Draft version (with a few typos corrected) of December 10, 2014

FUNDAMENTAL MATRIX SOLUTIONS OF PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS

LUCA DIECI AND LUCIANO LOPEZ

Abstract. We consider the fundamental matrix solution associated to piecewise smooth differential systems of Filippov type, in which the vector field varies discontinuously as solution trajectories reach one or more surfaces. We review the cases of transversal intersection and of sliding motion on one surface. We also consider the case when sliding motion takes place on the intersection of two or more surfaces. Numerical results are also given.

1. Introduction and Background

Our purpose in this paper is to survey definitions and properties of the fundamental matrix solution associated to piecewise smooth differential equations. Many of the results we give are available in the literature, but are not all readily available. Moreover, some of the extensions we consider herein, such as when there is sliding motion on intersection of surfaces, appear new.

We study differential equations with discontinuous right hand-side, and more precisely equations in which the right hand-side changes discontinuously as one or more surfaces are crossed. These surfaces are called *discontinuity or switching surfaces* and the systems under study are called *piecewise smooth dynamical systems* in [9].

We only consider the case of continuous solutions for an initial value problem associated to these systems (e.g., "impact" systems are not treated). The interesting case is what happens to a trajectory when it reaches a switching surface. Loosely speaking, there are two things which can occur: we may cross the surface, or we may stay on it, in which case Filippov's construction (see [11]) will define a vector field on the sliding surface and the motion will be called *sliding mode*.

Piecewise smooth differential equations appear pervasively in applications of various nature (see, e.g. [5, 14, 15, 17, 24]); for a significative sample of references in the context of control, see e.g. [25, 26, 27]; in the context of biological systems, see e.g. [7, 8, 15, 24]; in the context of mechanical systems, see e.g. [13, 20, 21]; see the classical references [4, 11, 26, 27] and the recent book [9] for a theoretical introduction to these systems, and finally see the recent book [1] for a review of numerical methods used on these systems.

In [2, 11, 18, 20, 21, 23] the authors consider fundamental matrix solutions of a nonsmooth system, and introduce the *saltation* matrix in order to take into account the lack of smoothness of the differential system as the solution crosses transversally a discontinuity surface. In [16], the fundamental matrix solutions of a nonsmooth system with two intersecting discontinuity surfaces has been studied in the case of solutions which cross the intersection transversally. The formal representation of the fundamental matrix solution has partly been used in defining the concept of Lyapunov exponents for non-smooth systems (see [19]). All of these cited results are relative to the case of a solution which crosses transversally a surface of discontinuity (or the intersection of two such surfaces). In the case of sliding motion on one surface of discontinuity, several authors have adapted to this case the concept of saltation matrix (see [11, 20, 21, 23]), and this methodology

1991 *Mathematics Subject Classification.* ....

Key words and phrases. Piecewise smooth dynamical systems, Filippov, sliding modes, fundamental matrix.

has proven valuable in practical experiments as well. But, in the case of sliding on the intersection of two (or more) surfaces, no clear extension has yet been proposed for the saltation matrix. A main goal of our work is to provide this extension.

1.1. **Background.** The simplest modelization of a non smooth system is when a (global) hypersurface partitions the state space in two regions:

$$x'(t) = f(x(t)) = \begin{cases} f_1(x(t)) \ , & x \in R_1 \ , \\ f_2(x(t)) \ , & x \in R_2 \ , \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \ . \tag{1.1}$$

The state space $\mathbb{R}^n$ is split into two subspaces R_1 and R_2 by a surface Σ such that $\mathbb{R}^n = R_1 \cup \Sigma \cup R_2$. The surface is defined by a scalar *indicator* (or *event*) function $h : \mathbb{R}^n \to \mathbb{R}$, so that the subspaces R_1 and R_2, and the hypersurface Σ, are implicitly characterized as

$$\Sigma = \{x \in \mathbb{R}^n |\ h(x) = 0\} \,, \quad R_1 = \{x \in \mathbb{R}^n |\ h(x) < 0\} \,, \quad R_2 = \{x \in \mathbb{R}^n |\ h(x) > 0\} \,. \tag{1.2}$$

In (1.1), we will assume that f_1 is $C^k, k \geq 1$, on $R_1 \cup \Sigma$ and f_2 is $C^k, k \geq 1$, on $R_2 \cup \Sigma$, but we will not assume that f_1, respectively f_2, extends smoothly also in R_2, respectively R_1. We will also assume that $h \in \mathcal{C}^k$, $k \geq 2$, and that $\nabla h(x) \neq 0$ for all $x \in \Sigma$. Thus, the unit normal to Σ, $n(x) = \frac{\nabla(h(x))}{\|\nabla(h(x))\|_2}$, exists and varies smoothly for all $x \in \Sigma$.

Obviously, in (1.1), $f(x(t))$ is not defined if $x(t)$ is on Σ and a standard way to overcome this difficulty is to consider the set valued extension $F(x)$ below:

$$x'(t) \in F(x(t)) = \begin{cases} f_1(x(t)) & x \in R_1 \\ \overline{\text{co}}\,\{f_1(x(t), f_2(x(t)\} & x \in \Sigma \\ f_2(x(t)) & x \in R_2 \end{cases} , \tag{1.3}$$

where $\overline{\text{co}}(A)$ denotes the smallest closed convex set containing A. In our particular case:

$$\overline{\text{co}}\,\{f_1, f_2\} = \{f_\Sigma \in \mathbb{R}^n \ : \ f_\Sigma = (1-\alpha) f_1 + \alpha f_2, \ \alpha \in [0,1]\} \ . \tag{1.4}$$

The extension (or convexification) of a discontinuous system (1.1) into a convex differential inclusion (1.3) is known as *Filippov convex method*. An absolutely continuous function $x : [0, \tau) \to \mathbb{R}^n$ is a *Filippov solution* of (1.1) if for almost all $t \in [0, \tau)$ it holds that $x'(t) \in F(x(t))$.

Now, suppose that $x_0 \in R_1$, and consider a trajectory of (1.1). As long as the trajectory remains in R_1, there is nothing special about this solution of (1.1). The interesting case is when we reach a point $x \in \Sigma$, a case we consider next.

So, let $x \in \Sigma$ and let $n(x)$ be the unit normal to Σ at x. Let $n^T(x) f_1(x)$ and $n^T(x) f_2(x)$ be the projections of $f_1(x)$ and $f_2(x)$ onto the normal to the hypersurface Σ. We have two main cases.
a) *Transversal Intersection.* In case in which, at $x \in \Sigma$, we have

$$[n^T(x) f_1(x)] \cdot [n^T(x) f_2(x)] > 0 \ , \tag{1.5}$$

then we will leave Σ. We will enter R_1, when $n^T(x) f_1(x) < 0$, and will enter R_2, when $n^T(x) f_1(x) > 0$. In the former case we will have (1.1) with $f = f_1$, in the latter case with $f = f_2$. Any solution of (1.1) reaching Σ at a time t_1, and having a transversal intersection there, exists and is unique.
b) *Attracting Sliding Mode.* An attracting sliding mode at Σ occurs if

$$[n^T(x) f_1(x)] > 0 \quad \text{and} \quad [n^T(x) f_2(x)] < 0, \qquad x \in \Sigma \ , \tag{1.6}$$

where the inequality signs depend of course on (1.2). When we have an attracting sliding mode at $x_0 \in \Sigma$, a solution trajectory which reaches x_0 does not leave Σ, and will therefore have to move along Σ. Filippov's theory provides an extension to the vector field on Σ, consistent with the interpretation in (1.4), giving rise to *sliding motion*. During the sliding motion the solution will continue along Σ with time derivative f_Σ given by:

$$f_\Sigma(x) = (1 - \alpha(x)) f_1(x) + \alpha(x) f_2(x) \tag{1.7}$$

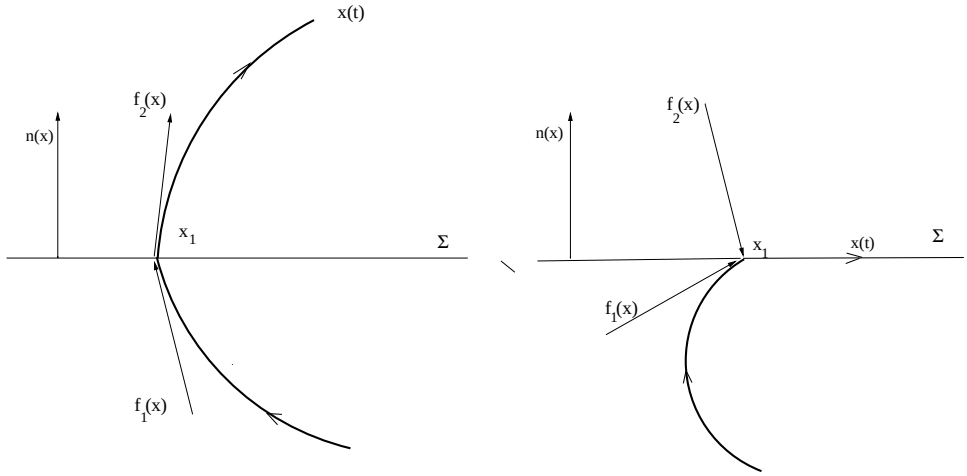

FIGURE 1. Transversal and sliding trajectory.

where $\alpha(x)$ is the value for which $f_\Sigma(x)$ lies in the tangent plane T_x of Σ at x, that is the value for which $n^T(x)f_\Sigma(x) = 0$. This gives

$$\alpha(x) = \frac{n^T(x)f_1(x)}{n^T(x)(f_1(x) - f_2(x))}. \tag{1.8}$$

In other words, an attractive (Filippov) sliding mode is the solution of

$$x'(t) \;=\; (1-\alpha(x))f_1(x) + \alpha(x)f_2(x)\ , \quad \alpha(x) = \frac{n^T(x)f_1(x)}{n^T(x)(f_1(x) - f_2(x))}\ . \tag{1.9}$$

Observe that a solution having an attracting sliding mode exists and is unique, in forward time. If the inner products in (1.6) are of opposite signs we have a *repulsive sliding mode* which does not lead to uniqueness of the solution (at any instant of time one may leave with f_1 or f_2). For this reason, we will not consider repulsive sliding motion in this work.

We are ready to recall (see [11]) a classical well-posedness result for solutions of (1.1), which includes the cases of transversal intersection and attracting sliding mode, and a combination of these:

Theorem 1.1. *Let $f_{1,2}$ be C^1 in $R_1 \cup \Sigma$ and $R_2 \cup \Sigma$, respectively, and h be C^2 on Σ. If, at any point $x \in \Sigma$ we have that at least one of $n^T(x)f_1(x) > 0$ and $n^T(x)f_2(x) < 0$ holds, then there exists a unique Filippov solution from each initial condition.*

We conclude this section with a remark about similarities between an attractive sliding mode on Σ (a solution of (1.9)), and a related differential-algebraic-equation (DAE) of index 2. The latter is written as the following system of n differential equations and 1 algebraic constraint, in $(n+1)$ unknowns, x and α:

$$\begin{cases} x'(t) \;=\; f_\Sigma(x,\alpha) = (1-\alpha)f_1(x) + \alpha f_2(x)\ , \\ h(x(t)) \;=\; 0\ . \end{cases} \tag{1.10}$$

Differentiating once the algebraic constraint, and using the form of f_Σ, one obtains

$$(\nabla h(x))^T x' = (\nabla h(x))^T \big[(1-\alpha)f_1(x) + \alpha f_2(x)\big] = 0\ ,$$

which gives precisely (1.9). However, this formal equivalence is not the full story. The sliding mode from (1.9) is just one possible mode of motion of the original piecewise smooth system, and trajectories of this system typically enter, slide on, and exit, the discontinuity surface repeatedly, whereas the DAE only defines constrained motion on Σ. Moreover, strictly speaking, (1.9) is only defined for $x \in \Sigma$, whereas the DAE (1.10) is defined, in the given form, for all values of x in a neighborhood of Σ. At the same time, this "equivalence" of sliding motion with the DAE (1.10) will provide important insight in interpreting the variational equation for the fundamental solution in the next two sections.

