$H(\xi_0) = 2 \begin{bmatrix} [(a-d)_{x_1}]^2 + 4(b_{x_1})^2 & (a-d)_{x_1}(a-d)_{x_2} + 4b_{x_1}b_{x_2} \\ (a-d)_{x_1}(a-d)_{x_2} + 4b_{x_1}b_{x_2} & [(a-d)_{x_2}]^2 + 4(b_{x_2})^2 \end{bmatrix}_{\xi_0} \, .$$

Therefore, $\det H(\xi_0) = 16\big[b_{x_1}(a-d)_{x_2} - b_{x_2}(a-d)_{x_1}\big]^2(\xi_0)$ and thus $\det H(\xi_0) = 0$ if and only if at ξ_0:

$$\begin{bmatrix}(a-d)_{x_1} & (a-d)_{x_2}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}b_{x_1}\\ b_{x_2}\end{bmatrix} = 0\,,$$

that is $\nabla(a-d)(\xi_0)$ is parallel to $\nabla b(\xi_0)$, and thus the Jacobian $\begin{bmatrix}\nabla(a-d)\\ \nabla b\end{bmatrix}_{\xi_0}$ will be singular, which we have excluded. We can conclude that $H(\xi_0)$ is positive definite, and that in the vicinity of ξ_0 the eigenvalues have the form $\lambda_{\pm} = \frac{(a+d)(x)}{2} \pm \frac{1}{2}\sqrt{\|z\|^2 + O(\|x-\xi_0\|^4)}$, where $z = H^{1/2}(\xi_0)\ (x-\xi_0)$. As a consequence, the eigenvalues behave like $\pm$ the norm $\|z\|$, which is a double cone.

In other words, the phenomenon which we presented in Example 1.2 is typical for functions of two parameters: We should always expect a lack of global differentiability at parameter values where singular values coalesce. Indeed, the situation insofar as smoothness of decompositions for 2-parameter matrix valued functions is somewhat discouraging, though coalescing of eigenvalues for symmetric functions of 2-parameters has been appreciated as an interesting and important phenomenon for a long time (see [16]). Indeed, although smoothness of decompositions cannot be expected, the problem of locating the parameters' values where eigenvalues (or singular values) coalesce is important, and it is this mindframe which we have adopted in the present work: To detect where singular values coalesce.

Remark 1.12. We stress that saying that "having a pair of eigenvalues of a symmetric function is a co-dimension 2 phenomenon" implies that generically a pair of eigenvalues of a symmetric function in two parameters will coalesce, but we have no idea of what will be the value of the eigenvalues when they coalesce. If we demanded, say, to have a pair of singular values coalescing and equal to a fixed value $\bar{\sigma} > 0$ this would have been a co-dimension 3 phenomenon, and generically one will need a function depending on 3 parameters in order to observe it. Perhaps surprisingly, to have a pair of coalescing singular values equal to 0 is actually a co-dimension 4 phenomenon, not 3; see Remark 2.16.

Our main result in this work is condensed in the final theorem we give, Theorem 3.13. This theorem is analogous to the Intermediate Value Theorem in Calculus, where one is able to infer that a continuous function has a zero in a closed interval whenever the function has opposite signs at the endpoints of this interval. In this work, we will have a two-dimensional analog of this result, retaining the original topological flavor: Oversimplifying it, Theorem 3.13 says that there is a coalescing inside a curve if a certain continuous function changes sign along the curve. We stress that the curve does not need to be a small loop near a coalescing point. The function which one needs to monitor is obtained by the continuous orthogonal factors in the decomposition of the two-parameter matrix function restricted to the curve. We will prove Theorem 3.13 proceeding by examining simpler cases first, and then showing that more complicated scenarios can be brought back to these simpler cases. Finally, we remark here that our theoretical result lends very nicely to a numerical approach to locate coalescing points, as we have reported in [7].

Remark 1.13. We recently became aware of some interesting results in the physical chemistry literature, which are ultimately connected to coalescing eigenvalues of matrices depending on parameters. For example, in [10], the authors observed that the eigenvectors of the 2×2 matrix function of Example 1.2 change sign under smooth continuation on a circle around the origin. This result was later generalized in [3], where the author introduced the so-called *geometric phase*, a phase $e^{i\gamma(C)}$ that is acquired by an eigenvector of a quantal system as the parameter-dependent (generally complex) Hamiltonian $\widehat{H}$ of the system is slowly varied around a closed circuit C in parameter space. This geometric phase reduces to the change of sign in [10] when the Hamiltonian is symmetric real valued and the circuit C encircles (and lies close to) a coalescing point for the eigenvalues (called a degeneracy of the spectrum of $\widehat{H}$ in [3]). A key result contained in our paper, namely Theorem 2.2 (but also 2.8), has a similar flavor to this result of [10]. However, we should point out that this specific result of ours (as well as all other results we give, for eigenvalues and singular values) are given in the most general and rigorous form, rest on solid mathematical proofs, and provide localization results for coalescing points which are of global nature, that is not valid only near the coalescing point. On the other hand, the arguments contained in [10], and [3] (and see also [2]), insofar as their application to coalescing eigenvalues of symmetric matrices, appear to be local and case specific.

2. One pair coalescing

We first tackle the case of coalescing eigenvalues of symmetric matrix functions. The situation of coalescing singular values of a general function will follow from this case.

Our first result is when $A(x)$ takes values in $\mathbb{R}^{2\times 2}$. This "simpler" case will turn out to be the stepping stone for the general case. Moreover, this 2×2 case already presents the key essential features, so we will be able to present the fundamental ideas in a transparent setting. First, we have this elementary result.

Lemma 2.1. *Let* $P = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ *be a given symmetric matrix. Then, this matrix has two identical eigenvalues if and only if* $P = \lambda I$.

Proof. Since the eigenvalues are given by $\frac{1}{2}\big(a + d \pm \sqrt{(a-d)^2 + 4b^2}\big)$, they coincide if and only if $a - d = 0$ and $b = 0$. □

We now have

Theorem 2.2 (Symmetric 2×2 case)**.** *Consider* $P \in \mathcal{C}^e(\Omega, \mathbb{R}^{2\times 2})$, $e \geq 1$, *symmetric. For all* $x \in \Omega$, *write*

$$P(x) = \begin{bmatrix} a(x) & b(x) \\ b(x) & d(x) \end{bmatrix} ,$$

and let λ_1 *and* λ_2 *be its two continuous eigenvalues, labeled so that* $\lambda_1(x) \geq \lambda_2(x)$ *for all* x *in* Ω. *Assume that there exists a unique point* $\xi_0 \in \Omega$ *where the eigenvalues coincide:* $\lambda_1(\xi_0) = \lambda_2(\xi_0)$. *Consider the* $\mathcal{C}^e$ *function* $F:\ \Omega \to \mathbb{R}^2$ *given by*

$$F(x) = \begin{bmatrix} a(x) - d(x) \\ b(x) \end{bmatrix} , \tag{2.1}$$

and assume that 0 *is a regular value for both functions* $a-d$ *and* b. *Then, consider the two* $\mathcal{C}^e$ *curves* Γ_1 *and* Γ_2 *through* ξ_0, *given by the zero-set of the components of* F*:* $\Gamma_1 = \{x \in \Omega :\ a(x) - d(x) = 0\}$, $\Gamma_2 = \{x \in \Omega :\ b(x) = 0\}$. *Assume that* Γ_1 *and* Γ_2 *intersect transversally at* ξ_0.

Let Γ *be a simple closed curve*[1] *enclosing the point* ξ_0, *and let it be parametrized as a* $\mathcal{C}^p$ $(p \geq 0)$ *function* γ *in the variable* t, *so that the function* $\gamma :\ t \in \mathbb{R} \to \Omega$ *is* $\mathcal{C}^p$ *and* 1*-periodic. Let* $m = \min(e, p)$, *and let* P_γ *be the* $\mathcal{C}^m$ *function* $P(\gamma(t))$, $t \in \mathbb{R}$. *Then, for all* $t \in \mathbb{R}$, $P_\gamma(t)$ *has the eigendecomposition*

$$P_\gamma(t) = V_\gamma(t)\Lambda_\gamma(t)V_\gamma^T(t)$$

such that:

(i) $\Lambda_\gamma \in \mathcal{C}_1^m(\mathbb{R}, \mathbb{R}^{2\times 2})$ *and diagonal:* $\Lambda_\gamma(t) = \begin{bmatrix} \lambda_1(\gamma(t)) & 0 \\ 0 & \lambda_2(\gamma(t)) \end{bmatrix}$ *for all* $t \in \mathbb{R}$;

(ii) $V_\gamma \in \mathcal{C}_2^m(\mathbb{R}, \mathbb{R}^{2\times 2})$ *real orthogonal, and* $V_\gamma(t+1) = -V_\gamma(t)$ *for all* $t \in \mathbb{R}$.

[1] Also called a Jordan curve

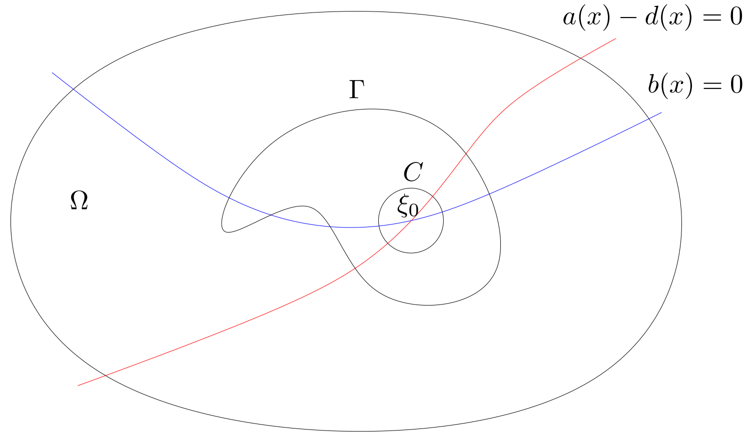

FIGURE 1. Transversal Intersection at ξ_0

Proof. We remark that, because of Lemma 2.1,

$$\lambda_1(x) = \lambda_2(x) \iff F(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

and, by hypothesis, ξ_0 is the unique root of $F(x)$ in Ω.

The proof will go as follows. First, we will prove that the asserted results hold true along a small circle C around ξ_0. Then, we will show that the same periodicity results hold when we continuously deform C into Γ.

Let C be a circle centered at ξ_0, of radius small enough so that the circle goes through each of Γ_1 and Γ_2 at exactly two distinct points (see Figure 1). This is possible since Γ_1 and Γ_2 intersect transversally at ξ_0. Further, let C be parametrized by a continuous 1-periodic function[2] ρ, $\rho(t+1) = \rho(t)$, for all $t \in \mathbb{R}$.

