SLIDING MOTION ON DISCONTINUITY SURFACES OF HIGH CO-DIMENSION. A CONSTRUCTION FOR SELECTING A FILIPPOV VECTOR FIELD

LUCA DIECI AND LUCIANO LOPEZ

Abstract. In this paper we consider the issue of sliding motion in Filippov systems on the intersection of two or more surfaces. To this end, we propose an extension of the Filippov sliding vector field on manifolds of co-dimension p, with $p \geq 2$. Our model passes through the use of a multivalued sign function reformulation. To justify our proposal, we will restrict to cases where the sliding manifold is attractive. For the case of co-dimension $p = 2$, we will distinguish between two types of attractive sliding manifold: "node-like" and "spiral-like". The case of node-like attractive manifold will be further extended to the case of $p \geq 3$. Finally, we compare our model to other existing methodologies on some examples.

1. Introduction

In this work we consider autonomous differential systems with discontinuous right-hand side, also called piecewise-smooth systems, PWS systems for short. PWS systems have been studied for a long time, and are presently receiving a great deal of attention from both analytical and numerical communities because of the ability to model complex systems of practical relevance. The books [1] and [14] present a recent account of numerical works on the subject, and the references [4, 5, 6, 7, 9, 10, 11, 12, 13, 17, 19, 23, 25, 24, 32, 34, 35] provide a representative sample of the many applications that PWS systems have in control engineering, mechanical engineering, and biological sciences.

The basic problem we consider can be written as follows. There is a discontinuous differential equation (ODE)

$$x' = f(x)\ ,\ \ f(x) = f_i(x)\ ,\ \ x \in R_i\ ,\ i = 1, \dots, m\ , \tag{1.1}$$

to be studied for t in some interval $[0, T]$, subject to initial condition $x(0) = x_0$. In (1.1), $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, whose closures cover $\mathbb{R}^n$: $\mathbb{R}^n = \overline{\bigcup_i R_i}$; also, f_i is smooth on $\overline{R}_i$ and $\mathbb{R}^n \setminus \bigcup_i R_i$ has zero (Lebesgue) measure. So, in each region R_i, we have a standard differential equation with smooth vector field f_i. On the boundaries of these regions, the vector field is not properly defined. A standard way to overcome this difficulty, and the one we consider in this work, is to work with the differential inclusion obtained by the so-called *Filippov convexification* method, see [18].

Filippov considers the set valued function

$$F(x) \ = \ \overline{\text{co}}\{\lim_{i \to \infty} f(x_i),\ x_i \to x,\ x_i \in R_i\}\ , \tag{1.2}$$

where $\text{co}(A)$ is the convex hull of A. In other words, $F(x)$ is the convex hull of the values of $f(x)$ obtained approaching x through (smooth) regions R_i. At this point, Filippov considers the

1991 *Mathematics Subject Classification.* 34A36, 65L.

Key words and phrases. Discontinuous dynamical systems, Filippov, sliding modes, intersection of surfaces, regularization.

differential inclusion obtained by replacing f with F:

$$x' \in F(x)\,. \tag{1.3}$$

A *Filippov solution* is a classical solution of this differential inclusion. That is, an absolutely continuous function $x:\ [0,T]\to\mathbb{R}^n$ such that $x'\in F(x(t))$ for almost all $t\in[0,T]$. Existence of Filippov solutions can be guaranteed with the help of the concept of upper semi-continuity of set-valued functions; we refer to [3, 18] for details. For completeness, we recall the basic result of Filippov (see [18]) "*If F is locally bounded, and the map $x\to F(x)$ is upper semicontinuous, then there exists a Filippov solution of $x'\in F(x)$, for any x_0.*" Uniqueness is more complicated, and cannot be settled simply by looking at properties of F. It is necessary to characterize what happens on the boundaries of the regions R_i's.

We now assume that the regions R_i's are separated (again, locally) by (hyper-)surfaces characterized as zero sets of smooth functions with linearly independent gradients. Moreover, we will also assume that if the separating surface, call it Σ, has co-dimension p, then (locally, in a neighborhood of Σ) there are 2^p regions R_i's and therefore 2^p vector fields f_i's. In other words, Σ will always be of the form

$$\Sigma=\{x\in\mathbb{R}^n\ :\ h(x)=0\,,\quad h:\ \mathbb{R}^n\to\mathbb{R}^p\}\,,$$