2. Fundamental matrix solution: Cross and/or slide on one surface

The fundamental matrix solution associated to the linearized system is a very useful tool in performing stability and bifurcation study of a smooth dynamical systems. It is natural to suspect that it should be a useful tool also for nonsmooth dynamical systems. In this and the next sections, we consider the fundamental matrix solutions for piecewise smooth systems. In this section, we look at the case in which we cross and/or slide on one surface. In the next section we also consider the case when the trajectory crosses and/or slides on the intersection of two surfaces.

2.1. **Smooth case.**

Consider the differential system:

$$x'(t) = f(x(t)) \ , \qquad x(0) = x_0 \in \mathbb{R}^n \ , \tag{2.1}$$

with f continuously differentiable. Denote with $\phi^t(x_0)$, for all $t \geq 0$, the solution of (2.1). The fundamental matrix solution is the derivative of the solution with respect to the initial condition. That is, if we let $\Phi(t,0)$ be the fundamental matrix solution, then

$$\Phi(t,0) \ = \ D_{x_0}\phi^t(x_0) \ , \tag{2.2}$$

so that $\Phi(t,0)$ satisfies

$$\begin{cases} \Phi'(t,0) = Df(\phi^t(x_0))\Phi(t,0) \ , \quad t \geq 0 \ , \\ \Phi(0,0) = I \ . \end{cases} \tag{2.3}$$

By construction, $\Phi(t,0)$ is continuous and nonsingular for all $t \geq 0$, and it enjoys several important properties. The following will be useful.

Composition property. If $t > t_1 > t_0$, then we have

$$\Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0) \ , \tag{2.4}$$

where $\Phi(t,s)$ solves the usual differential equation, (2.3) for $t \geq s$, and $\Phi(s,s) = I$.

Mapping property. Also, it is easy to see that:

$$f(\phi^t(x_0)) = \Phi(t,0)f(x_0) \ , \qquad \forall t \geq 0 \ , \tag{2.5}$$

and therefore (using (2.4)) the fundamental matrix solution carries vector fields into vector fields at later time. Using (2.2), (2.5) follows from the fact that its left and right hand sides solve the same, smooth, differential equation. Formula (2.5) is a key property of the fundamental matrix.

Magnification property. Another key property arises from very definition of the fundamental matrix, by expanding the solution about the initial condition. Namely, for two different initial conditions x_0 and y_0, one has

$$\phi^t(y_0) - \phi^t(x_0) = D_{x_0}\phi^t(x_0)(y_0 - x_0) + \mathcal{O}(\|y_0 - x_0\|^2) = \Phi(t,0)(y_0 - x_0) + \mathcal{O}(\|y_0 - x_0\|^2) \ , \tag{2.6}$$

that is –at first order– the fundamental matrix acts as magnification factor for nearby initial conditions.

Remark 2.1. Ideally, for non-smooth systems, one would like to continue having a fundamental matrix solution satisfying the three properties above. As we will see below, the properties expressed by (2.4), (2.5) and (2.6) will essentially continue to hold, for a properly defined fundamental matrix, though the formal definition of fundamental matrix as solution of (2.3) will not necessarily hold.

2.2. **Transversal crossing.** Now let us consider the discontinuous ODE:

$$x'(t) = f(x(t)) = \begin{cases} f_1(x(t)) \ , & x \in R_1 \ , \\ f_2(x(t)) \ , & x \in R_2 \ , \end{cases} \tag{2.7}$$

with $x(0) = x_0 \in R_1$.

Suppose that the solution $\phi^t(x_0)$ of (2.7) crosses Σ at $x_1 = \phi^{t_1}(x_0)$, that is

$$[n^T(x_1)f_1(x_1)] \cdot [n^T(x_1)f_2(x_1)] > 0 \ , \quad n^T(x_1)f_1(x_1) > 0 \ , \tag{2.8}$$

and that there are no further crossing of Σ for t up to t_2. Let us still denote by $\Phi(t,0)$ the "fundamental matrix solution" of the discontinuous system (2.7). Then, it must satisfy:

$$\begin{cases} \Phi(t,0) = \Phi_1(t,0) \ , & 0 \le t < t_1, \\ \Phi(t,t_1) = \Phi_2(t,t_1) \ , & t_1 < t \le t_2 \ , \end{cases} \tag{2.9}$$

where $\Phi_1(t,0)$ and $\Phi_2(t,t_1)$ are, respectively, the fundamental matrix solutions associated to

$$\begin{cases} x'(t) = f_1(x), \quad x(0) = x_0 \ : & \Phi_1'(t,0) = Df_1(\phi^t(x_0))\Phi_1(t,0) \ , \quad 0 \le t \le t_1 \ , \\ x'(t) = f_2(x), \quad x(t_1) = x_1 \ : & \Phi_2'(t,t_1) = Df_2(\phi^{t-t_1}(x_1))\Phi_2(t,t_1) \ , \quad t_1 \le t \le t_2 \ . \end{cases} \tag{2.10}$$

There is a discontinuous behavior at t_1 and we write:

$$\Phi(t_1^+,0) \ = \ S\Phi(t_1^-,0) \ , \tag{2.11}$$

where $\Phi(t_1^\pm,0) = \lim_{t\to t_1^\pm} \Phi(t,0)$.

The matrix S in (2.11) is called *jump* or *saltation* matrix, and it may be thought of as the fundamental matrix between t_1^- and t_1^+, that is $S = \Phi(t_1^+,t_1^-)$.

The correct expression for the saltation matrix, in this case of transversal intersection at x_1, was first proved in [2], and then in [11, 18, 20, 21, 23]. For completeness, here we review this derivation.

Lemma 2.2. *With above notation, we have*

$$S = I + (f_2(x_1) - f_1(x_1)) \cdot \frac{n^T(x_1)}{n^T(x_1)f_1(x_1)}, \qquad x_1 \in \Sigma \ . \tag{2.12}$$

Proof. Let $y_0 = x_0 + \Delta x_0$ be a perturbed initial condition, and suppose that $\phi^t(y_0)$ reaches the surface Σ at time $t_1 + \Delta t$. To fix ideas, suppose that $\Delta t > 0$.

We want to look at the first order expansion of the difference

$$\phi^{t_1+\Delta t}(x_0) - \phi^{t_1+\Delta t}(y_0) \, . \tag{2.13}$$

The following expansions hold:

(a) $\phi^{t_1+\Delta t}(x_0) = \phi^{t_1}(x_0) + f_2(\phi^{t_1}(x_0))\Delta t + \mathcal{O}(\Delta t^2) = x_1 + f_2(x_1)\Delta t + \mathcal{O}(\Delta t^2)$ and $\phi^{t_1+\Delta t}(y_0) = \phi^{t_1}(y_0) + f_1(\phi^{t_1}(y_0))\Delta t + \mathcal{O}(\Delta t^2) = \phi^{t_1}(y_0) + f_1(x_1)\Delta t + \text{h.o.t.}$, where "h.o.t." stands for higher order terms (in any combination of quadratic terms of Δt and $(\phi^{t_1}(y_0) - x_1)$);

(b) $0 = h(\phi^{t_1+\Delta t}(y_0)) = h\big(\phi^{t_1}(y_0) + f_1(\phi^{t_1}(y_0))\Delta t + \mathcal{O}(\Delta t^2)\big)$ and since $h(\phi^{t_1}(y_0)) = h(x_1) + (\nabla h(x_1))^T(\phi^{t_1}(y_0) - x_1) + \mathcal{O}(\|\phi^{t_1}(y_0) - x_1\|^2)$ we get

$0 = h(\phi^{t_1+\Delta t}(y_0)) = (\nabla h(x_1))^T(\phi^{t_1}(y_0) - x_1) + (\nabla h(x_1))^T f_1(x_1)\ \Delta t + \text{h.o.t.}$, from which at first order we get

$$\Delta t = -\frac{(\nabla h(x_1))^T}{(\nabla h(x_1))^T f_1(x_1)}\big(\phi^{t_1}(y_0) - x_1)\big)\,.$$

Now, at first order, from (a) we get

$$\phi^{t_1+\Delta t}(x_0) - \phi^{t_1+\Delta t}(y_0) = \phi^{t_1}(x_0) - \phi^{t_1}(y_0) + \big[f_2(x_1) - f_1(x_1)\big]\Delta t\,,$$

and using (b) we obtain the sought first order expansion

$$\phi^{t_1+\Delta t}(x_0) - \phi^{t_1+\Delta t}(y_0) = \Big[I + \big(f_2(x_1) - f_1(x_1)\big)\frac{\nabla h^T(x_1)}{\nabla h^T(x_1) f_1(x_1)}\Big]\big(\phi^{t_1}(x_0) - \phi^{t_1}(y_0)\big)\,, \tag{2.14}$$

so that (2.12) follows. The case of $\Delta t < 0$ is the same. □

Observe that

$$f_2(x_1) = S f_1(x_1)\ , \tag{2.15}$$

and that in the smooth case (i.e., $f_2(x_1) = f_1(x_1)$) it is simply $S = I$. Moreover, since $(\nabla h(x_1))^T f_2(x_1) \neq 0$, then S is non singular and

$$S^{-1} = I + (f_1(x_1) - f_2(x_1)) \cdot \frac{n^T(x_1)}{n^T(x_1) f_2(x_1)}\ .$$

We sum up by stressing that the matrix $\Phi(t, 0)$ is invertible for all $t \in [t_1, t_2]$ and thus we have continuous dependence on the initial conditions. The properties expressed by (2.4), (2.5), (2.6) hold, even though the fundamental matrix is not smooth. [Property (2.6) follows from (2.14), since $\phi^{t_1}(x_0) - \phi^{t_1}(y_0) = \Phi(t_1, 0)(x_0 - y_0) + \mathcal{O}(\|y_0 - x_0\|^2)$.]

To complete this section, below we summarize a simple spectral property of matrices which are rank-1 corrections of the identity.

Lemma 2.3. *Consider a matrix of the form*

$$S = I + ur^T\ , \tag{2.16}$$

where all vectors and matrices have dimension n. Then the eigenvalues of S are given by

$$\big\{1 + r^T u, 1, \ldots, 1\big\}\ .$$

When $r^T u = -1$, the eigenvector associated to the 0 eigenvalue is in the direction of u.

Proof. The proof is by direct verification. [Notice that the eigenvalues at 1 are a consequence of the fact that $S_{|r^\perp} = I$]. □

With notation from Lemma 2.3, in case of (2.12), we have

$$u = \frac{f_2(x_1) - f_1(x_1)}{n^T(x_1) f_1(x_1)}, \qquad r = n(x_1),$$

so that $r^T u + 1 \neq 0$, because of transversal intersection, confirming that S is invertible.