Now, let us consider $P_\rho(t) = P(\rho(t))$, $t \in \mathbb{R}$, which is continuous and 1-periodic, with distinct eigenvalues, so that its eigenvalues are necessarily 1-periodic. Also, the eigenvectors of $P_\rho(t)$, call them $V_\rho(t)$, are uniquely determined (for each t) up to sign. The first column of $V_\rho(\cdot)$ is given by an orthonormal basis for $\mathrm{Ker}(P_\rho(\cdot) - \lambda_1(\rho(\cdot))I)$. The function $(P_\rho(\cdot) - \lambda_1(\rho(\cdot))I)$ is a continuous 1-periodic constant rank matrix, for which the existence of a continuous periodic Schur decomposition, possibly with 2-periodic eigenvectors, was proved in [5]. We can therefore write:

$$\begin{bmatrix} P_\rho(t) - \lambda_1(\rho(t))I \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = (\lambda_2(\rho(t)) - \lambda_1(\rho(t))) \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} , \quad t \in \mathbb{R} \tag{2.2}$$

where the continuous scalar valued functions u_1, u_2 have period either 1 or 2. Let us use the notation $a_\rho(\cdot) = a(\rho(\cdot))$, and same for $b(\cdot)$ and $d(\cdot)$. Then, after a simple computation, equation (2.2) yields:

$$\begin{cases} \left(\frac{a_\rho(t) - d_\rho(t)}{2} + \frac{1}{2}\sqrt{(a_\rho(t) - d_\rho(t))^2 + 4b_\rho(t)^2} \right) u_1(t) = -b_\rho(t)\, u_2(t) \\ \left(\frac{d_\rho(t) - a_\rho(t)}{2} + \frac{1}{2}\sqrt{(a_\rho(t) - d_\rho(t))^2 + 4b_\rho(t)^2} \right) u_2(t) = -b_\rho(t)\, u_1(t) \end{cases} , \quad t \in \mathbb{R} .$$

From these last equations, it follows that u_1 (respectively, u_2) changes sign if and only if b_ρ goes through zero and $(a_\rho - d_\rho) > 0$ (respectively, $(a_\rho - d_\rho) < 0$). Therefore, each of the two functions u_1

[2] this parametrization does not need to be specified yet; see Remark 2.5

and u_2 changes sign only once over any interval of length 1. But no continuous function of period 1 can change sign only once over one period. Therefore, u_1 and u_2 must be 2-periodic functions and the periodicity assertions of the theorem follow relatively to the curve $\rho(t)$: $V_\rho \in \mathcal{C}_2(\mathbb{R}, \mathbb{R}^{2\times 2})$, where

$$V_\rho(t) = \begin{bmatrix} -u_2(t) & u_1(t) \\ u_1(t) & u_2(t) \end{bmatrix} \ , \ \text{or} \ \ V_\rho(t) = \begin{bmatrix} u_2(t) & u_1(t) \\ -u_1(t) & u_2(t) \end{bmatrix} \ , \ \ t \in \mathbb{R} \ .$$

Now, consider the case of Γ. As before, we notice that the existence of a $\mathcal{C}^m$ eigendecomposition for P along Γ, possibly with 2-periodic orthogonal factors, is known (e.g., see [5]). What we need to show is that the periodicity results that hold along the circle C also hold along the curve Γ.

Let us consider a homotopy $h(s,t)$, $(s,t) \in [0,1]\times[0,1]$ satisfying the following properties[3]:

(1) $h(s,t)$ is continuous in $(s,t) \in [0,1]\times[0,1]$ and for all $t \in [0,1]$: $h(0,t) = \rho(t)$, $h(1,t) = \gamma(t)$, and for any $s \in [0,1]$: $h(s,0) = h(s,1)$;

(2) h continuously (in s) deforms $\rho(\cdot)$ into $\gamma(\cdot)$ in such a way that –for each given $s \in (0,1)$– the simple closed curves $h(s,\cdot)$ are always contained in the interior of Γ and in the exterior of C.

Let us consider the function $P(h(s,t))$, $(s,t) \in [0,1]\times[0,1]$. $P(h(s,t))$ is continuous with distinct eigenvalues for all $(s,t) \in [0,1]\times[0,1]$. Therefore, by Theorem 1.3, we can write $P(h(s,t)) = V(s,t)\Lambda(s,t)V^T(s,t)$, where $\Lambda(s,t)$ and $V(s,t)$ are continuous, $\Lambda(s,t)$ is diagonal, and $V(s,t) = \begin{bmatrix} v_1(s,t) & v_2(s,t) \end{bmatrix}$ is real orthogonal. Let $f_k(s) = v_k^T(s,0)\, v_k(s,1)$, for $k = 1,2$. Since $h(s,0) = h(s,1)$ for all $s \in [0,1]$, we have that f_1 and f_2 take values in $\{-1,1\}$. Being continuous, they have to be constant over $[0,1]$. Therefore, we must have $f_1(0) = f_1(1) = -1$ and $f_2(0) = f_2(1) = -1$. This means that $V_\gamma(1) = -V_\gamma(0)$ and thus we obtain the asserted periodicity properties for V_γ. $\square$

Remark 2.3. The eigendecomposition of P_γ in Theorem 2.2 is essentially unique, within the class of $\mathcal{C}^m$ Schur decompositions. The degree of non-uniqueness is given by the ordering of the eigenvalues and by the signs of the columns of V_γ. In particular, for $\mathcal{C}^m$–decompositions, the statement about periodicity holds unchanged.

Remark 2.4. We stress that the assumption of transversality for the curves Γ_1 and Γ_2 at ξ_0 is generic. With abuse of notation, we say that ξ_0 is a *generic coalescing point of eigenvalues* when Γ_1 and Γ_2 intersect transversally at ξ_0.

Remark 2.5. The existence of the homotopy $h(s,t)$ is non-trivial[4]. Let us call G the open, simply connected, region enclosed by Γ, and let us call D the closed disk with boundary C. Then, using [1, Theorem 14.25], there exists a homeomorphism $f : \overline{G - D} \to A$, where A is the annulus $A = \{x \in \mathbb{R}^2 \ : \ 1 \le \|x\| \le 2\}$, so that f maps C and Γ into the circles of radius 1 and 2 in A, respectively, call them C_1 and C_2. Now, we let $f(\gamma(t))$ be a parametrization of C_2. We can think of this parametrization as having the form $2\begin{bmatrix} \cos(\alpha(t)) \\ \sin(\alpha(t)) \end{bmatrix}$ with α continuous and 1-periodic. We also parametrize C_1 as $\begin{bmatrix} \cos(\alpha(t)) \\ \sin(\alpha(t)) \end{bmatrix}$. Finally, we consider the pullback via f^{-1} of the linear homotopy in the annulus (i.e., $(1-s)C_1 + sC_2$), which will give the sought homotopy $h(s,t)$: $h(s,t) = f^{-1}\big((1+s)\begin{bmatrix} \cos(\alpha(t)) \\ \sin(\alpha(t)) \end{bmatrix}\big)$. [Notice that we can now qualify what parametrization we should choose for the circle C in the proof of Theorem 2.2: It is $\rho(t) = f^{-1}\big(\begin{bmatrix} \cos(\alpha(t)) \\ \sin(\alpha(t)) \end{bmatrix}\big)$.]

With the aid of Theorem 2.2, and of the reduction to block-diagonal form, we can now tackle the case of symmetric functions in $\mathbb{R}^{n\times n}$, whose eigenvalues coalesce at a unique point ξ_0. We will need the following definition (see Remark 2.4).

[3] See remark 2.5

[4] We thank Prof. M. Ghomi of Georgia Tech for clarifying to us the existence of this homotopy

Definition 2.6. Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$ be a symmetric function with continuous eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$, $x \in \Omega$, satisfying

$$\lambda_1(x) > \lambda_2(x) > \dots > \lambda_k(x) \geq \lambda_{k+1}(x) > \dots > \lambda_n(x) \ , \ \forall x \in \Omega \ ,$$

and

$$\lambda_k(x) = \lambda_{k+1}(x) \Longleftrightarrow x = \xi_0 \in \Omega \ .$$

Let R be a rectangular region $R \subseteq \Omega$ containing ξ_0 in its interior. Moreover, let

(1) $Q \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ be a $\mathcal{C}^e$ orthogonal function achieving the reduction of which in Theorem 1.3 (and see Remarks 1.5)

$$Q^T(x)A(x)Q(x) = \begin{bmatrix} \Lambda_1(x) & 0 & 0 \\ 0 & P(x) & 0 \\ 0 & 0 & \Lambda_2(x) \end{bmatrix} \ , \ \forall x \in R \ ,$$

where $\Lambda_1 \in \mathcal{C}^e(R, \mathbb{R}^{(k-1)\times(k-1)})$ and $\Lambda_2 \in \mathcal{C}^e(R, \mathbb{R}^{(n-k-1)\times(n-k-1)})$, such that, for all $x \in R$, $\Lambda_1(x) = \mathrm{diag}(\lambda_1(x), \dots, \lambda_{k-1}(x))$, and $\Lambda_2(x) = \mathrm{diag}(\lambda_{k+2}(x), \dots, \lambda_n(x))$. Moreover, $P \in \mathcal{C}^e(R, \mathbb{R}^{2\times 2})$ is symmetric, and $P(x)$ has eigenvalues $\lambda_k(x), \lambda_{k+1}(x)$ for each $x \in R$;

(2) for all $x \in R$, write $P(x) = \begin{bmatrix} a(x) & b(x) \\ b(x) & d(x) \end{bmatrix}$, and define the function F and the curves Γ_1 and Γ_2 as in Theorem 2.2, for $x \in R$.

Then, we call ξ_0 a *generic coalescing point of eigenvalues* in Ω, if the curves Γ_1 and Γ_2 intersect transversally at ξ_0.

Before proceeding, we must justify Definition 2.6, since the $\mathcal{C}^e$ function Q which does the block diagonalization of which in Theorem 1.3 is not unique. In particular, the function P in Definition 2.6 is not unique, and we need to argue that any other possible function would have shared the same property. This is true, and it is the content of the following result.

Theorem 2.7. *Let A be as in Definition 2.6 and let Q be a fixed $\mathcal{C}^e$ orthogonal function achieving the reduction of which in Theorem 1.3:*

$$Q^T(x)A(x)Q(x) = \begin{bmatrix} \Lambda_1(x) & 0 & 0 \\ 0 & P(x) & 0 \\ 0 & 0 & \Lambda_2(x) \end{bmatrix} \ , \ \forall x \in R \ ,$$

as in Definition 2.6. Write $P(x) = \begin{bmatrix} a(x) & b(x) \\ b(x) & d(x) \end{bmatrix}$. Let $U \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ be another orthogonal function achieving a similar block reduction:

$$U^T(x)A(x)U(x) = \begin{bmatrix} \Lambda_1(x) & 0 & 0 \\ 0 & \widetilde{P}(x) & 0 \\ 0 & 0 & \Lambda_2(x) \end{bmatrix} \ , \ \forall x \in R \ .$$

Write $\widetilde{P}(x) = \begin{bmatrix} \widetilde{a}(x) & \widetilde{b}(x) \\ \widetilde{b}(x) & \widetilde{d}(x) \end{bmatrix}$.

Then, if the curves $\Gamma_1 = \{x \in R : (a-d)(x) = 0\}$ and $\Gamma_2 = \{x \in R : b(x) = 0\}$ intersect transversally at ξ_0, so do the curves $\widetilde{\Gamma}_1 = \{x \in R : (\widetilde{a}-\widetilde{d})(x) = 0\}$ and $\widetilde{\Gamma}_2 = \{x \in R : \widetilde{b}(x) = 0\}$.

Proof. Obviously, P and $\widetilde{P}$ are similar, and thus have same eigenvalues, which coalesce only at ξ_0, so that $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ intersect at ξ_0, and only there. By using Theorem 1.6, we know that $\widetilde{P}(x) = V^T(x)P(x)V(x)$, where $V \in \mathcal{C}^e(R, \mathbb{R}^{2\times 2})$ is orthogonal.