where for all $x\in\Sigma$: $h(x)=\begin{bmatrix} h_1(x)\\ \vdots\\ h_p(x)\end{bmatrix}$, $\nabla h_j(x)\neq 0$, $j=1,\dots,p$, and the vectors $\{\nabla h_1(x),\dots,\nabla h_p(x)\}$ are linearly independent. Geometrically, we have p co-dimension 1 surfaces $\Sigma_1,\dots,\Sigma_p$, $\Sigma_j=\{x\in\mathbb{R}^n\ :\ h_j(x)=0\,,\quad h_j:\ \mathbb{R}^n\to\mathbb{R}\}$, $j=1,\dots,p$, and Σ is the co-dimension p surface given by the intersection of these surfaces. Moreover, in a neighborhood of Σ, there are 2^p regions R_i and inside each of these (1.1) holds, with vector fields f_i, $i=1,\dots,2^p$; see Figure 1. Filippov convexification for x in a neighborhood of Σ reads

$$x' \ \in\ F(x)=\sum_{i=1}^{2^p}\lambda_i(x)f_i(x)\ , \qquad \text{where}\qquad \lambda_i(x)\ge 0\ ,\ \text{and}\ \sum_{i=1}^{2^p}\lambda_i(x)=1\,. \tag{1.4}$$

A most important and interesting case is when for any initial condition near Σ the corresponding solution trajectories are attracted to Σ. In this case, which will be characterized below, a trajectory which arrives on Σ is forced to remain on Σ: The surface Σ is *globally attracting*. Filippov realized that in this situation the motion continues on Σ, giving rise to what he termed *sliding motion*. He further realized that as long as this motion takes place, necessarily it must occur with a vector field that lies in the tangent plane at $x\in\Sigma$, and therefore such vector field must be perpendicular to the gradients $\nabla h_j(x)$, $j=1,\dots,p$, for any $x\in\Sigma$. By this line of thought, during sliding motion, one is lead to consider a vector field $f_{\rm F}$ in the above convex-hull, which we will call *Filippov (sliding) vector field*, given by:

$$\begin{aligned}&\text{(a)}\quad x'=f_{\rm F}:=\sum_{i=1}^{2^p}\lambda_i(x)f_i(x)\ ,\ \text{with}\ \lambda_i(x)\ge 0\ ,\quad \sum_{i=1}^{2^p}\lambda_i(x)=1\,,\ \text{and}\\ &\text{(b)}\quad (\nabla h_j(x))^T f_{\rm F}(x)=0\ ,\ \text{for all}\ \ j=1,\dots,p\,.\end{aligned} \tag{1.5}$$

Moreover, the solution of this system, the λ_i's, ought to be smooth functions of x. But, for each $x\in\Sigma$, (1.5)-(b) defines a system of $(p+1)$ equations in 2^p unknowns (the λ_i's), and thus clearly (1.5)-(b) is an underdetermined system of equations for $p\ge 2$, and there is an ambiguity on how to select an appropriate Filippov vector field in these cases. Of course, this ambiguity had been observed by Filippov himself in [18], and there are special cases where this ambiguity does not arise (see [18] and Section 5). However, in general, there is no uniquely defined Filippov sliding vector. For the case of general value of p, under appropriate attractivity assumptions (the "nodal attractivity" conditions of the present work), in [16] we proposed a systematic approach to select an

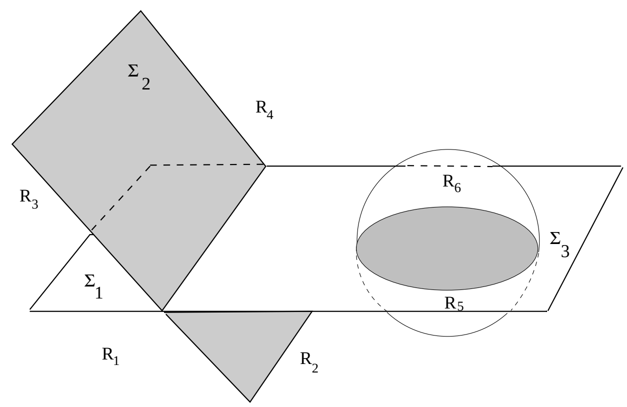

FIGURE 1. Regions and surfaces.

unique, well defined, sliding vector field. However, the choice we made in [16] does not necessarily reduce to a particular choice of a Filippov vector field of the form in (1.5), and this fact prompted us to reconsider the issue in this work.