2.3. **Sliding on a surface.** In case in which a solution of (2.7), upon reaching Σ, does not cross the surface, but slides on it (see (1.7) and (1.9)), then the saltation matrix takes a different form than in (2.12). As it is well known (see [11, p.119]), the appropriate generalization of (2.12) is

$$S = I + (f_\Sigma(x_1) - f_1(x_1)) \cdot \frac{n^T(x_1)}{n^T(x_1) f_1(x_1)} \ . \tag{2.17}$$

Remark 2.4. We observe that –for attractive sliding motion– the form of the saltation matrix is the same regardless of whether we are coming from region R_1 or R_2. This is simply because at $x(t_1)$ we have

$$I + (f_\Sigma(x_1) - f_1(x_1)) \cdot \frac{n^T(x_1)}{n^T(x_1) f_1(x_1)} \ = \ I + (f_\Sigma(x_1) - f_2(x_1)) \cdot \frac{n^T(x_1)}{n^T(x_1) f_2(x_1)} \, ,$$

upon using the expression for f_Σ in (1.7) (1.8).

Also, by Lemma 2.3, we observe that (2.17) is not invertible. In fact, in this case (in the notation of Lemma 2.3) we can write

$$u = \frac{f_\Sigma(x_1) - f_1(x_1)}{n^T(x_1) f_1(x_1)}, \qquad r = n(x_1),$$

so that $r^T u = -1$ and $Su = 0$. For later reference, we summarize this simple fact in the following Lemma.

Lemma 2.5. *With above notation, the kernel of S in* (2.17) *is spanned by the vector*

$$v = f_\Sigma(x_1) - f_1(x_1) \, .$$

In particular, since $f_\Sigma(x_1) \in T_{x_1}$, if $f_1(x_1) \notin T_{x_1}$, then $v \notin T_{x_1}$.

The singularity of the saltation matrix in case of sliding motion witnesses that motion on Σ will take place on a lower dimensional manifold and moreover that we cannot uniquely trace the orbit backward in time. As far as the fundamental matrix itself, during the sliding motion, this will obey the evolution of the linearized problem with respect to the sliding vector field. That is, suppose that the solution of (2.7) will be a sliding motion for $t_1 \le t \le t_2$. Then, for $t \in [t_1, t_2]$, the fundamental matrix will be the solution of

$$\Phi_s'(t, t_1) \ = \ Df_\Sigma(x, \alpha(x)) \Phi_s(t, t_1) \ , \ \ \Phi_s(t_1, t_1) = I \ , \tag{2.18}$$

where f_Σ is defined in (1.7) and α in (1.8).

Remark 2.6. By virtue of how we generalized the concept of fundamental matrix, properties (2.4), (2.5), (2.6), hold even in the case in which the trajectory reaches Σ and slides on it.

The expression (2.18) is insightfully arrived at by considering the point of view of the DAE (1.10). First, rewrite the DAE (1.10) as $F(z, z') = 0$, where $z = (x, \alpha)$, that is

$$F(z, z') = \begin{pmatrix} x' - (1-\alpha) f_1(x) - \alpha f_2(x) \\ h(x(t)) \end{pmatrix} \ . \tag{2.19}$$

Let $z_s = (x_s, \alpha_s)$ be the sliding trajectory (the solution of (1.7)-(1.9)) and consider linearizing (2.19) about z_s, that is take the first variation for the variable $z = z_s + v$ (see [6]). From (2.19), at first order we get:

$$[F_{z'}]_{z_s} v' + [F_z]_{z_s} v \ = \ 0 \ ,$$

that is

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} v' + \begin{pmatrix} -(1-\alpha) Df_1 - \alpha Df_2 & f_1 - f_2 \\ (\nabla h)^T & 0 \end{pmatrix}_{(x_s, \alpha_s)} v \ = \ 0 \, .$$

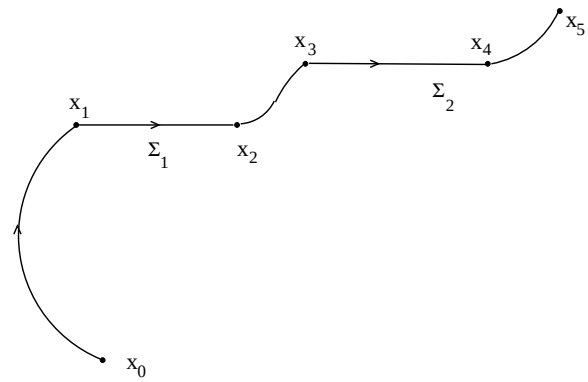

FIGURE 2. Sliding on different surfaces

Partitioning $v = (w, \beta)$, we obtain:

$$(\nabla h(x_s))^T w \;=\; 0 \;, \tag{2.20}$$

$$w' \;=\; Df_\Sigma(z_s)w \;, \quad \text{where} \quad Df_\Sigma(z_s) = [(1-\alpha)Df_1 + \alpha Df_2 - (f_1 - f_2)(\nabla\alpha)^T]_{z_s} \;, \tag{2.21}$$

which justifies (2.18).

Observe that (2.20) tells us that the corrections under the linearized flow are on the tangent plane to the surface. In fact, as a consequence of (2.21) and (2.20), if we let $y_1 \in T_{x_1}$, and set $y(t) = \Phi_s(t, t_1)y_1$, then $y(t) \in T_{\phi^t(x_1)}$. We summarize this observation in the form of a Lemma.

Lemma 2.7. *The fundamental matrix $\Phi_s(t, t_1)$, associated to linearization along a sliding trajectory on the surface Σ from x_1 to $\phi^t(x_1)$, brings the tangent space at x_1 into the tangent space at $\phi^t(x_1)$.*

2.3.1. *Finite loss of rank.* Suppose we have a trajectory, $\phi^t(x_0)$, which reaches Σ non-tangentially at time t_1, and exits the $(n-1)$ dimensional surface Σ tangentially at a later time t_2. Let x_1 and x_2 be the corresponding enter and exit points, so that (assuming for simplicity that the solution comes from, and returns in, the same region R_1) the solution arrives in Σ with vector field $f_1(x_1)$ and leaves Σ with vector field $f_\Sigma(x_2) = f_1(x_2)$. Let us also assume that the associated (Filippov) vector field never vanishes. Since the underlying fundamental matrix maps vector fields into vector fields (property (2.5)), we plainly have that the (non-smooth) fundamental matrix associated to this trajectory has always at least rank 1. In fact, this remains true even if the trajectory repeatedly enters (non-tangentially) and exits (tangentially) from Σ (see for instance [22]).

Below, we show that, under reasonable conditions, the fundamental matrix remains always of rank $(n-1)$, even if a trajectory progressively enters, slides, and exits, from a surface Σ of dimension $(n-1)$, regardless of how many sliding segments there are on Σ. In fact, a similar result holds when we slide on possibly different surfaces, as long as each of them is of dimension $(n-1)$, see Fig. 2. To prove the result, we will need that every vector in T_{x_2} (and not just the vector field itself) returns to Σ, under the linearized flow, in a different direction than that of the null space of the corresponding saltation matrix. This is guaranteed if the $(n-1)$ vectors in $\mathbb{R}^n$ spanning the tangent space cannot be used to obtain a vector in $\mathbb{R}^n$ in the direction of the 1-dim null space of the saltation matrix, a mild requirement.

Theorem 2.8. *Consider a trajectory $\phi^t(x_0)$ of (2.7) which slides on a surface of dimension $(n-1)$ (or on several different surfaces, each of dimension $(n-1)$), between the points $x_{2k+1} = \phi^{t_{2k+1}}(x_0)$*

and $x_{2k+2} = \phi^{t_{2k+2}}(x_0)$, for $k = 0, \dots, K$, $x_{2k+1} \neq x_{2k+2}$. Assume also that the trajectory exits tangentially (hence, smoothly) at the points x_{2k+2} and that it enters the surface(s) nontangentially at x_{2k+1}. Finally, let the trajectory be at x_0 at time $t = 0$ and at $x_N = \phi^{t_N}(x_0)$, at time $t_N > t_{2K+2}$, not on the surface. Let S_{2k+1} be the saltation matrices at the points x_{2k+1}, $k = 0, \dots, K$. Assume that $S_{2k+1}\Phi(t_{2k+1}, t_{2k})w \neq 0$ for any vector $w \in T_{x_{2k+2}}$, $k = 1, \dots, K$. Then, the fundamental matrix solution $\Phi(t_N, 0)$ has rank $(n-1)$, independently of K.

Proof. It suffices to consider the case of the solution sliding on the arcs (x_1, x_2) and (x_3, x_4), the general case will follow by a recursive argument. We can write

$$\Phi(t_5, 0) = \Phi(t_5, t_4)\Phi_s(t_4, t_3)S_3\Phi(t_3, t_2)\Phi_s(t_2, t_1)S_1\Phi(t_1, 0)\,,$$

where S_1 and S_3 are the two saltation matrices at x_1 and x_3, respectively, $\Phi_s(t_2, t_1)$ and $\Phi_s(t_4, t_3)$ are the two fundamental matrices on the sliding segments (x_1, x_2) and (x_3, x_4), respectively, and $\Phi(t_1, 0), \Phi(t_3, t_2), \Phi(t_5, t_4)$ are the fundamental matrices in between $[0, t_1^-), [t_2, t_3^-), [t_4, t_5^-)$. To establish the result, we need to show that the matrix $M = \Phi_s(t_4, t_3)S_3\Phi(t_3, t_2)\Phi_s(t_2, t_1)S_1\Phi(t_1, 0)$ is of rank $(n-1)$.

Let $V_{x_1} \in \mathbb{R}^{n\times(n-1)}$ be a matrix representation of a basis for T_{x_1}, and let v_1 be a vector in the null space of S_1. Since the trajectory enters x_1 non-tangentially, $v_1 \notin \mathrm{span}(V_{x_1})$. Represent $\Phi(t_1, 0)$ in the basis of $\mathbb{R}^n$ given by $[V_{x_1}\ ,\ v_1]$: $\Phi(t_1, 0) = [V_{x_1}\ ,\ v_1]B_1$, where $B_1 \in \mathbb{R}^{n\times n}$ is invertible. Notice that $S_1[V_{x_1}\ ,\ v_1] = [V_{x_1}\ ,\ 0]$ and recall that $\Phi_s(t_2, t_1)V_{x_1} = V_{x_2} \in \mathbb{R}^{n\times(n-1)}$ is a matrix representation for a basis of T_{x_2}. So, we have

$$M = \Phi_s(t_4, t_3)S_3\Phi(t_3, t_2)V_{x_2}[I_{n-1}\ ,\ 0]B_1\,.$$

Since B_1 and $\Phi_s(t_4, t_3)$ are invertible, we only need to show that $A = S_3\Phi(t_3, t_2)V_{x_2}[I_{n-1}\ ,\ 0]$ has rank $(n-1)$. Clearly, $A\begin{bmatrix}0\\1\end{bmatrix} = 0$. By contradiction, suppose there is another vector $w \in \mathbb{R}^n$ in the kernel of A, not in the direction of $\begin{bmatrix}0\\1\end{bmatrix}$. So, we can partition $w = \begin{bmatrix}w_1\\w_2\end{bmatrix}$ with $w_1 \in \mathbb{R}^{n-1}$, $w_1 \neq 0$. Then it follows that $0 = Aw = S_3\Phi(t_3, t_2)V_{x_2}w_1$, which means that $\Phi(t_3, t_2)$ brings a vector from T_{x_2} in the direction of the null space of S_3, contradicting our assumptions. □

As we will see in Section 3, there can be a more severe loss of rank, by 2, when we slide on the intersection of two surfaces of dimension $(n-1)$. In that case, the trajectory would be sliding on a manifold of dimension $(n-2)$, and the dimension of the manifold along which the trajectory slides impacts the dimension of the null space of the corresponding saltation matrix, hence the rank of the fundamental matrix.