Write $V(x) = [v_1(x), v_2(x)]$, for all $x \in R$. From the fact that $v_i^T(x)v_i(x) = 1$, $\forall x \in R$, it follows that

$$\text{(a)} \qquad v_i^T(x)\big(\partial_{x_j} v_i(x)\big) \;=\; 0 \;,\; i,j = 1,2,\; \forall x \in R\,.$$

Similarly, from the fact that $v_1^T(x)v_2(x) = 0$, $\forall x \in R$, it follows that

$$\text{(b)} \qquad v_1^T(x)\big(\partial_{x_j} v_2(x)\big) \;=\; -v_2^T(x)\big(\partial_{x_j} v_1(x)\big) \;,\; j = 1,2,\; \forall x \in R\,.$$

As a consequence of (a)-(b), and since $P(\xi_0)$ is a scalar multiple of the identity, direct computation shows that

$$\nabla\widetilde{b}(\xi_0) \;=\; \begin{bmatrix} v_1^T(\partial_{x_1}P)v_2 & v_1^T(\partial_{x_2}P)v_2 \end{bmatrix}_{\xi_0} ,$$

and

$$\nabla(\widetilde{a} - \widetilde{d})(\xi_0) \;=\; \begin{bmatrix} v_1^T(\partial_{x_1}P)v_1 - v_2^T(\partial_{x_1}P)v_2 & v_1^T(\partial_{x_2}P)v_1 - v_2^T(\partial_{x_2}P)v_2 \end{bmatrix}_{\xi_0} .$$

Now, since $V(\xi_0)$ is orthogonal, it must have one of the two forms (for some α):

$$\text{(i)} \quad V(\xi_0) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \qquad \text{or} \qquad \text{(ii)} \quad V(\xi_0) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} .$$

In case (i), the Jacobian matrix $\begin{bmatrix} \nabla\widetilde{b} \\ \nabla(\widetilde{a} - \widetilde{d}) \end{bmatrix}_{\xi_0}$ rewrites as

$$\begin{bmatrix} \nabla\widetilde{b} \\ \nabla(\widetilde{a} - \widetilde{d}) \end{bmatrix}_{\xi_0} \;=\; \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -2\sin(2\alpha) & 2\cos(2\alpha) \end{bmatrix} \begin{bmatrix} \partial_{x_1} b & \partial_{x_2} b \\ \frac{1}{2}\partial_{x_1}(a-d) & \frac{1}{2}\partial_{x_2}(a-d) \end{bmatrix}_{\xi_0} , \tag{2.3}$$

which is nonsingular since the Jacobian $\begin{bmatrix} \partial_{x_1} b & \partial_{x_2} b \\ \partial_{x_1}(a-d) & \partial_{x_2}(a-d) \end{bmatrix}_{\xi_0}$ is nonsingular (Γ_1 and Γ_2 intersect transversally at ξ_0). In case (ii), the Jacobian matrix $\begin{bmatrix} \nabla\widetilde{b} \\ \nabla(\widetilde{a} - \widetilde{d}) \end{bmatrix}_{\xi_0}$ rewrites as

$$\begin{bmatrix} \nabla\widetilde{b} \\ \nabla(\widetilde{a} - \widetilde{d}) \end{bmatrix}_{\xi_0} \;=\; \begin{bmatrix} -\cos(2\alpha) & \sin(2\alpha) \\ 2\sin(2\alpha) & 2\cos(2\alpha) \end{bmatrix} \begin{bmatrix} \partial_{x_1} b & \partial_{x_2} b \\ \frac{1}{2}\partial_{x_1}(a-d) & \frac{1}{2}\partial_{x_2}(a-d) \end{bmatrix}_{\xi_0} , \tag{2.4}$$

which is likewise nonsingular. Therefore, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ intersect transversally at ξ_0. □

Of course, for a coalescing point of eigenvalues to be a generic coalescing point is a generic property.

We can now give the following result for symmetric $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$.

Theorem 2.8. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$ be symmetric. Let $\lambda_1(x), \dots, \lambda_n(x)$, $x \in \Omega$, be its continuous eigenvalues. Suppose that*

$$\lambda_1(x) > \lambda_2(x) > \ldots > \lambda_k(x) \geq \lambda_{k+1}(x) > \ldots > \lambda_n(x) \;,\; \forall x \in \Omega \;,$$

and

$$\lambda_k(x) = \lambda_{k+1}(x) \Longleftrightarrow x = \xi_0 \in \Omega \;.$$

Let ξ_0 be a generic coalescing point.

Let Γ be a simple closed curve in Ω enclosing the point ξ_0, and let it be parametrized as a $\mathcal{C}^p$ ($p \geq 0$) function γ in the variable t, so that the function $\gamma:\; t \in \mathbb{R} \to \Omega$ is $\mathcal{C}^p$ and 1-periodic. Let $m = \min(e,p)$, and let A_γ be the $\mathcal{C}^m$ function $A(\gamma(t))$, $t \in \mathbb{R}$.

Then, for all $t \in \mathbb{R}$, $A(\gamma(t))$ has the eigendecomposition

$$A_\gamma(t) = U_\gamma(t)\Lambda_\gamma(t)U_\gamma^T(t)$$

satisfying the following conditions:

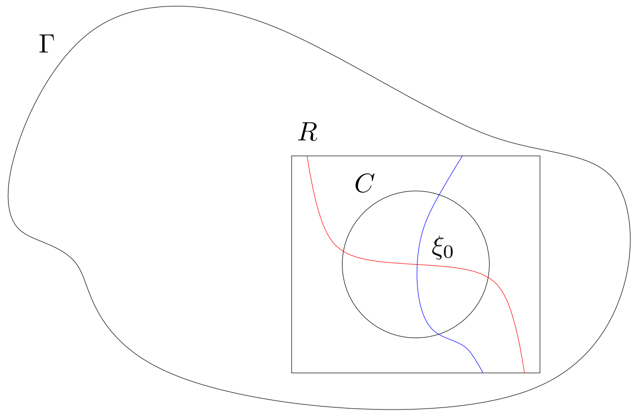

FIGURE 2. Generic coalescing at ξ_0.

(i) $\Lambda_\gamma \in \mathcal{C}_1^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *and diagonal:* $\Lambda_\gamma(t) = \mathrm{diag}(\lambda_1(\gamma(t)), \dots, \lambda_n(\gamma(t)))$, $\forall t \in \mathbb{R}$*;*
(ii) $U_\gamma \in \mathcal{C}_2^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *real orthogonal, and for all* $t \in \mathbb{R}$

$$U_\gamma(t+1) = U_\gamma(t)D \ , \quad D = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix} .$$

Proof. Consider a rectangle $R \subseteq \Omega$ around ξ_0, and consider a function $Q \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ giving the block decomposition

$$Q^T(x)A(x)Q(x) = \begin{bmatrix} \Lambda_1(x) & 0 & 0 \\ 0 & P(x) & 0 \\ 0 & 0 & \Lambda_2(x) \end{bmatrix} , \forall x \in R \ ,$$

as in Definition 2.6. Let C be a circle enclosing ξ_0 and contained in R, parametrized by a continuous 1-periodic function ρ, and let $P_\rho(t) = P(\rho(t))$, $t \in \mathbb{R}$; see Figure 2.

Let V_ρ be the orthogonal function of Theorem 2.2 associated to P_ρ, so that $V_\rho(t+1) = -V_\rho(t)$, for all t. Now, consider the following orthogonal continuous function

$$U_\rho(t) = Q(\rho(t)) \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & V_\rho(t) & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix} .$$

Since $Q(\rho(t+1)) = Q(\rho(t))$ for all t (see Corollary 1.4), we then have

$$U_\rho(t+1) \ = \ U_\rho(t) \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix} .$$

We now need to argue that the same periodicity properties hold for U_γ. We do this in the same way as what we did in Theorem 2.2. Take a homotopy $h(s,t)$, $(s,t) \in [0,1] \times [0,1]$, as in Theorem 2.2, and consider the function $A(h(s,t))$, $(s,t) \in [0,1] \times [0,1]$. $A(h(s,t))$ is continuous with distinct eigenvalues for all $(s,t) \in [0,1] \times [0,1]$, and so –by Theorem 1.3– we can write $A(h(s,t)) = V(s,t)\Lambda(s,t)V^T(s,t)$, where $\Lambda(s,t)$ and $V(s,t)$ are continuous, $\Lambda(s,t)$ is diagonal, and $V(s,t)$ is real orthogonal. Partition V by columns: $V(s,t) = \begin{bmatrix} v_1(s,t) & \cdots & v_n(s,t) \end{bmatrix}$. Let $f_j(s) = v_j^T(s,0)\, v_j(s,1)$, for $j = 1, \dots, n$. Since $h(s,0) = h(s,1)$ for all $s \in [0,1]$, we have that all

f_j's take values in $\{-1,1\}$. Being continuous, we must have $f_j(0)=f_j(1)=1$ for all $j\neq k,k+1$, and $f_j(0)=f_j(1)=-1$ for $j=k,k+1$, from which the result follows. □

Theorem 2.8 has an interesting geometric interpretation: the $\mathcal{C}^m$ eigenvectors of A_γ corresponding to the eigenvalues coalescing at ξ_0 (i.e., the k-th and $(k+1)$-st eigenvectors), "get upside down" as we complete one loop along the closed curve Γ. At the same time, it is worth stressing that the columns of U_γ (the eigenvectors) associated to eigenvalues which do not coalesce in Ω maintain period 1, as A_γ has. Indeed, with the same technique used in the proof of Theorem 2.8, it is easy to refine Corollary 1.4 to obtain the result below, which shows that the period of a continuous eigendecomposition along a simple curve Γ not containing coalescing points (on or) inside it, has period 1.

Corollary 2.9. *Let $A\in\mathcal{C}^e(\Omega,\mathbb{R}^{n\times n})$, and let let Γ be a simple closed curve in Ω, parametrized by the $\mathcal{C}^p$ and 1-periodic function γ. Let $m=\min(e,p)$, and let $A_\gamma\in\mathcal{C}^m_1(\mathbb{R},\mathbb{R}^{n\times n})$ be the function $A(\gamma(t))$, $t\in\mathbb{R}$. If there are no coalescing points inside Γ (nor on it), then any $\mathcal{C}^m$ eigendecomposition of A_γ is 1-periodic.*

Results similar to Theorem 2.2, Theorem 2.8, Theorem 2.7, and Corollary 2.9, hold true for the SVD of a matrix valued function A. To properly state these results, we need a preliminary result, similar to Theorem 1.3.

Theorem 2.10. *Let $R\subseteq\Omega$ be a rectangular region, and let $A\in\mathcal{C}^e(R,\mathbb{R}^{n\times n}), e\geq 0$. Suppose that the singular values of A can be labeled so that they belong to two disjoint sets for all $x\in\Omega$: $\sigma_1(x),\dots,\sigma_p(x)$ in $\Sigma_1(x)$ and $\sigma_{p+1}(x),\dots,\sigma_n(x)$ in $\Sigma_2(x)$, $\Sigma_1(x)\cap\Sigma_2(x)=\emptyset$, $\forall x\in R$. Then, there exist $U,V\in\mathcal{C}^e(R,\mathbb{R}^{n\times n})$, orthogonal, such that*

$$U^T(x)A(x)V(x)=:S=\begin{bmatrix}S_1(x) & 0\\ 0 & S_2(x)\end{bmatrix}\ ,\ \forall x\in R\ ,$$

where $S_1\in\mathcal{C}^e(R,\mathbb{R}^{p\times p})$, $S_2\in\mathcal{C}^e(R,\mathbb{R}^{(n-p)\times(n-p)})$, and the singular values of $S_i(x)$ are those in $\Sigma_i(x)$, for all $x\in R$ and $i=1,2$.

Proof. Using Theorem 1.3, for all $x\in R$ we have the two block-diagonalizations

$$U^T(x)A(x)A^T(x)U(x)=\begin{bmatrix}P_1(x) & 0\\ 0 & P_2(x)\end{bmatrix}\ , \tag{2.5}$$

$$V^T(x)A^T(x)A(x)V(x)=\begin{bmatrix}R_1(x) & 0\\ 0 & R_2(x)\end{bmatrix}\ , \tag{2.6}$$

where

(i) $U,V\in\mathcal{C}^e(R,\mathbb{R}^{n\times n})$ orthogonal,
(ii) $P_1,R_1\in\mathcal{C}^e(R,\mathbb{R}^{p\times p})$ symmetric, and for all $x\in R$ the eigenvalues of $P_1(x)$ and $R_1(x)$ are the squares of the singular values in $\Sigma_1(x)$;
(iii) $P_2,R_2\in\mathcal{C}^e(R,\mathbb{R}^{(n-p)\times(n-p)})$ symmetric, and for all $x\in R$ the eigenvalues of $P_2(x)$ and $R_2(x)$ are the squares of the singular values in $\Sigma_2(x)$.