In this paper, again under appropriate attractivity assumptions, for the case of sliding motion on surface(s) of co-dimension p, with $p > 1$, we will propose and justify a definition of the sliding vector field which is fully consistent with Filippov choice (1.5). For the case of $p = 1$, our present choice reduces to the classical choice of Filippov vector field. For the case of $p = 2$ and nodal attractivity (see Section 2), we recover the *sigmoid blending* choice of Alexander and Seidman, see [2]. As far as we could determine, the cases of $p = 2$ and spiral attractivity, and of $p > 2$, are new and amount to selecting a particular vector field in (1.5) by reducing the unknowns from 2^p to p. Complete details will be given for the cases of co-dimension $p = 2$, where we will give existence and uniqueness results and the construction of a unique, smoothly varying, vector field on Σ. We will further prove existence for $p \geq 3$.

Problems with sliding motion on surfaces of co-dimension $p \geq 2$ are not only of mathematical interest, but arise also naturally in mechanical systems (see Example 5.1), and in control applications whenever there are multiple discontinuous control variables. There are two main advantages of selecting a sliding vector field (1.5), rather than dealing with the more complicated differential inclusion (1.4): (a) it is simple to develop numerical methods during the sliding regime (e.g., as done in [16]), and (b) it becomes possible to carry out a non-ambiguous study of the dynamics of the system (e.g., see Examples 5.2, 5.3, 5.4).

Remark 1.1. It must be appreciated that in a general situation one may have solution trajectories that slide on (parts of) surfaces of different co-dimension, and enter and exits such surfaces repeatedly. Therefore, a robust simulation of these types of problem will require to be able to detect when a different regime is reached and to select the appropriate vector fields in these different regimes. To simplify matters, and also as an alternative to define a vector field in agreement with (1.5) during a sliding regime, several authors have studied the possibility of regularizing the problem so to avoid having to deal with a discontinuous system in the first place. These studies have been so far restricted to the case of a discontinuity surface of co-dimension 1. For example, see [26, 27] for possible regularization in practical cases, see the works of Teixeira and coworkers, [30, 29, 28], for a systematic exploration from the point of view of singular perturbation theory, and see the recent work of Fusco and Guglielmi for regularization through a certain averaging process, [20]. In spite of the potential simplifications of having a regularized problem, all of the proposed choices above ultimately modify the vector field in a neighborhood of the discontinuity surface, and this may possibly lead to undesired dynamical behavior (e.g., solutions may oscillate around the discontinuity surface). Partly because of this, we are not presently interested in regularization techniques as a mean to solve the original problem. However, since in all cases above the regularized problem depends on a small parameter ϵ, the limiting process (as $\epsilon \to 0$) is of interest. For the cases of co-dimension 1, it is to be expected that any well designed regularization should reproduce (in the $\epsilon \to 0$ limit) the uniquely defined Filippov vector field. For the cases of higher co-dimension, this study is being undertaken in [15].

A plan of the paper is as follows. In the remainder of this introduction we review the case of sliding motion on a co-dimension 1 surface (the classical case of Filippov). We will also revisit this case by adopting a rewriting introduced in [1], and which will be conducive to appropriate generalizations in Sections 2 and 3 where we deal in details with the cases of co-dimensions 2 and 3, respectively. In Section 4 we discuss the case of $p > 3$. Section 5 is dedicated to several examples which illustrate the theoretical results.

1.1. **Co-dimension 1 case.** To clarify the previous setup, consider the simplest case, when the state space is (at least locally) split into two regions by a co-dimension 1 surface. Thus, we have

$$x'(t) = f(x(t)) = \begin{cases} f_1(x(t)) \ , & x \in R_1 \ , \\ f_2(x(t)) \ , & x \in R_2 \ , \end{cases} \tag{1.6}$$

with $x(0) = x_0 \in \mathbb{R}^n$. The regions R_1 and R_2 are separated by a co-dimension one surface Σ, defined as the zero-set of a smooth scalar valued function $h : \mathbb{R}^n \to \mathbb{R}$, the so-called *event* function. So, the regions R_1, R_2, and the surface Σ, are characterized as

$$\Sigma = \{x \in \mathbb{R}^n |\ h(x) = 0\}, \quad R_1 = \{x \in \mathbb{R}^n |\ h(x) < 0\}, \quad R_2 = \{x \in \mathbb{R}^n |\ h(x) > 0\}. \tag{1.7}$$

In (1.6), we assume that f_1 is $C^k, k \geq 1$, on $R_1 \cup \Sigma$, that f_2 is $C^k, k \geq 1$, on $R_2 \cup \Sigma$, and that $h \in \mathcal{C}^k$, $k \geq 2$, for x on, and in a neighborhood of, Σ, and that the gradient $h_x(x) \neq 0$ for all $x \in \Sigma$. In other words, Σ is a true surface with well defined and smoothly varying unit normal n at all points $x \in \Sigma$, perpendicular to the tangent plane $T_x(\Sigma)$: $n(x) = \frac{h_x(x)}{\|h_x(x)\|}$.