2.4. **Periodic sliding case: 1 surface.** Now let us consider the case in which a solution of (2.7) is periodic. We are interested in two cases.

2.4.1. *Case 1.* First take $\phi^t(x_0)$ to be a periodic solution of minimal period $\tau > 0$, which starts off Σ, then enters in Σ at a point x_1, has attractive sliding motion on it, leaves Σ at x_2, re-enters at x_1 and repeats this cycle. Without loss of generality, we suppose that the initial condition x_0 belongs to R_1, we let $x_1 = x(t_1)$ the point where the trajectory hits Σ, and by $x_2 = x(t_2)$ the point where the trajectory leaves Σ to entry R_1, with vector field f_1 (that is, we will leave Σ smoothly).

From 0 to t_1 the solution evolves in R_1 and $\Phi_1(t, 0)$ is the associated fundamental matrix solution, from t_1 to t_2 the solution evolves on Σ and $\Phi_S(t, t_1^+)$ is the associated fundamental matrix solution, and from t_2 to τ the solution is again in R_1 with $\Phi_1(t, t_2)$ the associated fundamental matrix solution. More complicated scenarios, with several segments of sliding motion, are generalization

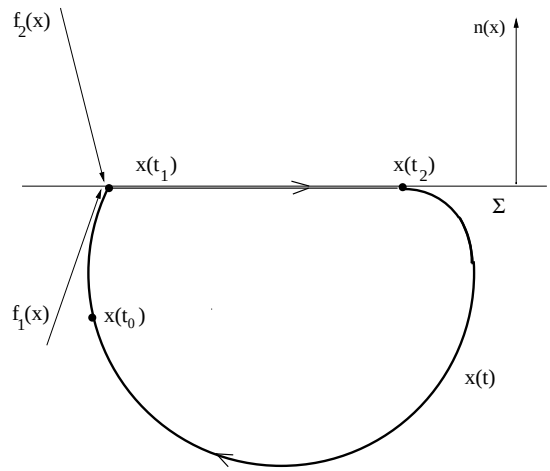

FIGURE 3. Periodic sliding trajectory

of the present one. Recall that the Filippov's system is:

$$x' = f(x) = \begin{cases} f_1(x) \ , & x \in R_1 \ , \\ f_\Sigma(x) \ , & x \in \Sigma \ , \\ f_2(x) \ , & x \in R_2 \ , \end{cases} \tag{2.22}$$

where $f_\Sigma(x)$ is given by (1.7), (1.8).

Denote with $\Phi(t,0)$ the "fundamental matrix solution" (for $t \in (0,\tau)$) of the discontinuous system (2.22); we want to look at the "monodromy" matrix $\Phi(\tau,0)$. An obvious property that the monodromy must have is that

$$f(\phi^\tau(x_0)) = \Phi(\tau,0)f(x_0) \ ,$$

that is it must have an eigenvalue equal to 1.

Now, from (2.5) it follows that:

$$\begin{cases} f_1(x_1) = \Phi_1(t_1^-,0))f_1(x_0) \\ f_\Sigma(x_2) = \Phi_\Sigma(t_2,t_1)f_\Sigma(x_1) \\ f_1(x(\tau)) = \Phi_1(\tau,t_2)f_1(x_2) \ . \end{cases}$$

So, the fundamental matrix solution $\Phi(t,0)$ is discontinuous at t_1 and:

$$\Phi(t_1^+,0) = S\Phi(t_1^-,0) \tag{2.23}$$

where the saltation matrix is given by (2.17).

From (2.17), observing that $f_1(x_2) = f_\Sigma(x_2)$, we have:

$$f_1(x(\tau)) = \Phi_1(\tau,t_2)f_1(x_2) = \Phi_1(\tau,t_2)\Phi_\Sigma(t_2,t_1^+)f_\Sigma(x_1) =$$

$$= \Phi_1(\tau,t_2)\Phi_\Sigma(t_2,t_1^+)Sf_1(x_1) = \Phi_1(\tau,t_2)\Phi_\Sigma(t_2,t_1^+)S\Phi_1(t_1^-,0)f_1(x_0) \ ,$$

so that the monodromy matrix has the form

$$\Phi(\tau,0) = \Phi_1(\tau,t_2)\Phi_\Sigma(t_2,t_1^+)S\Phi_1(t_1^-,0) \ . \tag{2.24}$$

By virtue of the discussion in the previous sections, we immediately have:

Theorem 2.9. *The monodromy matrix in* (2.24) *has one eigenvalue equals to* 1 *because of periodicity, and one eigenvalue equals to* 0*, because of the singularity of the saltation matrix* S*. The remaining eigenvalues measure stability of the periodic orbit.*

2.4.2. *Case 2.* As a second case, consider the situation in which the periodic solution is an attracting sliding mode. In other words, we have a periodic solution on the surface Σ:

$$x_0 \in \Sigma \ , \ \ \phi^t(x_0) \in \Sigma \ , \ \text{for all}\, t \ , \ \ \phi^\tau(x_0) = x_0 \, .$$

In this case the fundamental matrix satisfies (2.18) and remains always invertible, although the motion is taking place on a $(n-1)$-dimensional manifold. This is easily explained by recalling that –in this case– the fundamental matrix makes vectors in the tangent plane evolve into vectors in the tangent plane. As a consequence, the monodromy matrix will not only have an eigenvalue equal to 1 as always, relative to the vector field itself, but will also have an invariant subspace relative to all vectors in the tangent plane.

Theorem 2.10. *Let* $\phi^t(x_0)$ *be an attracting sliding mode, periodic of period* τ*. Let* $\Phi(\tau, 0)$ *be monodromy matrix associated to the linearized problem for* $\phi^t(x_0)$*,* $t \in [0, \tau]$*. Then,* $\Phi(\tau, 0)$ *has an invariant subspace of dimension* $(n-1)$ *associated to the tangent plane at* x_0*. The remaining eigenvalue measures the rate of attractivity from directions normal to the surface.*

Proof. Let Y_0 be a basis for the tangent plane at x_0 and let $Y_0^\perp$ be its orthogonal complement (a vector in the direction of $(\nabla h(x_0))^T$ in this case). Then $\Phi(t, 0)Y_0 \ = \ Y_{\phi^t(x_0)}$ are vectors in the tangent plane at the point $\phi^t(x_0)$, and therefore since the tangent plane at $\phi^\tau(x_0)$ is the same as the tangent plane at x_0, we must have that $Y_{\phi^t(x_0)} = Y_0 C$ where $C \in \mathbb{R}^{n-1,n-1}$. In other words, we must have

$$\Phi(\tau, 0)[Y_0, \ Y_0^\perp] \ = \ [Y_0, \ Y_0^\perp] \begin{bmatrix} C & b \\ 0 & a \end{bmatrix}$$

and the result follows. □

Remark 2.11. With above notation, the value of a measures the effect of perturbations off Σ, whereas the eigenvalues of C measure the effect of perturbations with respect to solutions which remain on Σ. The value of a is measuring the stability of the periodic orbit of the DAE system which describes the motion on Σ.

Example 2.12. (See Galvanetto and Bishop, [13]). Consider the dynamics of a simple oscillator consisting of a block of mass m, supported by a moving belt. The block is connected to a fixed support by a linear elastic spring and by a linear dashpot. The surface between the block and the belt is rough so that the belt exerts a friction force on the block. Energy is continuously introduced into the system by the motion of the belt, which moves at constant velocity (the driving velocity v) and it is transferred to the block by means of the static friction force that allows for the build-up of potential energy within the spring. The structural damping generated by the dashpot continuously dissipates energy. The dynamic friction force may either introduce energy into the system, if it has the same sign as the velocity of the block (this is usually the case at the beginning of a slip phase), or dissipate energy, if the signs of the friction force and of the velocity are different. Mechanical systems of this type are referred to as stick-slip since there are times when there is no relative motion between the block and the belt (stick phase) and others in which the block slips. A mechanical system of this type may be described, in its simplest form, by means of the following differential equation

$$mx'' + cx' + kx = f(x' - v) \ , \tag{2.25}$$

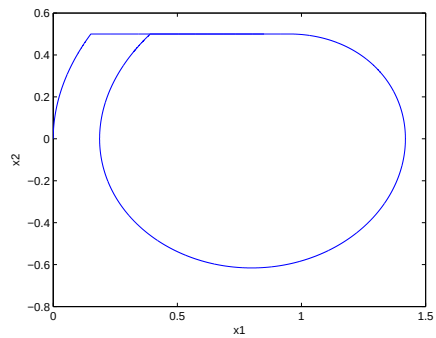

FIGURE 4. Periodic sliding trajectory.

where $x(t)$ is the displacement of the oscillator from the position in which the spring assumes its natural length, x_0 the initial displacement, m is the mass, c is the damping coefficient, k the stiffness of the spring, $f(x'-v)$ is the friction force of the block which we assume of the form:

$$f(x'-v) = \begin{cases} \frac{1-\delta}{1-\gamma(x'-v)} + \delta + \eta(x'-v)^2, & \text{for } v > x', \\ \frac{-(1-\delta)}{1+\gamma(x'-v)} - \delta - \eta(x'-v)^2 & \text{for } v < x', \end{cases} \tag{2.26}$$

where δ, γ, η are constants.

Letting $x_1 = x$, $x_2 = x'$, $h(x_1, x_2) = x_2 - v$, we can rewrite (2.25) in form of the Filippov differential system (1.1), with the extension (2.22). Here, we have

$$f_1(x_1, x_2) = \begin{bmatrix} x_2 \\ -kx_1 - \frac{c}{m}x_2 + \frac{1-\delta}{1-\gamma(x_2-v)} + \delta + \eta(x_2-v)^2 \end{bmatrix},$$

$$f_2(x_1, x_2) = \begin{bmatrix} x_2 \\ -kx_1 - \frac{c}{m}x_2 + \frac{-(1-\delta)}{1+\gamma(x_2-v)} - \delta - \eta(x_2-v)^2 \end{bmatrix}.$$

In Fig. 4 we show the solution computed by the technique we presented in [10]. Values of the constants are $x_0 = [0, 0]$, $m = 1$, $k = 1$, $c = 0.1$, $\delta = 0.5$, $\gamma = 1$, $\eta = 0.001$, $v = 0.5$. The monodromy matrix has the form (2.24) and, as expected, the numerically computed monodromy matrix has an eigenvalue 0 (to machine precision) and one eigenvalue (approximately) equal to 1. In fact, starting with $x_0 \cong (0.9503, 0.49999)$ on the limit cycle, after one period ($\tau = 6.5$), a first order method provides a fundamental matrix with eigenvalues given by 0 and 0.9997.