For all $x\in R$, write

$$U^T(x)A(x)V(x)=\begin{bmatrix}S_1(x) & X_{12}(x)\\ X_{21}(x) & S_2(x)\end{bmatrix}\ ,$$

where the partitioning is inherited by the right hand sides of (2.5) and (2.6). Manipulation of (2.5) and (2.6) yields

$$U^T(x)A(x)A^T(x)A(x)V(x)=$$

$$= \begin{bmatrix} P_1(x) & 0 \\ 0 & P_2(x) \end{bmatrix} U^T(x)A(x)V(x) =$$
$$= U^T(x)A(x)V(x) \begin{bmatrix} R_1(x) & 0 \\ 0 & R_2(x) \end{bmatrix} ,$$

which implies that

$$P_1(x)X_{12}(x) = X_{12}(x)R_2(x)$$
$$P_2(x)X_{21}(x) = X_{21}(x)R_1(x) ,$$

for all x in R. By hypothesis, the spectra of P_1 and R_2, and of P_2 and R_1, are disjoint in R, and hence we must have $X_{12}(x) = 0$, $X_{21}(x) = 0$. □

Remarks 2.11.

(1) In general, just as in the case of Theorems 1.3 and 1.6, the block SVD of Theorem 2.10 is not unique. A result similar to Theorem 1.6 holds now as well, the proof being nearly identical to that of Theorem 1.6, and (with obvious notation) it can be phrased as follows.
Under the same assumptions of Theorem 2.10, suppose that U and $V \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$, are given, orthogonal, functions (guaranteed to exist by Theorem 2.10) such that

$$U^T(x)A(x)V(x) =: S = \begin{bmatrix} S_1(x) & 0 \\ 0 & S_2(x) \end{bmatrix} , \forall x \in R ,$$

where the singular values of S_1 (respectively S_2) are those in Σ_1 (respectively Σ_2), for all $x \in R$. Then, any other $\mathcal{C}^e$ block SVD of A in R, in two groups corresponding to the singular values in Σ_1, Σ_2, must have the form:

$$U(x) \begin{bmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{bmatrix} , \quad V(x) \begin{bmatrix} Z_1(x) & 0 \\ 0 & Z_2(x) \end{bmatrix} ,$$

where the $\mathcal{C}^e$ functions W_1, Z_1, and W_2, Z_2, are orthogonal, taking values in $\mathbb{R}^{p\times p}$ and $\mathbb{R}^{(n-p)\times(n-p)}$ respectively.

(2) Of course, Theorem 2.10 can be refined to any number of groups of singular values which remain disjoint in R. In particular, if all singular values are distinct, S is diagonal. Moreover, if the singular values are distinct, then the $\mathcal{C}^e$ functions U and V are essentially unique: the degree of non-uniqueness is solely determined by the ordering of the diagonal entries of S and by joint changes of signs for the columns of U, V. Finally, an obvious analog of Corollary 1.4 holds now as well.

We can now tackle the case of coalescing singular values. It is again convenient to first consider the (2×2) case. To begin with, we have the following simple result.

Lemma 2.12. *Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a given matrix. Then, this matrix has two identical nonzero singular values if and only if $B = \sigma Q$, where $Q \in \mathbb{R}^{2\times 2}$ is an orthogonal matrix (and $\sigma \neq 0$).*

Proof. Let $B = U\Sigma V^T$ be the SVD of B. Then, if $\Sigma = \sigma I$, we clearly have $B = \sigma Q$ with $Q = UV^T$. The converse is obvious. □

Naturally, if both singular values are 0, then $B = 0$ in Lemma 2.12, which is a trivial multiple of an orthogonal matrix as well, but –as it turns out– the case of both singular values equal to 0 is different and our results do not cover this case, see Remark 2.16.

We now have the following result about coalescing singular values.

Theorem 2.13 (Signed SVD: 2×2 case)**.** *Consider $B \in \mathcal{C}^e(\Omega, \mathbb{R}^{2\times 2})$, $e \geq 1$. For all $x \in \Omega$, write*

$$B(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} ,$$

and let σ_1 and σ_2 be its two continuous singular values, labeled so that $\sigma_1(x) \geq \sigma_2(x) \geq 0$ for all x in Ω. Assume that there exists a unique point $\xi_0 \in \Omega$ where these singular values coincide, $\sigma_1(\xi_0) = \sigma_2(\xi_0)$. Consider the $\mathcal{C}^e$ functions $F, G :\ \Omega \to \mathbb{R}^2$ given by

$$F(x) = \begin{bmatrix} a^2(x) + c^2(x) - b^2(x) - d^2(x) \\ a(x)b(x) + c(x)d(x) \end{bmatrix} ,\ G(x) = \begin{bmatrix} a^2(x) + b^2(x) - c^2(x) - d^2(x) \\ a(x)c(x) + b(x)d(x) \end{bmatrix} , \tag{2.7}$$

and assume that 0 is a regular value for the scalar valued functions given by the 1st and the 2nd components of F and G. Then, consider the $\mathcal{C}^e$ curves Γ_1, Γ_2, Γ_3, and Γ_4, given by the zero-set of the components of F and G: $\Gamma_1 = \{x \in \Omega :\ a^2(x) + c^2(x) - b^2(x) - d^2(x) = 0\}$, $\Gamma_2 = \{x \in \Omega :\ a(x)b(x) + c(x)d(x) = 0\}$, and $\Gamma_3 = \{x \in \Omega :\ a^2(x) + b^2(x) - c^2(x) - d^2(x) = 0\}$, $\Gamma_4 = \{x \in \Omega :\ a(x)c(x) + b(x)d(x) = 0\}$. Assume that the pair of curves Γ_1 and Γ_2, and also[5] the pair Γ_3 and Γ_4, intersect transversally at ξ_0.

Let Γ be a simple closed curve enclosing the point ξ_0, and let it be parametrized as a $\mathcal{C}^p$ ($p \geq 0$) function γ in the variable t, so that the function $\gamma :\ t \in \mathbb{R} \to \Omega$ is $\mathcal{C}^p$ and 1-periodic. Let $m = \min(e, p)$, and let B_γ be the $\mathcal{C}^m_1$ function $B(\gamma(t))$, $t \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$, $B_\gamma(t)$ has the signed singular value decomposition

$$B_\gamma(t) = Q_\gamma(t)\Sigma_\gamma(t)Z_\gamma^T(t)$$

such that:

- (i) *$\Sigma_\gamma \in \mathcal{C}^m_1(\mathbb{R}, \mathbb{R}^{2\times 2})$ and diagonal, $\Sigma_\gamma(t) = \begin{bmatrix} s_1(\gamma(t)) & 0 \\ 0 & s_2(\gamma(t)) \end{bmatrix}$ and $|s_i(\gamma(t))| = \sigma_i(\gamma(t))$, for $i = 1, 2$, and for all $t \in \mathbb{R}$;*
- (ii) *$Q_\gamma, Z_\gamma \in \mathcal{C}^m_2(\mathbb{R}, \mathbb{R}^{2\times 2})$ real orthogonal, and $Q_\gamma(t + 1) = -Q_\gamma(t)$, $Z_\gamma(t + 1) = -Z_\gamma(t)$, for all $t \in \mathbb{R}$.*

Proof. Because of Lemma 2.12, we have

$$\sigma_1(x) = \sigma_2(x) \Longleftrightarrow F(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } G(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

By hypothesis, ξ_0 is the unique root of $F(x)$ and $G(x)$ in Ω, and Γ_1 and Γ_2, and Γ_3 and Γ_4, intersect transversally at ξ_0.

Next, consider the functions B^TB and BB^T. By the assumption on the singular values of B, the eigenvalues of B^TB (which are the same as those of BB^T) in Ω coincide only at ξ_0. Moreover, Theorem 2.2 applies to B^TB and BB^T. Let Z_γ and Q_γ be the two orthogonal functions of which in Theorem 2.2 relative to B^TB and BB^T, respectively, so that for each $t \in \mathbb{R}$ we have

$$Q^T(\gamma(t))B(\gamma(t))B^T(\gamma(t))Q(\gamma(t)) = \begin{bmatrix} \sigma_1^2(t) & 0 \\ 0 & \sigma_2^2(t) \end{bmatrix} , \quad \text{and}$$

$$Z^T(\gamma(t))B^T(\gamma(t))B(\gamma(t))Z(\gamma(t)) = \begin{bmatrix} \sigma_1^2(t) & 0 \\ 0 & \sigma_2^2(t) \end{bmatrix} , \quad \text{where}$$

$\sigma_1^2(\gamma(t)) > \sigma_2^2(\gamma(t))$, for all $t \in \mathbb{R}$, $Q_\gamma, Z_\gamma \in \mathcal{C}^m_2(\mathbb{R}, \mathbb{R}^{2\times 2})$, and $Q_\gamma(t+1) = -Q_\gamma(t)$ and $Z_\gamma(t+1) = -Z_\gamma(t)$.

[5]But see Remark 2.15

Next, consider the function $Q_\gamma^T B_\gamma Z_\gamma$, which is in $\mathcal{C}^m(\mathbb{R}, \mathbb{R}^{2\times 2})$. For all $t \in \mathbb{R}$, write

$$Q^T(\gamma(t))B(\gamma(t))Z(\gamma(t)) = \begin{bmatrix} s_1(\gamma(t)) & z_{12}(\gamma(t)) \\ z_{21}(\gamma(t)) & s_2(\gamma(t)) \end{bmatrix} .$$

Similarly to the proof of Theorem 2.10, we must have $z_{12}(\gamma(t)) = 0$, $z_{21}(\gamma(t)) = 0$, for all t. So, we have the decomposition, for all $t \in \mathbb{R}$,

$$Q^T(\gamma(t))B(\gamma(t))Z(\gamma(t)) = \begin{bmatrix} s_1(\gamma(t)) & 0 \\ 0 & s_2(\gamma(t)) \end{bmatrix} ,$$

where $s_1(\gamma(t)) \neq s_2(\gamma(t))$ for all $t \in \mathbb{R}$, and $|s_1(\gamma(t))| = \sigma_1(\gamma(t)) > |s_2(\gamma(t))| = \sigma_2(\gamma(t))$. Now, for all $t \in \mathbb{R}$, we have $B_\gamma(t+1) = B_\gamma(t)$, $Q_\gamma(t+1) = -Q_\gamma(t)$, $Z_\gamma(t+1) = -Z_\gamma(t)$, and therefore the functions s_1 and s_2 are 1-periodic. □

Remark 2.14. The decomposition of B_γ in Theorem 2.13 is essentially unique, within the class of $\mathcal{C}^m$ decompositions. The degree of non-uniqueness is given by the ordering of the diagonal and by joint (and global) choices of signs for the columns of Q_γ and Z_γ. In particular, for $\mathcal{C}^m$–decompositions, the statement about periodicity of the functions Q_γ and Z_γ holds unchanged. It is also worth noticing that the functions s_1 and s_2 do not necessarily remain positive along Γ: In fact, if B_γ loses rank, singular value(s) will usually change sign. If we had insisted on having a decomposition with positive singular values, in cases where B_γ lost rank, we would have not been able to retain the $\mathcal{C}^m$ factors Q_γ and Z_γ. Loosely speaking, by insisting on having $\mathcal{C}^m$ orthogonal factors, then the diagonal functions s_1, s_2, must follow their course, and we cannot demand that they are positive in case B_γ loses rank.

Remark 2.15. In Theorem 2.13, we have assumed that the pair Γ_1, Γ_2, and the pair Γ_3, Γ_4, intersected transversally at ξ_0. In reality, it suffices to assume that one pair of these curves does so, and transversality of the other pair will follow. To verify this statement, we could appeal to Lemma A-1 in a similar way to what we do in Section 3.2.2. More directly, a lengthy but otherwise simple computation shows that

$$\det \begin{bmatrix} \nabla\big(a^2(x) + c^2(x) - b^2(x) - d^2(x)\big) \\ \nabla\big(a(x)b(x) + c(x)d(x)\big) \end{bmatrix}_{\xi_0} \; = \; \pm \det \begin{bmatrix} \nabla\big(a^2(x) + b^2(x) - c^2(x) - d^2(x)\big) \\ \nabla\big(a(x)c(x) + b(x)d(x)\big) \end{bmatrix}_{\xi_0} .$$

Remark 2.16. We observe that the assumption of transversal intersection of (say) Γ_1 and Γ_2 at the coalescing point ξ_0 rules out the possibility that the singular values be 0 there. This is actually not unexpected, since the request of having a pair of coalescing singular values equal to 0 is a codimension 4 phenomenon: $a = b = c = d = 0$. But, in fact, more is true. The very assumption in Theorem 2.13, that 0 be a regular value for the scalar valued functions given by the 1st and the 2nd components of F and G there, implies that we cannot have a pair of coalescing singular values equal to 0, as otherwise there would be no properly defined tangents at all. In other words, our assumptions –here and later on– for coalescing singular values do not allow the coalescing pair of singular values to be 0.

Remark 2.17. Similarly to Remark 2.4, the assumption of transversality for the curves Γ_1 and Γ_2 (or Γ_3 and Γ_4) at ξ_0 is generic. We will say that ξ_0 is a *generic coalescing point of singular values.*

Using Theorem 2.13, and Theorem 2.10, we can now tackle the case of a general function in $\mathbb{R}^{n\times n}$, with singular values coalescing at a unique point ξ_0. We first give a definition similar to Definition 2.6.

Definition 2.18. Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$ have continuous singular values $\sigma_1(x), \ldots, \sigma_n(x)$, $x \in \Omega$, satisfying

$$\sigma_1(x) > \sigma_2(x) > \ldots > \sigma_k(x) \geq \sigma_{k+1}(x) > \ldots > \sigma_n(x) \ , \forall x \in \Omega \ ,$$

and

$$\sigma_k(x) = \sigma_{k+1}(x) \Longleftrightarrow x = \xi_0 \in \Omega \ .$$

Let R be a rectangular region $R \subseteq \Omega$ containing ξ_0 in its interior. Moreover, let

(1) $U, V \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ be $\mathcal{C}^e$ orthogonal functions achieving the reduction of which in Theorem 2.10 and Remark 2.11-(2):

$$U^T(x)A(x)V(x) = \begin{bmatrix} \Sigma_1(x) & 0 & 0 \\ 0 & B(x) & 0 \\ 0 & 0 & \Sigma_2(x) \end{bmatrix} \ , \forall x \in R \ ,$$

where $\Sigma_1 \in \mathcal{C}^e(R, \mathbb{R}^{(k-1)\times(k-1)})$ and $\Sigma_2 \in \mathcal{C}^e(R, \mathbb{R}^{(n-k-1)\times(n-k-1)})$, such that, for all $x \in R$, $\Sigma_1(x) = \mathrm{diag}(\sigma_1(x), \ldots, \sigma_{k-1}(x))$, and $\Sigma_2(x) = \mathrm{diag}(\sigma_{k+2}(x), \ldots, \sigma_n(x))$. Moreover, $B \in \mathcal{C}^e(R, \mathbb{R}^{2\times 2})$ has singular values $\sigma_k(x), \sigma_{k+1}(x)$ for each $x \in R$;

(2) for all $x \in R$, write $B(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$, and define the function F (and G) and the curves Γ_1 and Γ_2 (Γ_3 and Γ_4) as in Theorem 2.13, for $x \in R$.

Then, we call ξ_0 a *generic coalescing point of singular values* in Ω, if the curves Γ_1 and Γ_2 (equivalently, Γ_3 and Γ_4) intersect transversally at ξ_0.

It is again true that Definition 2.18 holds true regardless of which $\mathcal{C}^e$ transformations U and V we have used to simplify the structure of A. In other words, a result similar to Theorem 2.7 holds here as well. In light of Remark 2.11-(1), this is a consequence of the following result, whose proof is essentially identical to that of Theorem 2.7 and therefore omitted.

Theorem 2.19. *Let $B \in \mathcal{C}^e(\Omega, \mathbb{R}^{2\times 2})$ have singular values $\sigma_k(x), \sigma_{k+1}(x)$ for each $x \in \Omega$, and let ξ_0 be the only point where these two singular values coalesce. Write $B(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$, and define the functions F (and G) as in Theorem 2.13, satisfying the same assumptions therein, and the curves Γ_1, Γ_2, (and Γ_3, Γ_4), as in Theorem 2.13. Let $W, Z \in \mathcal{C}^e(\Omega, \mathbb{R}^{2\times 2})$ be orthogonal, and for all $x \in \Omega$ let*

$$\widetilde{B}(x) = W^T(x)B(x)Z(x) = \begin{bmatrix} \widetilde{a}(x) & \widetilde{b}(x) \\ \widetilde{c}(x) & \widetilde{d}(x) \end{bmatrix} \ .$$

Define the curves $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$, (and $\widetilde{\Gamma}_3$, $\widetilde{\Gamma}_4$), similarly to how $\Gamma_i, i = 1, \ldots, 4$, are defined; e.g., $\widetilde{\Gamma}_1 = \{x \in \Omega : \widetilde{a}^2(x) + \widetilde{c}^2(x) - \widetilde{b}^2(x) - \widetilde{d}^2(x) = 0\}$, $\widetilde{\Gamma}_2 = \{x \in \Omega : \widetilde{a}(x)\widetilde{b}(x) + \widetilde{c}(x)\widetilde{d}(x) = 0\}$. Then, if Γ_1 and Γ_2 (equivalently, Γ_3 and Γ_4) intersect transversally at ξ_0, so do $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ ($\widetilde{\Gamma}_3$ and $\widetilde{\Gamma}_4$).

We are now ready for the general case of $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$.

Theorem 2.20. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$. For all $x \in \Omega$, assume that its continuous singular values, $\sigma_1(x), \ldots, \sigma_n(x)$, satisfy*

$$\sigma_1(x) > \sigma_2(x) > \ldots > \sigma_k(x) \geq \sigma_{k+1}(x) > \ldots > \sigma_n(x) \ , \forall x \in \Omega \ ,$$

and

$$\sigma_k(x) = \sigma_{k+1}(x) \Longleftrightarrow x = \xi_0 \ .$$

Let ξ_0 be a generic coalescing point.

Define Γ and γ as in Theorem 2.13, and let A_γ be the $\mathcal{C}^m_1$ function $A(\gamma(t))$, $t \in \mathbb{R}$.

Then, for all $t \in \mathbb{R}$, $A_\gamma(t)$ has the (signed) singular value decomposition

$$A_\gamma(t) = U_\gamma(t)\Sigma_\gamma(t)V_\gamma^T(t)$$

satisfying the following conditions:

(i) $\Sigma_\gamma \in \mathcal{C}_1^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *and diagonal:* $\Sigma_\gamma(t) = \operatorname{diag}(s_1(\gamma(t)), \dots, s_n(\gamma(t)))$, *and* $|s_i(\gamma(t))| = \sigma_i(\gamma(t))$, *for* $i = 1\dots, n$, $\forall t \in \mathbb{R}$;

(ii) $U_\gamma, V_\gamma \in \mathcal{C}_2^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *real orthogonal, and for all* $t \in \mathbb{R}$

$$U_\gamma(t+1) = U_\gamma(t)D\ ,\quad V_\gamma(t+1) = V_\gamma(t)D\ ,\qquad D = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}\ .$$

Proof. The proof follows similar steps as that of Theorem 2.8. First, consider a rectangular region $R \subseteq \Omega$, containing ξ_0. In R, we have a block-SVD

$$\widehat{U}^T(x)A(x)\widehat{V}(x) = \begin{bmatrix} \Sigma_1(x) & 0 & 0 \\ 0 & B(x) & 0 \\ 0 & 0 & \Sigma_2(x) \end{bmatrix}\ ,\ \forall x \in R\ ,$$

as in Definition 2.18. Then, we take a circle C enclosing ξ_0 and contained in R. Let C be parametrized by a continuous 1-periodic function ρ, and let $B_\rho(t) = B(\rho(t))$, $t \in \mathbb{R}$. Let Q_ρ and Z_ρ be the functions of Theorem 2.13 associated to B_ρ. Then, along C, the stated periodicity result holds, since we have

$$U_\rho(t) = \widehat{U}(\rho(t))\begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & Q_\rho(t) & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix} \qquad \text{and} \qquad V_\rho(t) = \widehat{V}(\rho(t))\begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & Z_\rho(t) & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}\ ,$$

and (see Corollary 1.4) $\widehat{U}(\rho(t+1)) = \widehat{U}(\rho(t))$ and $\widehat{V}(\rho(t+1)) = \widehat{V}(\rho(t))$ for all t.

Finally, to argue that the same periodicity results hold along Γ, we use the homotopy $h(s,t)$ –which we already used in Theorems 2.2 and 2.8– to carry U_ρ and V_ρ into U_γ and V_γ. That is, we take the continuous function, which has distinct singular values, $A(h(s,t))$, $(s,t) \in [0,1]\times[0,1]$. Using Theorem 2.10, we write $A(h(s,t)) = U(s,t)\Sigma(s,t)V^T(s,t)$, where all factors are continuous, U and V are orthogonal, and Σ is diagonal, $\Sigma(s,t) = \operatorname{diag}\big(s_i(s,t), i = 1,\dots,n\big)$ with $|s_i| = \sigma_i$. Partition $U(s,t)$ and $V(s,t)$ by columns: $U(s,t) = \begin{bmatrix} u_1(s,t) & \cdots & u_n(s,t)\end{bmatrix}$, $V(s,t) = \begin{bmatrix} v_1(s,t) & \cdots & v_n(s,t)\end{bmatrix}$, and consider the functions (for $s \in [0,1]$) $g_j(s) = u_j^T(s,0)u_j(s,1)$ and $f_j(s) = v_j^T(s,0)v_j(s,1)$ for all $j = 1,\dots,n$. Using continuity, we get that $g_j(0) = g_j(1) = f_j(0) = f_j(1) = 1$ for all $j \neq k, k+1$, and $g_j(0) = g_j(1) = f_j(0) = f_j(1) = -1$ for all $j = k, k+1$, and the proof is complete. □

Remark 2.21. Similarly to Theorem 2.8, also now we have that the $\mathcal{C}^m$ singular vectors of A_γ corresponding to the singular values coalescing at ξ_0 "get upside down" as we complete one loop along the closed curve Γ. Since the singular values of A_γ are distinct, the singular vectors of any $\mathcal{C}^m$ SVD of A_γ cannot increase in period, unless the associated singular values coalesce inside Γ. Finally, we notice that, in Theorem 2.20, if the function A_γ had full rank, then we could have obtained that $s_i = \sigma_i$, and positive.

3. Generalizations and main result

So far, we have studied cases in which a unique pair of eigenvalues of a symmetric function, or singular values of a general function, underwent a generic coalescing at a point ξ_0. Genericity was related to the way that two smooth curves intersected each other. In this section, we move a step further and consider cases in which several eigenvalues, respectively singular values, coalesce in a region Ω. First, we will give –and prove– results for the eigendecomposition of a symmetric matrix valued function. Then, we will state an analogous result for the SVD, but omit the proof since it will be a transparent generalization of the symmetric case.

3.1. **Main Results: Generic Cases.** Before proceeding, we characterize a generic coalescing point, when this is not the only coalescing point in Ω.

Definition 3.1. Let $\xi_0 \in \Omega$ be a point where two eigenvalues of the symmetric function $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$, $e \geq 1$, coalesce. Then, we call ξ_0 a *generic coalescing point of eigenvalues* if there exists an open simply connected region $\Omega_0 \subseteq \Omega$, inside which ξ_0 is the unique point where eigenvalues coalesce, and ξ_0 is a generic coalescing point of eigenvalues in Ω_0.

In a nearly identical way, we define a generic coalescing point relatively to the singular values.

Definition 3.2. Let $\xi_0 \in \Omega$ be a point where two singular values of the function $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$, $e \geq 1$, coalesce. Then, we call ξ_0 a *generic coalescing point of singular values* if there exists an open simply connected region $\Omega_0 \subseteq \Omega$, inside which ξ_0 is the unique point where singular values of A coalesce, and ξ_0 is a generic coalescing point of singular values in Ω_0.

We are now ready to provide the sought after generalizations. First we will consider the symmetric eigenproblem.