3. Fundamental matrix solution: Cross and/or slide on two surfaces

Now, suppose that the state space is split into four regions R_1, R_2, R_3 and R_4 by two intersecting hypersurfaces Σ_1 and Σ_2 which are defined by the scalar functions $h_1 : \mathbb{R}^n \to \mathbb{R}$ and $h_2 : \mathbb{R}^n \to \mathbb{R}$, that is:

$$R_1 = \{x \in \mathbb{R}^n \mid h_1(x) < 0,\ h_2(x) < 0\}, \quad R_2 = \{x \in \mathbb{R}^n \mid h_1(x) < 0,\ h_2(x) > 0\},$$

$$R_3 = \{x \in \mathbb{R}^n \mid h_1(x) > 0,\ h_2(x) > 0\}, \quad R_4 = \{x \in \mathbb{R}^n \mid h_1(x) > 0,\ h_2(x) < 0\},$$

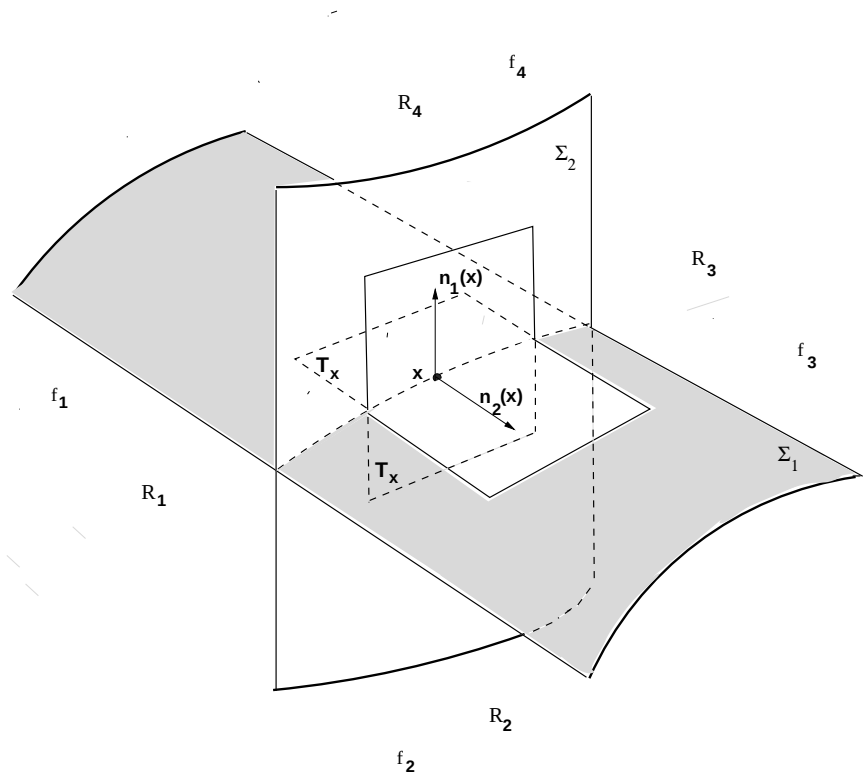

FIGURE 5. The intersection of two sliding surfaces.

(see Fig. 5). Consider the system with discontinuous right-hand side:

$$x'(t) = f(x(t)) = \begin{cases} f_1(x) \ , & x \in R_1 \ , \\ f_2(x) \ , & x \in R_2 \ , \\ f_3(x) \ , & x \in R_3 \ , \\ f_4(x) \ , & x \in R_4 \ , \end{cases} \tag{3.1}$$

with initial point $x(0) = x_0 \in \mathbb{R}^n$. The functions $h_1(x)$ and $h_2(x)$ are assumed to be $\mathcal{C}^k$ functions ($k \geq 2$) and moreover $\nabla h_1(x) \neq 0$, for all $x \in \Sigma_1$, and $\nabla h_2(x) \neq 0$, for all $x \in \Sigma_2$. So, we have well defined unit normals $n_1(x)$ and $n_2(x)$ to the tangent planes $T_x(\Sigma_1)$ and $T_x(\Sigma_2)$, respectively. We will also assume that $n_1(x)$ and $n_2(x)$ are linearly independent for all $x \in \Sigma_1 \cap \Sigma_2$.

Conditions guaranteeing attracting sliding on $\Sigma_1 \cap \Sigma_2$ are given by:

$$\begin{cases} n_1^T(x) f_1(x) > 0 \ , & n_2^T(x) f_1(x) > 0 \ , \\ n_1^T(x) f_2(x) > 0 \ , & n_2^T(x) f_2(x) < 0 \ , \\ n_1^T(x) f_3(x) < 0 \ , & n_2^T(x) f_3(x) < 0 \ , \\ n_1^T(x) f_4(x) < 0 \ , & n_2^T(x) f_4(x) > 0 \ , \end{cases} \tag{3.2}$$

while crossing conditions at $x \in \Sigma_1 \cap \Sigma_2$ are given by:

$$\begin{cases} n_1^T(x)f_1(x) > 0 \ , & n_2^T(x)f_1(x) > 0 \ , \\ n_1^T(x)f_2(x) > 0 \ , & n_2^T(x)f_2(x) > 0 \ , \\ n_1^T(x)f_3(x) > 0 \ , & n_2^T(x)f_3(x) > 0 \ , \\ n_1^T(x)f_4(x) > 0 \ , & n_2^T(x)f_4(x) > 0 \ , \end{cases} \tag{3.3}$$

or similar conditions with opposite signs.

It is well known that in general Filippov's theory is not able to define uniquely the vector field on the intersection $\Sigma_1 \cap \Sigma_2$ of the discontinuity surfaces Σ_1 and Σ_2, except in particular cases (see [11, 27]). Recently, two different approaches, the first one based on sigmoid blending techniques (see [3, 9]), and the second one based on geometric considerations (see [10]), have been proposed to eliminate the ambiguity of the Filippov's theory. In both cases, one can unambiguously define a solution of (3.1) also in case in which the solution slides on the intersection $\Sigma_1 \cap \Sigma_2$. Let us call $f_\cap$ the chosen vector field on the intersection; for our numerical experiments, we have used the choice of [10].

Notice that the ambiguity to uniquely define a vector field in the Filippov sense on the intersection is simply a consequence of the fact that Filippov convexification method would require to have

$$f_\cap = \sum_{i=1}^{4} \lambda_i(x) f_i(x) \ , \ \ \lambda_i \geq 0 \ .$$

To the convexity constraint $\sum_{i=1}^4 \lambda_1 = 1$, one can add two more equations coming from the requirement that $f_\cap$ lies on the tangent plane:

$$n_1^T(x)f_\cap(x) = 0 \ , \quad n_2^T(x)f_\cap(x) = 0 \ ,$$

and clearly we have a generally underdetermined system to solve for the λ_i's. Of course, this same ambiguity shows up when taking the point of view of a DAE modeling sliding motion on the intersection. In this case, for $x \in \Sigma_1 \cap \Sigma_2$, considering the $(n+4)$ variables $(x, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and a general sliding vector field $f_\cap$ on the intersection, we would need to satisfy

$$\begin{cases} x'(t) \ = \ f_\Sigma(x, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \ , \\ \qquad h_1(x(t)) \ = \ 0 \ , \\ \qquad h_2(x(t)) \ = \ 0, \\ \qquad \sum_{i=1}^4 \lambda_i \ = \ 1 \ . \end{cases} \tag{3.4}$$

However, unlike (1.10), this is not an index-2 DAE, and in fact there is no properly defined index for (3.4).

We now proceed to define appropriate generalizations for the fundamental matrix solution when the solution trajectory crosses or slides on the intersection of two surfaces. Results in the crossing case exist (see below), but we have found no work on the fundamental matrix solution when there is sliding motion on the intersection of two surfaces.

In what follows, we will use the following notation for the Filippov vector field in a regime of attractive sliding motion along one of the two surfaces Σ_1 or Σ_2:

$$f_{\Sigma_r}^{i,j}(\bar{x}) = (1 - \alpha(\bar{x}))f_i(\bar{x}) + \alpha(\bar{x})f_j(\bar{x}) \, , \quad \alpha(\bar{x}) = \frac{n_r^T(\overline{x})f_i(\overline{x})}{n_r^T(\overline{x})(f_i(\overline{x}) - f_j(\bar{x}))} \tag{3.5}$$

for $i, j = 1, 2, 3, 4$, $r = 1, 2$, with $\bar{x} \in \Sigma_r$.

(i) *From Outside to one surface.* The case in which, starting with an initial point (say) in R_1, a trajectory crosses Σ_1 towards R_4 at a point $\overline{x} = x(\overline{t})$, or slides on Σ_1, is identical to the cases

previously discussed (see (2.11), (2.12) and (2.23), (2.17)). The saltation matrix, which links the components of the fundamental matrix solution at the discontinuity point $\overline{x}$, will be given by:

$$S_1 = I + (g(\overline{x}) - f_1(\overline{x})) \cdot \frac{n_1^T(\overline{x})}{n_1^T(\overline{x}) f_1(\overline{x})} \ , \qquad \text{with } n_1^T(\overline{x}) f_1(\overline{x}) \neq 0, \tag{3.6}$$

and S_1 brings the vector field $f_1(\overline{x})$ into $g(\overline{x}) = f_4(\overline{x})$ when the trajectory crosses Σ_1, or into the Filippov's vector $g(\overline{x}) = f^{1,4}_{\Sigma_1}(\bar{x})$ of the tangent space $T_{\overline{x}}(\Sigma_1)$ when the trajectory slides on Σ_1. We have a similar form of the saltation matrix in the other cases, for instance when we cross Σ_1 from R_2 towards R_3, or when we cross Σ_2 from R_4 towards R_3 or from R_1 towards R_2, etc..
(ii) *From one surface to the intersection.* We now suppose that while sliding on Σ_1 with vector field $f^{1,4}_{\Sigma_1}$, the trajectory arrives at a point $\overline{x}$ which is on Σ_2, hence on the intersection, non-tangentially with respect to Σ_2. In principle, there are several possibilities: (a) We cross the intersection, while remaining on Σ_1, (b) we exit both surfaces tangentially with respect to Σ_1, (c) we leave Σ_1 tangentially and remain on Σ_2, or (d) we remain on the intersection. All of these cases are effectively similar to those previously considered. The saltation matrix is given by:

$$S_{\Sigma_1} = I + (g(\overline{x}) - f^{1,4}_{\Sigma_1}(\bar{x})) \cdot \frac{n_2^T(\overline{x})}{n_2^T(\overline{x}) f^{1,4}_{\Sigma_1}(\overline{x})}, \qquad \text{with } n_2^T(\overline{x}) f^{1,4}_{\Sigma_1}(\overline{x}) \neq 0 \ , \tag{3.7}$$

where

(a) $g(\overline{x}) = f^{2,3}_{\Sigma_1}(\bar{x})$;
(b) $g(\overline{x}) = f_2(\overline{x})$ or $g(\overline{x}) = f_3(\overline{x})$ (and $f^{1,4}_{\Sigma_1}(\overline{x}) = g(\overline{x})$);
(c) $g(\overline{x}) = f^{1,2}_{\Sigma_2}(\bar{x})$ or $g(\overline{x}) = f^{3,4}_{\Sigma_2}(\bar{x})$ (and $f^{1,4}_{\Sigma_1}(\overline{x}) = g(\overline{x})$);
(d) $g(\overline{x}) = f_{\cap}(\overline{x})$.

It is important to emphasize that –in all cases (i) and (ii) above– we still have an extention of the concept of fundamental solution matrix, as before. This is no longer true in the case we consider next.

3.1. **From outside to the intersection.** A most interesting case is when a trajectory $\phi^t(x_0)$, with a value x_0 in (say) R_1, at time t_1 hits the intersection $\Sigma_1 \cap \Sigma_2$, that is $x_1 := \phi^{t_1}(x_0) \in \Sigma_1 \cap \Sigma_2$. We will assume that this happens non-tangentially with respect to both Σ_1 and Σ_2; more precisely, we assume that $(\nabla h_1(x_1))^T f_1(x_1) \neq 0$ and $(\nabla h_2(x_1))^T f_1(x_1) \neq 0$. In this situation, the interesting cases are when the trajectory either crosses the intersection and continues with vector field $f_3(x_1)$, or slides on the intersection; in other words, we will assume that either (3.3) or (3.2) hold at x_1. The case in which the trajectory after arriving on the intersection begins sliding on only one surface is actually simpler, and it is handled similarly to the situations previously examined: there is going to be just one jump matrix in this case, like that in (2.17).