Theorem 3.3. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$, $e \geq 1$, be symmetric. For all $x \in \Omega$, let $\lambda_1(x), \dots, \lambda_n(x)$, be its continuous, ordered, eigenvalues, which are distinct except at two generic coalescing points ξ_0 and ξ_1. Let k_1 and k_2 be two distinct indices such that*

$$\lambda_{k_1}(x) = \lambda_{k_1+1}(x) \Longleftrightarrow x = \xi_0 \ , \ \lambda_{k_2}(x) = \lambda_{k_2+1}(x) \Longleftrightarrow x = \xi_1 \ .$$

Without loss of generality, we can take $k_2 > k_1$.

Let Γ be a simple closed curve enclosing the points ξ_0 and ξ_1, and let it be parametrized as a $\mathcal{C}^p$ ($p \geq 0$) function γ in the variable t, so that the function $\gamma : \ t \in \mathbb{R} \to \Omega$ is $\mathcal{C}^p$ and 1-periodic. Let $m = \min(e, p)$, and let A_γ be the $\mathcal{C}^m$ function $A(\gamma(t))$, $t \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$, $A(\gamma(t))$ has the eigendecomposition

$$A_\gamma(t) = U_\gamma(t)\Lambda_\gamma(t)U_\gamma^T(t)$$

such that:

(i) $\Lambda_\gamma \in \mathcal{C}_1^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *and diagonal:* $\Lambda_\gamma(t) = \operatorname{diag}\big(\lambda_1(\gamma(t)), \dots, \lambda_n(\gamma(t))\big)$, *for all* $t \in \mathbb{R}$;

(ii) $U_\gamma \in \mathcal{C}_2^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *real orthogonal, and* $U_\gamma(t+1) = U_\gamma(t)\ D$ *for all* $t \in \mathbb{R}$, *where* $D = D_0 D_1$ *with*

   (a) $D_0 = \operatorname{diag}(d_j^{(0)},\ j = 1, \dots, n)$, $d_j^{(0)} = 1$, *for* $j \neq k_1, k_1+1$, *and* $d_j^{(0)} = -1$, *for* $j = k_1, k_1+1$;

   (b) $D_1 = \operatorname{diag}(d_j^{(1)},\ j = 1, \dots, n)$, $d_j^{(1)} = 1$, *for* $j \neq k_2, k_2+1$, *and* $d_j^{(1)} = -1$, *for* $j = k_2, k_2+1$.

Proof. By assumption, the eigenvalues of A_γ are distinct, and thus the fact that an eigendecomposition of A_γ can be taken of class $\mathcal{C}^m$ is well known (e.g., see [5, 6]). The issue to determine is the periodicity of the eigendecomposition, since the results in [5] do not qualify if the eigendecomposition will be 1-periodic or 2-periodic. We need to look at the behavior of the $\mathcal{C}^m$ eigenvectors of A_γ corresponding to the eigenvalues coalescing inside Γ as we complete a loop along Γ.

Let u_i denote the eigenvector of A_γ corresponding to the eigenvalue λ_i. Let α be a simple curve, which stays inside the region bounded by Γ, connecting two distinct points on Γ, $y_0 = \gamma(t_0)$ to $y_1 = \gamma(t_1)$, with $t_0, t_1 \in [0, 1)$, such that α leaves ξ_0 and ξ_1 on opposite sides (see Figure 3). This is possible, since ξ_0 and ξ_1 are distinct.

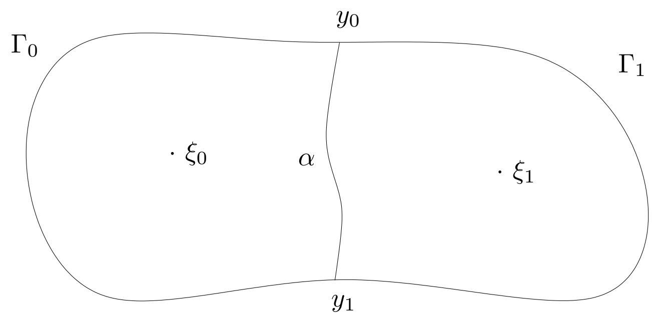

FIGURE 3. Reference picture for proof of Theorem 3.3

With abuse of notation, the symbol $\cup$ will denote the obvious operation of taking the "union" of two curves, whenever this makes sense. Thus, we can let $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 having y_0 and y_1 as endpoints. Further, let us require α to be such that $\Gamma_1 \cup \alpha$ is parametrized as a continuous closed curve encircling ξ_1 and $\Gamma_0 \cup (-\alpha)$ as a continuous closed curve encircling ξ_0; here, $-\alpha$ is the same curve transversed in the opposite direction (from y_1 to y_0). Since $\xi_0 \neq \xi_1$, we can always assume that $\Gamma_0 \cup (-\alpha)$ and $\Gamma_1 \cup \alpha$ are enclosed in two domains, $\Omega_0 \subseteq \Omega$ and $\Omega_1 \subseteq \Omega$ respectively, inside which the situation of Definition 3.1 applies, relatively to ξ_0 and ξ_1 respectively.

Now, let us consider a $\mathcal{C}^m$ eigendecomposition of A_γ along Γ, starting from and coming back to y_0 (we make the loop only once). Let us denote the eigenvectors of A_γ at the beginning of this loop as u_i^0 and those at the end of the loop as u_i^1, $i = 1, \dots, n$. We are interested in comparing u_i^0 with u_i^1, for all $i = 1, \dots, n$. Since the curve α does not contain any coalescing point, the vectors u_i^1's would be the same as if, instead of following the curve Γ, we were to follow the path $\Gamma_0 \cup (-\alpha \cup \alpha) \cup \Gamma_1$. This last path can be interpreted as the union of the two closed loops $\Gamma_0 \cup (-\alpha)$ and $\alpha \cup \Gamma_1$. Let us denote the eigenvectors of A_γ at the end of the first loop as $u_i^{1/2}$, $i = 1, \dots, n$. Applying Theorem 2.8 to A along the closed curve $\Gamma_0 \cup (-\alpha)$, we conclude that $u_i^0 = -u_i^{1/2}$ for $i = k_1, k_1 + 1$ and $u_i^0 = u_i^{1/2}$ for all other indices. Similarly, we can conclude that $u_i^{1/2} = -u_i^1$ for $i = k_2, k_2 + 1$ and $u_i^{1/2} = u_i^1$ for all other indices. Putting everything together, we have proven the result. $\square$

Remark 3.4. Two special cases are worth being singled out.

(1) In case $k_2 = k_1 + 1$, then from Theorem 3.3 we would get

$$D = \begin{bmatrix} I_{k_1-1} & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & I_{n-k_1-2} \end{bmatrix}.$$

(2) In case ξ_0 and ξ_1 were generic coalescing points where the same pair of eigenvalues coalesced, the same reasoning used to prove Theorem 3.3 would have rendered a 1-periodic, $\mathcal{C}^m$ factor U_γ.

Remark 3.4 and the argument used in Theorem 3.3 can be extended to prove the following theorem, in which any combination of coalescing of eigenvalues at generic points is allowed.

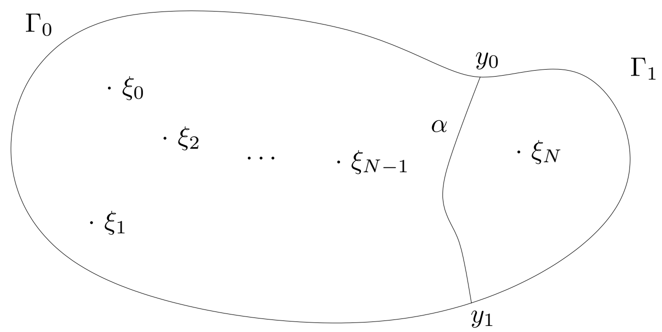

FIGURE 4. Reference picture for proof of Theorem 3.5

Theorem 3.5. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$, $e \geq 1$, be symmetric and let $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$ be its continuous eigenvalues. Suppose that, for every $k = 1, \ldots, n-1$,*

$$\lambda_k(x) = \lambda_{k+1}(x)$$

at d_k distinct generic coalescing points of eigenvalues in Ω, so that there are $\sum_{k=1}^{n-1} d_k$ such points. Let Γ be a simple closed curve enclosing all of these distinct generic coalescing points of eigenvalues. Let Γ be parametrized as a $\mathcal{C}^p$ ($p \geq 0$) function γ in the variable t, so that the function $\gamma: \ t \in \mathbb{R} \to \Omega$ is $\mathcal{C}^p$ and 1-periodic. Let $m = \min(e, p)$ and $A_\gamma \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}^{n\times n})$ defined as $A_\gamma(t) = A(\gamma(t))$, for all $t \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$,

$$A_\gamma(t) = U_\gamma(t)\Lambda_\gamma(t)U_\gamma^T(t)$$

such that:

(i) $\Lambda_\gamma \in \mathcal{C}_1^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *is diagonal:* $\Lambda_\gamma(t) = \operatorname{diag}\big(\lambda_1(\gamma(t)), \ldots, \lambda_n(\gamma(t))\big)$, *for all $t \in \mathbb{R}$;*

(ii) $U_\gamma \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}^{n\times n})$ *orthogonal, with*

$$U_\gamma(t+1) = U_\gamma(t)\ D\ ,\ \forall t \in \mathbb{R}\ ,$$

where D is a diagonal matrix of ± 1 given as follows:

$$D_{11} = (-1)^{d_1},\quad D_{kk} = (-1)^{d_{k-1}+d_k} \text{ for } k = 2, \ldots, n-1,\quad D_{nn} = (-1)^{d_{n-1}}\ .$$

In particular, if $D = I_n$, then U_γ is 1-periodic, otherwise U_γ is 2-periodic.

Proof. The proof is by induction on the number of coalescing points. We know that the result is true for 1 and 2 coalescing points. Thus, we will assume the result to be true for $N-1$ distinct generic coalescing points, and will show it for N distinct generic coalescing points, where $N = \sum_{k=1}^{n-1} d_k$.

Since the coalescing points are distinct, we can always separate one of them, call it ξ_N, from the other $N-1$, with a curve α not containing coalescing points, and which stays inside the region bounded by Γ, joining two distinct points on Γ, y_0 and y_1 (see Figure 4). Let j, $1 \leq j \leq n-1$, be the index for which $\lambda_j(\xi_N) = \lambda_{j+1}(\xi_N)$.

Now, with a construction identical to the one we used in the proof of Theorem 3.3, and with the same notation we used there, consider a smooth eigendecomposition of A_γ along Γ, starting from and coming back to y_0 (the loop is done once). Denote the matrix of eigenvectors of A_γ at the beginning of this loop as U_0 and that at the end of the loop as U_1. Since the curve α does not contain any coalescing point, the matrix U_1 would be the same as if, instead of following the curve Γ, we were to follow the path $(\Gamma_0 \cup (-\alpha)) \cup (\alpha \cup \Gamma_1)$. Denote the matrix of eigenvectors of A_γ at the end of the first loop by $U_{1/2}$. Using the induction hypothesis along the closed curve $\Gamma_0 \cup (-\alpha)$, we have

$$U_0 = U_{1/2}\widehat{D}\,,$$

where $\widehat{D}$ is a diagonal matrix $\widehat{D} = \text{diag}(\widehat{d}_1, \dots \widehat{d}_n)$, with $\widehat{d}_k = d_k$, for all $k \neq j$, and $\widehat{d}_j = d_j - 1$.

By looking at what happens on the second loop, by virtue of Theorem 2.8, we have that all columns of $U_{1/2}$ coincide with those of U_1, except for the j-th and $(j+1)$-st ones which have changed in sign. Putting everything together, we have $U_0 = U_1 D$ with D as given in the statement of the Theorem. □

A result similar to Theorem 3.5 holds for the singular value decomposition of a general function in a domain where singular values coalesce at generic coalescing points of singular values. The precise statement follows without proof, since the proof is a clear adaptation of the previous ones.

Theorem 3.6. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$, $e \geq 1$. Let $\sigma_1(x) \geq \ldots \geq \sigma_n(x)$ be its continuous singular values and suppose that, for every $k = 1, \dots, n-1$,*

$$\sigma_k(x) = \sigma_{k+1}(x)$$

at d_k distinct generic coalescing points of singular values in Ω. Let Γ be a simple closed curve enclosing all these coalescing points, parametrized by the 1-periodic function $\gamma \in \mathcal{C}^p_1(\mathbb{R}, \Omega)$. Let $m = \min(e, p)$ and $A_\gamma \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}^{n\times n})$ defined as $A_\gamma(t) = A(\gamma(t))$, for all $t \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$,

$$A_\gamma(t) = U_\gamma(t) S_\gamma(t) V_\gamma^T(t)$$

such that:

(i) *$S_\gamma \in \mathcal{C}^m_1(\mathbb{R}, \mathbb{R}^{n\times n})$ diagonal: $\Sigma_\gamma(t) = \text{diag}\big(s_1(\gamma(t)), \dots, s_n(\gamma(t))\big)$, for all $t \in \mathbb{R}$, and $|s_i(\gamma(t))| = \sigma_i(\gamma(t))$, for all $i = 1, \dots, n$, and all $t \in \mathbb{R}$;*

(ii) *$U_\gamma, V_\gamma \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}^{n\times n})$ orthogonal, with*

$$U_\gamma(t+1) = U_\gamma(t)\ D\ ,\ \ V_\gamma(t+1) = V_\gamma(t)\ D\ ,\quad \forall t \in \mathbb{R}\ ,$$

where D is as in Theorem 3.5.

3.2. **Nongeneric coalescing and crossing.** At this point, we observe that our results have been "built upon" Theorem 2.2. The assumptions in that theorem were motivated by the realization that when the curves $\Gamma_1 = \{x : (a-d)(x) = 0\}$ and $\Gamma_2 = \{x : b(x) = 0\}$ intersect each other, they are expected to do so transversally. This is the generic case. If not, we would need to have three conditions satisfied at a point ξ_0 where eigenvalues coalesce: $(a-d)(\xi_0) = 0$, $b(\xi_0) = 0$, and $\det \begin{bmatrix} \nabla(a-d) \\ \nabla b \end{bmatrix}_{\xi_0} = 0$. This means that non-transversal intersection has co-dimension 3 and thus is a nongeneric property for functions of 2 parameters.

However, closer inspection of the proof of Theorem 2.2 reveals that transversal intersection of Γ_1 and Γ_2 is not strictly necessary. What we really need in Theorem 2.2 is that the curves Γ_1 and Γ_2 *cross each other at* ξ_0. Let us clarify what we mean by this.

3.2.1. *Crossing: Symmetric eigenproblem.* First, consider the case of symmetric $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{2\times 2})$, $e \geq 1$. Notation is the same as in Theorem 2.2.

As usual, we assume that 0 be a regular value for the functions $a-d$ and b, so that the curves $\Gamma_{1,2}$ through ξ_0 are $\mathcal{C}^e$ curves, with non-vanishing gradients along the curves. Consider the two vectors $\nabla(a-d)(\xi_0)$ and $\nabla b(\xi_0)$, both of which are non zero. If these vectors are independent, then the two curves intersect transversally in the sense of Definition 1.10. If they are not independent, then we must have $\nabla(a-d)(\xi_0) = \kappa \nabla b(\xi_0)$ for some constant $\kappa \neq 0$. In particular, the first or the second component of these two vectors, or both, must be different from 0. By virtue of the implicit function theorem, this means that –locally, near ξ_0– both Γ_1 and Γ_2 may be parametrized in terms of the same variable: x_1 if the second component of the tangent vectors is not 0, or x_2 if the first is not 0. Without loss of generality, we can assume that $b_{x_2}(\xi_0) \neq 0$ and write, near ξ_0, $\Gamma_2 = \{x = \xi_0 + \big(t,\ g(t)\big)\}$, for t in a neighborhood of the origin, say $t \in (-\eta,\ \eta)$, $\eta > 0$. Define $\varphi(t) = (a-d)\,\big(\xi_0 + \big(t,\ g(t)\big)\big)$, $t \in (-\eta, \eta)$. By construction, $g(0) = \varphi(0) = 0$. Again appealing to the implicit function theorem, g, and hence φ, is a $\mathcal{C}^e$ function of t. Also, notice that $g'(t) = -b_{x_1}/b_{x_2}$, at points $x = \xi_0 + (t, g(t))$, $t \in (-\eta, \eta)$. In particular, since $\nabla(a-d)(\xi_0) = \kappa \nabla b(\xi_0)$, we have $\varphi'(0) = 0$.

Definition 3.7. We say that Γ_1 and Γ_2 *cross each other at* ξ_0 if they intersect transversally in the sense of Definition 1.10, or the function φ above changes sign as t goes through 0.

We observe that, by using the above characterization of curves crossing, instead of their transversal intersection, the construction and proof of Theorem 2.2 hold unchanged, as it is plainly seen.

Now, assuming enough differentiability for the function A, and hence for φ, one can characterize more precisely the order of contact of the curves Γ_1 and Γ_2. In the discussion that follows, the degree of differentiability e of $A \in \mathcal{C}^e$, will be assumed to be sufficiently high so that all derivative terms we take make sense. We reiterate that $\varphi(0) = 0 = \varphi'(0)$ in case Γ_1 and Γ_2 do not intersect transversally.

Definition 3.8 (Order of Contact)**.** Γ_1 and Γ_2 have a contact of order $p = 0$ at ξ_0 if they intersect transversally there. Also, Γ_1 and Γ_2 have a contact of order $p \geq 1$ at ξ_0 if $\varphi^{(j)}(0) = 0$, $0 \leq j \leq p$, but $\varphi^{(p+1)}(0) \neq 0$.

Definition 3.8 allows us to formally define multiplicity of coalescing for the eigenvalues themselves, by tracing it back to the order of contact of the curves Γ_1 and Γ_2 at ξ_0. More precisely, we have:

Definition 3.9 (Multiplicity of Coalescing of Eigenvalues)**.** We say that the eigenvalues have a coalescing of multiplicity $p \geq 1$ at ξ_0 if the curves Γ_1 and Γ_2 have a contact of order $p-1$ at ξ_0, in the sense of Definition 3.8.

Remark 3.10 (Crossing Equals Even Order of Contact)**.** According to Definitions 3.7 and 3.9, Γ_1 and Γ_2 cross each other at ξ_0 if there exist an index $k \geq 1$ for which $\varphi^{(j)}(0) = 0$, $0 \leq j \leq 2k$, but $\varphi^{(2k+1)}(0) \neq 0$. In other words, crossing is the same as even order of contact.

Next, we want to extend the above concepts to the case of a symmetric function $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$. Informally, we want to apply the (2×2) case locally. We have the following immediate adaptation of Definitions 2.6 and 3.1, with the same notation used there.

Definition 3.11. Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$ be a symmetric function with ordered eigenvalues $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$, for all $x \in \Omega$. Let $\xi_0 \in \Omega$ be a coalescing point, that is, for some k, $1 \leq k \leq n$, $\lambda_k(\xi_0) = \lambda_{k+1}(\xi_0)$. Let R be a sufficiently small rectangular region $R \subseteq \Omega$ containing ξ_0 in its interior, and such that ξ_0 is the only coalescing point in R. Let $Q \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ be a $\mathcal{C}^e$ orthogonal function achieving the reduction:

$$Q^T(x)A(x)Q(x) = \begin{bmatrix} \Lambda_1(x) & 0 & 0 \\ 0 & P(x) & 0 \\ 0 & 0 & \Lambda_2(x) \end{bmatrix}\ ,\ \forall x \in R\ ,$$

where $P \in \mathcal{C}^e(R, \mathbb{R}^{2\times 2})$ is symmetric with eigenvalues $\lambda_k(x), \lambda_{k+1}(x)$, and Λ_1 and Λ_2 are diagonal functions containing the other eigenvalues. Let $P(x) = \begin{bmatrix} a(x) & b(x) \\ b(x) & d(x) \end{bmatrix}$, and let the curves Γ_1 and Γ_2 be as in the (2×2) case. Then, we say that the curves Γ_1 and Γ_2 cross each other at ξ_0 if they do so according to Definition 3.7. Finally, with the obvious extension of Definitions 3.8 and 3.9 to this case, we will say that eigenvalues have a coalescing of multiplicity $p \geq 1$ at ξ_0 if the curves Γ_1 and Γ_2 have a contact of order $p-1$ there.

To justify Definition 3.11, we need to show that –in this $(n \times n)$ case– the above concept of order of contact is independent of the choice of $\mathcal{C}^e$ transformation Q which achieves the block-diagonalization of Definition 3.11. This is actually correct, see below. First of all, because of Theorem 2.7, we know that transversal intersection (hence, contact of order 0) is independent of the choice of the $\mathcal{C}^e$ transformation Q. In general, we have the following result.

Theorem 3.12. *Let A, ξ_0, R, Q and P be as in Definition 3.11. Let $U \in \mathcal{C}^e(R, \mathbb{R}^{n\times n})$ be another orthogonal function achieving a similar block reduction:*

$$U^T(x)A(x)U(x) = \begin{bmatrix} \Lambda_1(x) & 0 & 0 \\ 0 & \widetilde{P}(x) & 0 \\ 0 & 0 & \Lambda_2(x) \end{bmatrix} \ , \forall x \in R \ .$$

Write $\widetilde{P}(x) = \begin{bmatrix} \widetilde{a}(x) & \widetilde{b}(x) \\ \widetilde{b}(x) & \widetilde{d}(x) \end{bmatrix}$. Then, if the curves $\Gamma_1 = \{x \in R : (a-d)(x) = 0\}$ and $\Gamma_2 = \{x \in R : b(x) = 0\}$ have a contact of order $p \geq 1$ at ξ_0, so do the curves $\widetilde{\Gamma}_1 = \{x \in R : (\widetilde{a} - \widetilde{d})(x) = 0\}$ and $\widetilde{\Gamma}_2 = \{x \in R : \widetilde{b}(x) = 0\}$.

Proof. We begin by observing that, from (2.3-2.4) in Theorem 2.7, it follows that ∇b and $\nabla \widetilde{b}$ are parallel to each other in case Γ_1 and Γ_2 have a non-transversal intersection at ξ_0. Therefore, by the implicit function theorem, near ξ_0 the curves Γ_2 and $\widetilde{\Gamma}_2$ can be parameterized in terms of the same variable. So, without loss of generality, we can assume that they are parameterized in terms of x_1: $\Gamma_2 = \{x = \xi_0 + (t,\ g(t))\}$, $\widetilde{\Gamma}_2 = \{x = \xi_0 + (t,\ \widetilde{g}(t))\}$, for $t \in (-\eta,\ \eta)$, $\eta > 0$. Let $\varphi(t) = (a-d)\left(\xi_0 + (t,\ g(t))\right)$ and $\widetilde{\varphi}(t) = (\widetilde{a} - \widetilde{d})\left(\xi_0 + (t,\ \widetilde{g}(t))\right)$. By hypothesis, $t = 0$ is a zero of multiplicity p for φ. We want to show that the same is true for $\widetilde{\varphi}$. By virtue of Theorem 2.7, ξ_0 must be a point of non-transversal intersection also for $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, so that $\widetilde{\varphi}(0) = \widetilde{\varphi}'(0) = 0$. Let us suppose, by contradiction, that $t = 0$ is a zero of multiplicity q, $q < p$, for $\widetilde{\varphi}$. Recall that, for all $x \in R$, P and $\widetilde{P}$ are orthogonally similar (see Theorem 1.6): $\widetilde{P}(x) = V^T(x)P(x)V(x)$, with orthogonal $V \in \mathcal{C}^e(R, \mathbb{R}^{2\times 2})$.