The case in which the trajectory crosses the intersection was first considered by Ivanov in [16]. As far as we know, the case where we slide on the intersection was not previosuly considered. Similarly to what we did in Lemma 2.2, we want to look at what happens to trajectories, $\phi^t(y_0)$, for perturbed initial condition $y_0 = x_0 + \Delta x$. But now there are new issues to consider since the perturbed trajectory $\phi^t(y_0)$ will not necessarily arrive at the intersection $\Sigma_1 \cap \Sigma_2$ from R_1, unlike the case we considered in Lemma 2.2 where the perturbed trajectory reached Σ from R_1 just like the unperturbed one.

What we have now, relatively to $\phi^t(y_0)$, are two possibly different times $\Delta_1 t$ and $\Delta_2 t$ where

$$h_1(\phi^{t_1+\Delta_1 t}(y_0)) \ = 0 \ , \quad h_2(\phi^{t_1+\Delta_2 t}(y_0)) \ = 0 \, .$$

Assume (with no loss of generality) that both $\Delta_1 t$ and $\Delta_2 t$ are positive.

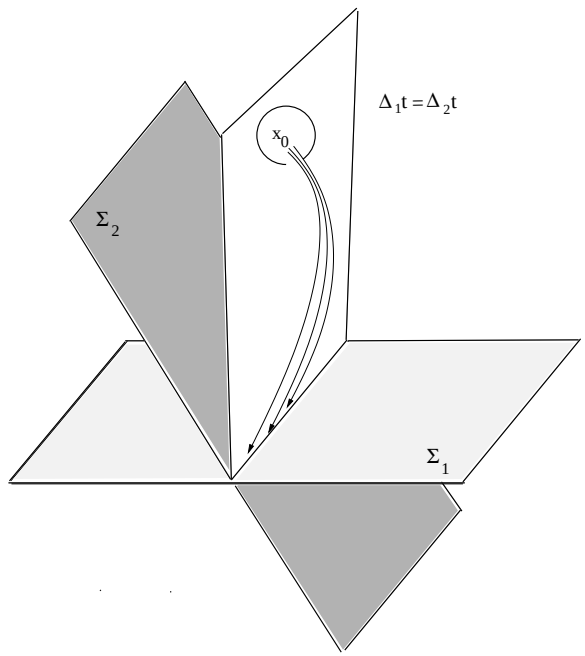

FIGURE 6. The intersection of two sliding surfaces.

Expanding $h_1(\phi^{t_1+\Delta_1 t}(y_0))$ and $h_2(\phi^{t_1+\Delta_1 t}(y_0))$ in a similar way to what we did in Lemma 2.2, retaining first order terms and solving for $\Delta_1 t$ and $\Delta_2 t$ we obtain

$$\Delta_1 t = -\frac{(\nabla h_1(x_1))^T\big(\phi^{t_1}(y_0)-x_1\big)}{(\nabla h_1(x_1))^T f_1(x_1)} \neq 0\ , \quad \text{and} \quad \Delta_2 t = -\frac{(\nabla h_2(x_1))^T\big(\phi^{t_1}(y_0)-x_1\big)}{(\nabla h_2(x_1))^T f_1(x_1)} \neq 0\ . \tag{3.8}$$

If $\Delta_1 t < \Delta_2 t$, then $\phi^t(y_0)$ first meets Σ_1 and then meets Σ_2. If $\Delta_2 t < \Delta_1 t$, then $\phi^t(y_0)$ first meets Σ_2 and then meets Σ_1. Finally, if $\Delta_1 t = \Delta_2 t$, then $\phi^t(y_0)$ meets Σ_1 and Σ_2 simultaneously, that is it lands directly on the intersection. Accordingly, we split the region R_1, locally, near the intersection point x_1, into three regions,

$$R_1 \ = \ R_1^1 \cup R_1^0 \cup R_1^2\,,$$

defined as follows:

(0) $y_0 \in R_1^0$, if $y_0 \in R_1$ and the trajectory $\phi^t(y_0)$ meets first directly the intersection surface $\Sigma_1 \cap \Sigma_2$;
(1) $y_0 \in R_1^1$, if $y_0 \in R_1$ and the trajectory $\phi^t(y_0)$ meets first the surface Σ_1;
(2) $y_0 \in R_1^2$, if $y_0 \in R_1$ and the trajectory $\phi^t(y_0)$ meets first the surface Σ_2.

Effectively, the region R_1^0 is the (local) backward set of points on the intersection itself; since the intersection is a $(n-2)$-dimensional manifold, it follows that R_1^0 is an $(n-1)$-dimensional manifold, invariant under the flow; see Fig. 6. Because of uniqueness of solutions with data in R_1, actually each of R_1^0, R_1^1 and R_1^2 is invariant under the flow.

In all cases above, at first order we have

$$\phi^{t_1+\Delta t}(x_0) - \phi^{t_1+\Delta t}(y_0) = S\ \big(\phi^{t_1}(x_0) - \phi^{t_1}(y_0)\big)$$

where the specific form of the jump matrix S depends on whether $\Delta_1 t \lesseqgtr \Delta_2 t$ and on whether $\phi^t(x_0)$ crosses the intersection at x_1 (going from R_1 into R_3), or begins sliding on it. In accordance

with the above, and our discussion in the previous section, we have the cases below for the saltation matrix S.

3.1.1. *Cross the intersection.* Here $\phi^t(x_0)$ crosses the intersection at x_1. In the three cases below, (3.9)-(3.10)-(3.11), the matrix S is invertible, though the specific form of S differs.

Case $\Delta_1 t < \Delta_2 t$. The perturbed trajectory first crosses Σ_1 then Σ_2. As a consequence, S is the product of two saltation matrices:

$$S = \left[I + (f_3(x_1) - f_4(x_1)) \cdot \frac{n_2^T(x_1)}{n_2^T(x_1) f_4(x_1)}\right] \left[I + (f_4(x_1) - f_1(x_1)) \cdot \frac{n_1^T(x_1)}{n_1^T(x_1) f_1(x_1)}\right] . \tag{3.9}$$

Case $\Delta_2 t < \Delta_1 t$. Now the perturbed trajectory first crosses Σ_2 then Σ_1, and again S is the product of two saltation matrices:

$$S = \left[I + (f_3(x_1) - f_2(x_1)) \cdot \frac{n_1^T(x_1)}{n_1^T(x_1) f_2(x_1)}\right] \left[I + (f_2(x_1) - f_1(x_1)) \cdot \frac{n_2^T(x_1)}{n_2^T(x_1) f_1(x_1)}\right] . \tag{3.10}$$

Case $\Delta_1 t = \Delta_2 t$. The perturbed trajectory passes from R_1 to R_3 at the intersection. We have only one jump matrix with two equivalent rewritings, in the sense that for either form below

$$\begin{aligned} S &= \left[I + (f_3(x_1) - f_1(x_1)) \cdot \frac{n_1^T(x_1)}{n_1^T(x_1) f_1(x_1)}\right] \quad \text{or} \\ S &= \left[I + (f_3(x_1) - f_1(x_1)) \cdot \frac{n_2^T(x_1)}{n_2^T(x_1) f_1(x_1)}\right] , \end{aligned} \tag{3.11}$$

we have $\phi^{t_1^+}(x_0) - \phi^{t_1^+}(y_0) = S\left(\phi^{t_1^-}(x_0) - \phi^{t_1^-}(y_0)\right)$; the two equivalent rewritings are a consequence of (3.8).

3.1.2. *Slide on the intersection.* We now extend the above to the case in which $\phi^t(x_0)$, upon reaching $\Sigma_1 \cap \Sigma_2$ at x_1, begins sliding on $\Sigma_1 \cap \Sigma_2$.

Case $\Delta_1 t < \Delta_2 t$. The perturbed trajectory first arrives at Σ_1, where it has sliding behavior, then at Σ_2 (hence at the intersection). As a consequence, S is the product of two saltation matrices:

$$S = \left[I + (f_\cap(x_1) - f_{\Sigma_1}^{1,4}(x_1)) \cdot \frac{n_2^T(x_1)}{n_2^T(x_1) f_{\Sigma_1}^{1,4}(x_1)}\right] \left[I + (f_{\Sigma_1}^{1,4}(x_1) - f_1(x_1)) \cdot \frac{n_1^T(x_1)}{n_1^T(x_1) f_1(x_1)}\right] , \tag{3.12}$$

where $f_{\Sigma_1}^{1,4}(\bar{x})$ is the sliding vector field on Σ_1, which feels the fields $f_1(\bar{x})$ and $f_4(\bar{x})$, that is (3.5).

Case $\Delta_2 t < \Delta_1 t$. Now the perturbed trajectory first arrives at Σ_2, on which it has sliding behavior, then at Σ_1 (hence at the intersection), and again S is the product of two saltation matrices:

$$S = \left[I + (f_\cap(x_1) - f_{\Sigma_2}^{1,2}(x_1)) \cdot \frac{n_1^T(x_1)}{n_1^T(x_1) f_{\Sigma_2}^{1,2}(x_1)}\right] \left[I + (f_{\Sigma_2}^{1,2}(x_1) - f_1(x_1)) \cdot \frac{n_2^T(x_1)}{n_2^T(x_1) f_1(x_1)}\right] , \tag{3.13}$$

where $f_{\Sigma_2}^{1,2}(\bar{x})$ is the sliding vector field on Σ_2, which feels the fields $f_1(\bar{x})$ and $f_2(\bar{x})$.

Case $\Delta_1 t = \Delta_2 t$. The perturbed trajectory arrives directly on the intersection. We have only one jump matrix with two equivalent rewritings, in the sense that for either form below

$$\begin{aligned} S &= \left[I + (f_\cap(x_1) - f_1(x_1)) \cdot \frac{n_1^T(x_1)}{n_1^T(x_1) f_1(x_1)}\right] \quad \text{or} \\ S &= \left[I + (f_\cap(x_1) - f_1(x_1)) \cdot \frac{n_2^T(x_1)}{n_2^T(x_1) f_1(x_1)}\right] , \end{aligned} \tag{3.14}$$

we have $\phi^{t_1^+}(x_0) - \phi^{t_1^+}(y_0) = S\,\big(\phi^{t_1^-}(x_0) - \phi^{t_1^-}(y_0)\big)$.

Remark 3.1. *Interpretation.* It is important to understand what the above different rewritings mean. In both cases of crossing and sliding on the intersection, the main implication of the above different forms for the saltation matric(-ces) is that we **do not have** a properly (nor uniquely) defined concept of a **fundamental matrix solution**. What we have is an expression for a matrix which shows how initial perturbations are magnified, but the expression itself depends on where the perturbed initial point y_0 is with respect to the unperturbed initial value x_0, that is on whether y_0 is in R_1^0, R_1^1, or R_1^2. This effectively inhibits the extension of the concept of fundamental matrix solution to the case in which linearization occurs along a trajectory which from outside the surfaces goes directly to the intersection.

The above notwithstanding, some important features are maintained also in this case, in particular about the rank of the saltation matrix(-ces) in (3.12) and (3.13).

Lemma 3.2. *Consider the matrices $S^{(1)}$ and $S^{(2)}$ in* (3.12) *and* (3.13) *rewritten as follows:*

$$S^{(1)} = S_2^{(1)} S_1^{(1)} \ , \quad S_1^{(1)} = I + a_1 n_1^T \ , \ \ S_2^{(1)} = I + c_1 n_2^T \ ,$$

and

$$S^{(2)} = S_1^{(2)} S_2^{(2)} \ , \quad S_1^{(2)} = I + a_2 n_1^T \ , \ \ S_2^{(1)} = I + c_2 n_2^T \ ,$$

where

$$a_1 = \frac{f_{\Sigma_1}^{1,4}(x_1) - f_1(x_1)}{n_1^T(x_1) f_1(x_1)} \ , \ \ c_1 = \frac{f_\cap(x_1) - f_{\Sigma_1}^{1,4}(x_1)}{n_2^T(x_1) f_{\Sigma_1}^{1,4}(x_1)} \ ,$$

and

$$a_2 = \frac{f_\cap(x_1) - f_{\Sigma_2}^{1,2}(x_1)}{n_1^T(x_1) f_{\Sigma_2}^{1,2}(x_1)} \ , \ \ c_2 = \frac{f_{\Sigma_2}^{1,2}(x_1) - f_1(x_1)}{n_2^T(x_1) f_1(x_1)} \ .$$

Then, both $S^{(1)}$ and $S^{(2)}$ have $(n-2)$ eigenvalues equal to 1 *and a two dimensional kernel.*

Proof. The fact that there are $(n-2)$ eigenvalues equal to 1 is a direct consequence of the fact that both $S^{(1)}$ and $S^{(2)}$ leave invariant the tangent plane to the intersection at x_1. As far as the 2-d kernel is concerned, a direct verification shows that $\{a_1, c_1\}$ are both in the kernel of $S^{(1)}$ and likewise $\{a_2, c_2\}$ are in the kernel of $S^{(2)}$. □

Remark 3.3. The double loss of rank witnesses that the motion is taking place on an $(n-2)$-dimensional manifold. Interestingly, the saltation matrix in (3.14) is of rank $(n-1)$; this is a consequence of the fact that it only acts on data coming from R_1^0 which itself is a manifold of dimension $(n-1)$ and not n.

The last result we give is about motion which is periodic on the intersection, similarly to what we did in Section 2.4.

3.2. **Periodic sliding case on the intersection of two surfaces.** Consider the case when we have attractive periodic sliding motion restricted to the intersection:

$$x_0 \in \Sigma_1 \cap \Sigma_2 \ , \ \ \phi^t(x_0) \in \Sigma_1 \cap \Sigma_2 \ , \ \text{for all}\, t \ , \ \ \phi^\tau(x_0) = x_0 \, .$$

Now there is a properly defined fundamental matrix satisfying

$$\Phi_s'(t,0) \ = \ Df_\cap(x, \lambda(x)) \Phi_s(t,0) \ , \ \ \Phi_s(0,0) = I \ , \tag{3.15}$$

where $f_\cap$ is the vector field on the intersection, for example defined as in [10]; as our notation suggests, we stress that this vector field $f_\cap$ depends on a vector valued function λ defined for $x \in \Sigma_1 \cap \Sigma_2$ and which we also need to assume being smooth. Now, (3.15) defines an invertible

fundamental matrix, even though the motion is taking place on a $(n-2)$-dimensional manifold. As usual, this is because the fundamental matrix maps vectors of the tangent plane into vectors of the tangent plane.

Theorem 3.4. *Let $\phi^t(x_0)$ be an attracting sliding mode, periodic of period τ. Let $\Phi_s(\tau,0)$ be monodromy matrix associated to the linearized problem for $\phi^t(x_0)$, $t\in[0,\tau]$. Then, $\Phi(\tau,0)$ has an invariant subspace of dimension $(n-2)$ associated to the tangent plane at x_0. The remaining eigenvalues measure the attractivity rates from directions normal to the surface.*

Proof. The proof is essentially the same as that of Theorem 2.10. Let Y_0 be a basis for the tangent plane at x_0 and let $Y_0^\perp$ be its orthogonal complement. Therefore, $Y_{\phi^t(x_0)} = Y_0 C$ where $C\in\mathbb{R}^{n-2,n-2}$ and

$$\Phi(\tau,0)[Y_0,\ Y_0^\perp] \ = \ [Y_0,\ Y_0^\perp]\begin{bmatrix} C & B \\ 0 & A\end{bmatrix} .$$

□

Remark 3.5. The eigenvalues of C measure the stability properties with respect to perturbations on Σ, whereas the eigenvalues of A measure the effect of perturbations off Σ. But, again, this statement has to be interpreted with a grain of salt: We are assuming that on the intersection there is attracting sliding motion, which means that solutions of the nonsmooth system near the periodic orbit approach this periodic trajectory itself. At the same time, the eigenvalues of A are measuring the stability of the periodic orbit of the differential-algebraic equation which describes the motion on the intersection.

The next example illustrates Theorem 3.4.

Example 3.6. Let us consider the three-dimensional differential system $x'(t)=f(x(t))$, $x_0=x(0)$, where $x=(x_1,x_2,x_3)$ and $f(x(t))$ is a discontinuous vector field with respect to the two surfaces:

$$\Sigma_1=\{x|\ h_1(x)=x_1+x_2+x_3-1=0\}\ ,\ \Sigma_2=\{x|\ h_2(x)=2x_1^2+2(x_2-x_3)^2-1=0\}\ .$$

Σ_1 is a plane and Σ_2 is a cylinder in $\mathbb{R}^3$. Here, $f(x)=A(c_1,c_2)x+b(c_2)$ where:

$$A(c_1,c_2)=\begin{pmatrix} 1-2c_1 & 2 & -2 \\ -2+c_1 & -1-c_1 & c_1 \\ c_1 & -2+c_1 & 1-c_1\end{pmatrix}, \qquad b(c_2)=\begin{pmatrix}0\\ c_2\\ c_2\end{pmatrix}\ ,$$

and

$$c_1=\begin{cases}0, & h_2(x)<0\\ 1, & h_2(x)>0\end{cases}\ ,\quad c_2=\begin{cases}0, & h_1(x)>0\\ 1, & h_1(x)<0\end{cases}\ .$$

Thus, we have four different vector fields in the four regions of $\mathbb{R}^3$ isolated by the two surfaces Σ_1 and Σ_2. On the intersecting surface $\Sigma_1\cap\Sigma_2$ the conditions for attracting sliding are satisfied.

In Fig. 7 we have reported the numerical solution, on the time interval $[0,10]$, with initial value in region R_1, together with its limit cycle.

As far as the fundamental matrix is concerned, if x_0 is a point in region R_1, $x(t)$ is a point of the trajectory on the limit cycle, by using the same arguments we have used to prove (3.20), we can show that:

$$\Phi(t,0)=\Phi_\cap(t,t_2^+)S_\cap\Phi_{\Sigma_1}(t_2^-,t_1^+)S_1\Phi_1(t_1^-,0)\ , \tag{3.16}$$

where $\Phi_1(t,0)$ for $t\in[0,t_1)$ denotes the fundamental matrix solution of the differential problem in R_1 starting with x_0 and vector field given by $f_1(x)$; S_1 is the saltation matrix in (3.6) which moves the vector field $f_1(x_1)$ into Filippov vector $f^{1,4}_{\Sigma_1}(x_1)$ (defined in (3.5)) of the tangent space $T_{x_1}(\Sigma_1)$; $\Phi_{\Sigma_1}(t,t_1)$, for $t\in[t_1,t_2)$, denotes the fundamental matrix solution of the differential problem on Σ_1 starting with x_1 and vector field given by the Filippov vector $f^{1,4}_{\Sigma_1}(x)$; $S_{\Sigma_1\cap\Sigma_2}$ is the saltation

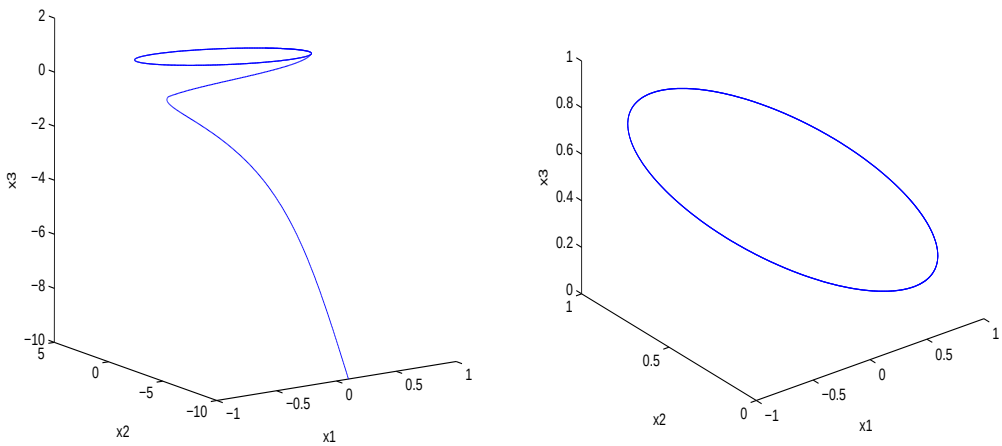

FIGURE 7. Numerical solution and limit cycle.

matrix in (3.7)-(d) which moves the vector $f_{\Sigma_1}(x_2)$ into a vector of the tangent space of $\Sigma_1 \cap \Sigma_2$ at x_2; $\Phi_\cap(t, t_2)$, for $t \in [t_2, t_3)$, denotes the fundamental matrix solution of the differential problem on $\Sigma_1 \cap \Sigma_2$ with initial point x_2 and vector field given by $f_\cap(x)$.

The fundamental matrix in (3.16) has two eigenvalues equal to 0, because of the different saltation matrices S_1 and $S_\cap$. In fact, starting with $x_0 = (0.1, -10, -10)$, a first order numerical method provides a fundamental matrix (given by (3.16)) at $t = 6.23$ with eigenvalues: 0.0, 0.0, 7.1.

On the other hand, if we start with a point $x_0 = x(0)$ on the limit cycle, the fundamental matrix after one period τ will be invertible, and given by

$$\Phi(\tau, 0) = \Phi_\cap(\tau, 0), \tag{3.17}$$

which is a matrix with one eigenvalue equal to 1, and the other eigenvalues measuring the rates of attractivity/repulsivity towards the limit cycle. Indeed, starting with $x_0 \cong (0.7029, 0.1866, 0.1103)$ on the limit cycle, using a first order method a numerical simulation provides the following eigenvalues for the fundamental matrix (3.17) after one period $\tau = 3.142$: 0.085, 1.770, 1.0, which shows that the periodic orbit of the underlying differential-algebraic equation is hyperbolic.

The final example below highlights the case of a periodic orbit with a portion of the trajectory sliding on the intersection of two surfaces, the remaining parts either sliding on one single surface or not sliding on any surface at all; in this case, the effect of the saltation matrices becomes apparent.

Example 3.7. (See Galvanetto, [12]). Now, let us consider a nonsmooth dynamical system the solution of which slides on the intersection of two surfaces and the fundamental matrix solution of which has the studied behaviour. In [12] the author studies a mechanical system composed by two blocks on a moving belt. The velocity of the belt is constant and is called the driving velocity v. Each block is connected to a fixed support and to the other block by elastic springs. The surface between the blocks and the belt is rough so that the belt exerts a dry friction force on each block that sticks on the belt to the point where the elastic forces due to the springs exceed the maximum static force. At this point the blocks start slipping and the slipping motion will continue to the point where the velocity of the block will equal that of the belt and the elastic forces will be equilibrated by the static friction force. The continuous repetition of this type of

motions generates a stick-slip oscillation. This mechanical system may be described in its simplest form by the following set of differential equations:

$$\begin{cases} m_1 x_1'' = -k_1 x_1 - k_{12}(x_1 - x_2) + f_{k1}(x_1' - v) \ , \\ m_2 x_2'' = -k_2 x_2 - k_{12}(x_2 - x_1) + f_{k2}(x_2' - v) \ , \end{cases} \tag{3.18}$$

where $x_i(t)$ is the displacement, m_i is the mass, $f_{ki}(x_i' - v)$ the kinetic friction force of the $i-$th block, k_1, k_2, k_{12} suitable constants. The kinetic force has the form $f_{k2}(x' - v) = \beta f_{k1}(x' - v)$ with:

$$f_{k1}(x' - v) = \begin{cases} \frac{1-\delta}{1-\gamma(x'-v)} + \delta + \eta(x' - v)^2, & \text{for } v > x', \\ \\ \frac{-(1-\delta)}{1-\gamma(x'-v)} - \delta - \eta(x' - v)^2 & \text{for } v < x', \end{cases} \tag{3.19}$$

where $\beta, \delta, \gamma, \eta$ are suitable constants.