Since $t = 0$ is a zero of multiplicity p for φ, we have: $\varphi^{(j)}(0) = 0$, $0 \leq j \leq p$, but $\varphi^{(p+1)}(0) \neq 0$, and we can write $\varphi(t) = t^{p+1}\eta(t)$, with $\eta(t) \neq 0$ for all t in a small neighborhood of the origin. Now, for $\epsilon > 0$, define the function $\rho(\epsilon, t)$ from

$$\prod_{i=1}^{p+1} \left(t - \frac{\varepsilon}{i}\right) = t^{p+1} + \rho(\varepsilon, t) \ ,$$

and consider the function $\varphi_\varepsilon(t) = \varphi(t) + \rho(\varepsilon, t)\eta(t)$. By construction, φ_ε has $p+1$ distinct zeros for $t \in (-\eta, \eta)$, for all $\varepsilon \neq 0$ sufficiently small, say $0 < \varepsilon < \bar{\varepsilon}$. Hence, by additively perturbing $P(\cdot)$ with the symmetric matrix function $E(x, \varepsilon) = \text{diag}\left(\begin{bmatrix} \rho(\varepsilon, x_1)\eta(x_1) & 0 \end{bmatrix}\right)$, we obtain a matrix $P_\varepsilon(x) = P(x) + E(x, \varepsilon)$ which has p distinct transversal coalescing points in a small neighborhood of ξ_0 (one for each zero of φ_ε), for all $\varepsilon \neq 0$ sufficiently small.

Next, define $\widetilde{P}_\varepsilon = V^T P_\varepsilon V$ and modify in an obvious way $\widetilde{\varphi}$ into $\widetilde{\varphi}_\varepsilon$. The zeros of φ_ε and $\widetilde{\varphi}_\varepsilon$ correspond to coalescing points for the eigenvalues of P_ε and $\widetilde{P}_\varepsilon$ respectively. Since P_ε and $\widetilde{P}_\varepsilon$ are similar, φ_ε and $\widetilde{\varphi}_\varepsilon$ have exactly the same $p+1$ distinct zeros for $t \in (-\eta, \eta)$ and $\varepsilon \in (0, \bar{\varepsilon})$, which are all approaching 0 as ε goes to 0. Therefore, in the limit as ε approaches 0, we can conclude that $\widetilde{\varphi}$ has a zero of multiplicity at least $p > q$ for $t = 0$, which contradicts our earlier assumption. The case $q > p$ needs not be discussed, as the roles of φ and $\widetilde{\varphi}$ can be interchanged. □

3.2.2. *Crossing: SVD.* We need to extend the results relative to non-generic crossing we saw for the symmetric eigenproblem to the case of singular values. Recall that, see Remark 2.16, the pair of singular values which will coalesce cannot be 0. Thus, there is a neighborhood of the coalescing singular values where the function A is invertible. Now, appealing to Lemma A-1 which we present in the appendix, we can write $A = QP$, with Q orthogonal, P symmetric and positive definite, and both as smooth as A in a neighborhood of a coalescing point. Obviously, the singular values are the same as the eigenvalues of P, and the functions AA^T and A^TA are smoothly similar via Q. As a consequence, we can immediately extend Definitions 3.7, 3.8, 3.9, and 3.11, to the case of singular values.

Finally, it is simple to appreciate that, with the obvious modifications for the singular values, all of Theorems 2.8, 2.13 and 2.20 hold unchanged in their proofs and conclusions by replacing the assumption of generic coalescing point with that of coalescing point of odd multiplicity for the eigenvalues (singular values), as appropriate. Likewise, by replacing in Definitions 3.1 (and similar modifications in 3.2) the request that ξ_0 be a "generic coalescing point" with that of "coalescing point of odd multiplicity for the eigenvalues (singular values)", also Theorems 3.3, 3.5, and 3.6, hold unchanged.

3.3. **Main Result: Generalization.** The last part of this Section is devoted to our main Theorem, which is an important and useful consequence of the previous results, chiefly Lemma 1.7. With the generalization afforded by the considerations of Section 3.2, Theorems 3.5 and 3.6 "played backwards" give the following result which forms the basis for algorithms ([7]) which attempt to locate coalescing points.

Theorem 3.13. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$, $e \geq 1$, symmetric, with continuous and ordered eigenvalues (respectively, $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$ have continuous and ordered singular values). Let Γ be a simple closed curve in Ω, with no coalescing point for the eigenvalues (respectively, singular values) of A on it. Let γ be a $\mathcal{C}^p$, 1-periodic parametrization for Γ, let $m = \min(e,p)$, and let $A_\gamma \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}^{n\times n})$, U_γ (and V_γ) be defined as in Theorems 3.5 (and 3.6). Finally, let $U_0 = U_\gamma(0)$ and $U_1 = U_\gamma(1)$, and define $D = U_0^T U_1$.*

Next, let $2q$ be the (even) number of indices k_i, $k_1 < k_2 < \cdots < k_{2q}$, for which $D_{k_ik_i} = -1$. Let us group these indices in pairs $(k_1, k_2), \dots, (k_{2q-1}, k_{2q})$. Then, λ_k and λ_{k+1} (respectively, σ_k and σ_{k+1}) coalesce at least once inside the region encircled by Γ, if $k_{2j-1} \leq k < k_{2j}$ for some $j = 1, \dots, q$.

Remarks 3.14. Even though Theorem 3.13 above does not need a proof, some points need to be clarified.

(a) In the statement, we claim that the number of indices k_i for which $D_{k_ik_i} = -1$ is even. Here we justify the claim only for the eigenproblem, the SVD case being similar. Let A, Γ, γ, U_γ, Λ_γ and D be as usual. Then we have $A_\gamma(t) = U_\gamma(t)\Lambda_\gamma(t)U_\gamma^T(t)$, with $U_\gamma(t+1) = U_\gamma(t)\ D$. Being $U_\gamma(t)$ continuous for all t, we have that its determinant must be constantly equal either to 1 or to -1. Now, since $D = U_\gamma^T(0)U_\gamma(1)$, we have that $\det(D) = 1$, hence the claim is justified.

(b) In the conclusion of Theorem 3.13, we cannot rule out the possibility that the eigenvalues (singular values) coalesce infinitely many times, though this is certainly not a generic situation. At the same time, the case of infinitely many coalescing points may as well go undetected. To clarify, see Example 3.15 below.

(c) We stress that we expect all coalescing points to have multiplicity 1 (this is the generic property). In this case, we can illustrate Theorem 3.13 as follows. Suppose we have $n \geq 4$, and D with $D_{11} = -1 = D_{44}$, all other D_{ii}'s being 1. Then, we expect that inside the region encircled by Γ, the pairs (λ_1, λ_2), (λ_2, λ_3), and (λ_3, λ_4), have coalesced.

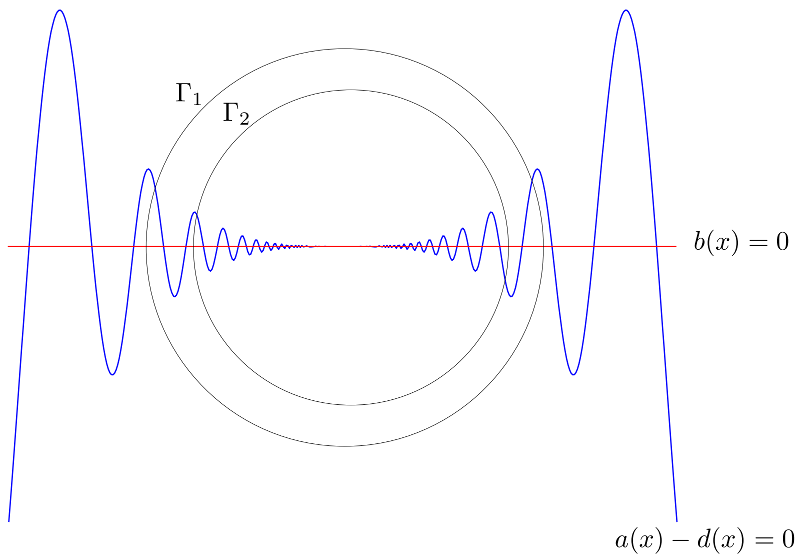

FIGURE 5. Infinitely many intersections: period 1 or 2.

(d) Unfortunately, even in case of generic coalescing points, it is impossible to distinguish, by the arguments above, whether some pair of eigenvalues (respectively, singular values) coalesce an even number of times or do not coalesce at all inside Γ.
(e) We emphasize that one may have values -1 in the matrix D of Theorem 3.13, while also having *degenerate* coalescing points inside Γ. That is, points where the underlying assumptions on which our constructions ultimately rest (namely, those of the Regular Value Theorem 1.8) are violated; see Example 3.16.
(f) Example 3.16 notwithstanding, if all coalescing points occur as intersections of smooth curves with well defined tangents, as defined through the Regular Value Theorem, then in Theorem 3.13 we can quantify more precisely the number of times that the eigenvalues, respectively singular values, coalesce. More precisely, with the notation of Theorem 3.13, we can say that λ_k and λ_{k+1} (respectively, σ_k and σ_{k+1}) will have coalesced an odd number of times, counting multiplicities, or infinitely many times, inside the region encircled by Γ.

Example 3.15. Take $x = (x_1, x_2)$, and consider the $\mathcal{C}^1$ function

$$A(x) = \begin{bmatrix} x_2 & x_2 \\ x_2 & x_1^3 \sin(1/x_1) \end{bmatrix} ,$$

whose eigenvalues coalesce on the x_1-axis (and only there), infinitely many times. Let Γ be a simple closed curve containing the origin inside. With usual notation, we may have that U_γ is 1-periodic: $U_1 = U_0$ (e.g., along Γ_1) or 2-periodic: $U_1 = -U_0$ (e.g., along Γ_2). See Figure 5.

Example 3.16. [6] Take $x = (x_1, x_2)$, and consider the function

$$A(x) = \begin{bmatrix} (x_1 - 1)(x_1^2 + x_2^2) & x_2 \\ x_2 & 0 \end{bmatrix} .$$

The eigenvalues coalesce at the origin and at the point $(1, 0)$. Taking Γ to be a sufficiently large circle around the origin, one obtains $D = -I$, and can thus infer the existence of a point of intersection. In this case, the point $(1, 0)$ has been responsible for the values of -1 in D, whereas the origin is a degenerate coalescing point and has no impact on the matrix D. Notice that the value 0 is not a regular value for the function $(x_1 - 1)(x_1^2 + x_2^2)$.

APPENDIX

The result below was used in Section 3.2.2. Although the proof is simple, the result may be of independent interest.

Lemma A-1. *Let $A \in \mathcal{C}^e(\Omega, \mathbb{R}^{n\times n})$. Assume that $A_0 := A(\xi_0)$ is invertible. Then, there is an open neighborhood of ξ_0, call it Ω_0, where A admits the polar factorization $A = QP$, where $Q \in \mathcal{C}^e(\Omega_0, \mathbb{R}^{n\times n})$ is orthogonal, and $P \in \mathcal{C}^e(\Omega_0, \mathbb{R}^{n\times n})$ is symmetric positive definite.*

Proof. It is well known that the matrix A_0, being invertible, admits a unique polar factorization $A_0 = Q_0 P_0$, with Q_0 orthogonal and P_0 symmetric positive definite; e.g., see [9]. The matrix P_0 is the unique positive definite square root of $A_0^T A_0$. Since the function A is invertible in a neighborhood Ω_0 of ξ_0, then it admits a unique polar factorization at each given $\xi \in \Omega_0$. That the factors are smooth in a neighborhood of ξ_0 is a consequence of the implicit function theorem. To witness, consider the matrix equation for the square root P: $F(P) := P^2 - A^T A = 0$. The Frechét derivative of F at P_0 is the linear operator $F'(P_0) : X \to P_0 X + X P_0$, which is invertible, since P_0 is positive definite. Therefore, there is a unique $\mathcal{C}^e$ (symmetric positive definite) square root P of $A^T A$ in a neighborhood of ξ_0, and passing through P_0, and hence a unique $\mathcal{C}^e$ polar factorization, by letting $Q = AP^{-1}$. □

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