If we set $x_1' = x_3$ and $x_2' = x_4$ we may rewrite the differential system as a Filippov differential system (3.18, 3.19) in $\mathbb{R}^4$, with two discontinuity surfaces Σ_1 and Σ_2, characterized as the 0-sets of the functions h_1 and h_2, respectively, where $h_1(x) = x_3 - v$, and $h_2(x) = x_4 - v$. In Fig. 8 we report the numerical solution of this nonsmooth differential system together with its limit cycle for $m = 1$, $k_1 = k_2 = k_{12} = 1$, $\delta = 0$, $\gamma = 3$, $\eta = 0$, $v = 0.295$, $\beta = 1.301$. We have reported the coordinates (x_1, x_3, x_4) of the numerical solution on the time interval $[0, 80]$ with initial point given by the origin of $\mathbb{R}^4$.

Now, let us consider a periodic solution of (3.18, 3.19) of minimal period τ. Suppose that the initial point of $x_0 = x(0)$ of such a periodic solution lies in the region R_1 (that is $h_1(x_0) < 0$ and $h_2(x_0) < 0$). Denote by $x_1 = x(t_1)$ the point where the trajectory hits Σ_1, by $x_2 = x(t_2)$ the point where the trajectory hits the intersection $\Sigma_1 \cap \Sigma_2$, $x_3 = x(t_3)$ the point where the trajectory leaves Σ_2 to entry the region where $h_1(x) = 0$ and $h_2(x) < 0$, and $x_4 = x(t_4)$ the point where the trajectory leaves Σ_1 also to entry R_1. As far as the fundamental matrix is concerned, by using the same arguments we have used to prove (2.24), we can show that:

$$\Phi(\tau, 0) = \Phi_1(\tau, t_4)\Phi_{\Sigma_1}(t_4, t_3)\Phi_{\cap}(t_3, t_2^+)S_{\cap}\Phi_{\Sigma_1}(t_2^-, t_1^+)S_1\Phi_1(t_1^-, 0) \ , \tag{3.20}$$

where $\Phi_1(t, 0)$ for $t \in [0, t_1)$ denotes the fundamental matrix solution of the differential problem in R_1 starting with x_0 and vector field given by $f_1(x)$; S_1 is the saltation matrix in (3.6) which moves the vector field $f_1(x_1)$ into Filippov vector $f^{1,4}_{\Sigma_1}(\bar{x}_1)$ in (3.5) of the tangent space $T_{x_1}(\Sigma_1)$; $\Phi_{\Sigma_1}(t, t_1)$, for $t \in [t_1, t_2)$, denotes the fundamental matrix solution of the differential problem on Σ_1 starting with x_1 and vector field given by the Filippov vector $f^{1,4}_{\Sigma_1}(x)$; $S_{\cap}$ is the saltation matrix in (3.7)-(d) which moves the vector $f_{\Sigma_1}(x_2)$ into a vector of the tangent space of $\Sigma_1 \cap \Sigma_2$ at x_2; $\Phi_{\cap}(t, t_2)$, for $t \in [t_2, t_3)$, denotes the fundamental matrix solution of the differential problem on $\Sigma_1 \cap \Sigma_2$ with initial point x_2 and vector field given by $f_{\cap}(x)$; $\Phi_{\Sigma_1}(t, t_3)$, for $t \in [t_3, t_4]$, is the fundamental matrix solution of the differential problem on Σ_1 starting with x_3 and vector field given by the Filippov vector $f^{1,4}_{\Sigma_1}(x)$; $\Phi_1(t, t_4)$, for $t \in [t_4, \tau]$, denotes the fundamental matrix solution of the differential problem in R_1 starting with x_4 and vector field given by $f_1(x)$.

Theorem 3.8. *The fundamental matrix in* (3.20) *has one eigenvalue equals to* 1 *because of periodicity, two eigenvalues equal to* 0*, because of the singularity of the saltation matrices* S *and* $S_{\cap}$*, and the remaining eigenvalue characterizes stability of the orbit.*

Starting with the value $x_0 \cong (1.222, 1.333, 0.272, 0.148)$ on the limit cycle, after one period $\tau \cong 5$, a simple numerical simulation using a first order method gives the following eigenvalues for the fundamental matrix (3.20): 0.916, 0.168, 0.0, 0.0, implying that the periodic orbit is stable.

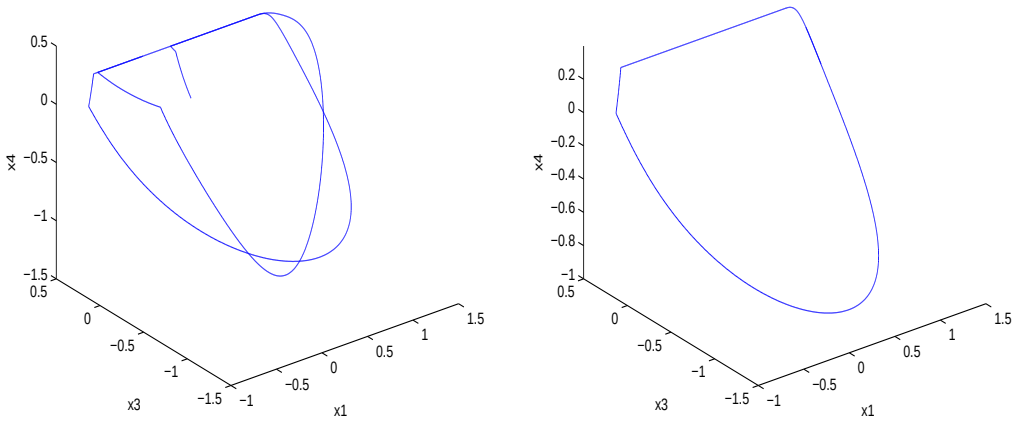

FIGURE 8. Numerical solution and limit cycle.

REFERENCES

[1] V. Acary and B. Brogliato. *Numerical Methods for Nonsmooth Dynamical Systems. Applications in Mechanics and Electonics*. Lecture Notes In Applied and Computational Mechanics. Springer-Verlag, Berlin, 2008.

[2] M.A. Aizerman and F.R. Gantmacher. On the stability of periodic motion. *Journal of Applied Mathematics*, pages 1065–1078, 1958.

[3] J.C. Alexander and T. Seidman. Sliding modes in intersecting switching surfaces, I: Blending. *Houston J. Math.*, 24:545–569, 1998.

[4] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, Berlin, 1984.

[5] G. Bartolini, F. Parodi, V.I. A. Utkin, and T. Zolezzi. The simplex method for nonlinear sliding mode control. *Mathematical Problems in Engineering*, 4:461–487, 1999.

[6] S. Campbell. Linearization of DAEs along trajectories. *Z. Angew. Math. Phys.*, 46:70–84, 1995.

[7] R. Casey, H. de Jong, and Gouze J.-L. Piecewise-linear models of genetics regulatory networks: Equilibria and their stability. *J.Math. Biol.*, 52:27–56, 2006.

[8] H. de Jong, J.-L. Gouze, C. Hernandez, M. Page, T. Sari, and J. Geiselmann. Qualitative simulation of genetic regulatory networks using piecewise-linear models. *Bulletin of Mathematical Biology*, 66:301–340, 2004.

[9] M. di Bernardo, C.J. Budd, A.R. Champneys, and P. Kowalczyk. *Piecewise-smooth Dynamical Systems. Theory and Applications*. Applied Mathematical Sciences 163. Springer-Verlag, Berlin, 2008.

[10] L. Dieci and L. Lopez. Sliding motion in Filippov differential systems: Theoretical results and a computational approach. *SIAM J. Numer. Anal.*, to appear, 2009. Available at http://www.math.gatech.edu/ dieci/preps/DL-Fili.pdf.

[11] A.F. Filippov. *Differential Equations with Discontinuous Right-Hand Sides*. Mathematics and Its Applications, Kluwer Academic, Dordrecht, 1988.

[12] U. Galvanetto. Some discontinuous bifurcation in a two-block stick-slip system. *Journal of Sound and Vibration*, 248:653–669, 2001.

[13] U. Galvanetto and S.R. Bishop. Dynamics of a simple damped oscillator undergoing stick-splip vibrations. *Meccanica*, 34:337–347, 2000.

[14] M.P. Glazos, S. Hui, and S.H Zak. Sliding modes in solving convex programming problems. *SIAM J. Control Optimization*, 36:680–697, 1998.

[15] J.-L. Gouze and T. Sari. A class of piecewise linear differential equations arsing in biological models. *Dynamical Systems*, 17:299–319, 2002.

[16] A.P. Ivanov. The stability of periodic solutions of discontinuous systems that intersect several surfaces of discontinuity. *J. Appl. Math. Mechs*, 62:677–685, 1998.

[17] K.H. Johansson, A.E. Barabanov, and K.J. Astrom. Limit cycles with chattering in relay feedback systems. *IEEE Transations on Automatic Control*, 247:1414–1423, 2002.

[18] P. Kucucka. Jumps of the fundamental solution matrix in discontinuous systems and applications. *Nonlinear Analysis*, 66:2529–2525, 2007.
[19] M. Kunze and T. Kuepper. Nonsmooth dynamical systems: An overview. In B. Fiedler, editor, *Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems*, pages 431–452. Springer-Verlag, 2001.
[20] R.I. Leine. Bifurcations in Discontinuous Mechanical Systems of Filippov's type. PhD thesis, Techn. Univ. Eindhoven, The Netherlands, 2000.
[21] R.I. Leine and H. Nijmeijer. *Dynamics and Bifurcations in Non-Smooth Mechanical Systems*. Lecture Notes in Applied and Computational Mechanics, Vol. 18, Springer-Verlag, Berlin, 2004.
[22] G.A. Leonov. Pendulum with positive and negative dry friction. Continuum of homoclinic orbits. *International Journal of Bifurcation and Chaos*, 5:251–254, 1995.
[23] P.C. Mueller. Calculation of Lyapunov exponents for dynamic systems with discontinuities. *Chaos, Solitons and Fractals*, 5:167–1681, 1995.
[24] E. Plathe and S. Kjoglum. Analysis and genetic proporties of gene regulatory networks with graded response functions. *Physica D*, 201:150–176, 2005.
[25] V.I. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22:212–222, 1977.
[26] V.I. Utkin. *Sliding Modes and Their Application in Variable Structure Systems*. MIR Publisher, Moskow, 1978.
[27] V.I. Utkin. *Sliding Mode in Control and Optimization*. Springer, Berlin, 1992.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 U.S.A.
E-mail address: dieci@math.gatech.edu

Dip. di Matematica, Universitá di Bari, Via Orabona 4, 70125 Italy
E-mail address: lopezl@dm.uniba.it