Filippov convexification gives the set valued function

$$x'(t) \in F(x(t)) = \begin{cases} f_1(x(t)) & x \in R_1 \\ \overline{\text{co}}\,\{f_1(x(t), f_2(x(t)\} & x \in \Sigma \\ f_2(x(t)) & x \in R_2 \end{cases} , \tag{1.8}$$

where

$$\overline{\text{co}}\,\{f_1, f_2\} = \{(1-\alpha)f_1 + \alpha f_2,\ \alpha \in [0,1]\} \ . \tag{1.9}$$

We recall a fundamental result of Filippov on existence and uniqueness of solutions in the present case.

Theorem 1.2. [18] *Let f_1 and f_2 be $\mathcal{C}^1$ in $R_1 \cup \Sigma$, respectively on $R_2 \cap \Sigma$, and let h be $\mathcal{C}^2$ in a neighborhood of Σ. If, at any point $x \in \Sigma$, we have that at least one of $n^T(x)f_1(x) > 0$ and $n^T(x)f_2(x) < 0$ holds, then there exists a unique Filippov solution from each initial condition.*

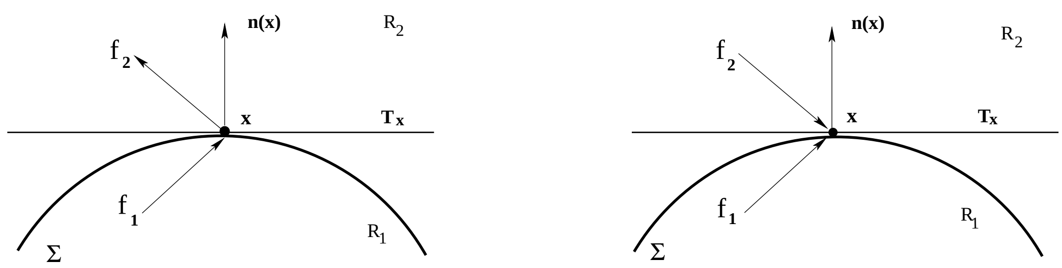

FIGURE 2. Discontinuity surface

The cases allowed by Theorem 1.2 are those of *transversal intersection*, *attractive sliding mode*, see Figure 2, and smoothly leaving Σ to enter in the region R_1 or R_2.

(a) *Transversal Intersection.* When, at $x \in \Sigma$, we have

$$[n^T(x)f_1(x)] \cdot [n^T(x)f_2(x)] > 0 \ , \tag{1.10}$$

then we will leave Σ. We will enter R_1, when $n^T(x)f_1(x) < 0$, and will enter R_2, when $n^T(x)f_1(x) > 0$. Any solution reaching Σ at a time t_1, and having a transversal intersection there, exists and is unique, in forward time.

(b) *Attracting Sliding Mode.* An attracting sliding mode at Σ occurs if

$$[n^T(x)f_1(x)] > 0 \quad \text{and} \quad [n^T(x)f_2(x)] < 0, \qquad x \in \Sigma \ , \tag{1.11}$$

where the inequality signs depend of course on (1.7). When we have an attracting sliding mode at $x_0 \in \Sigma$, a solution trajectory which reaches x_0 cannot leave Σ. According to Filippov vector field, *sliding motion* on Σ will take place with the smooth vector field

$$f_{\mathrm{F}}(x) = (1-\alpha(x))f_1(x) + \alpha(x)f_2(x) \ , \tag{1.12}$$

where $\alpha(x)$ is such that $n^T(x)f_{\mathrm{F}}(x) = 0$, and therefore

$$\alpha(x) = \frac{n^T(x)f_1(x)}{n^T(x)(f_1(x) - f_2(x))} \ . \tag{1.13}$$

(c) *Smooth exits.* Either (i) $n^T(x)f_1(x) = 0$ and $n^T(x)f_2(x) < 0$, or (ii) $n^T(x)f_1(x) > 0$ and $n^T(x)f_2(x) = 0$. In other words, one of the vector fields f_1 or f_2 is already in the tangent plane. In this case, we expect to leave Σ and enter into R_1 in case (i) or R_2 in case (ii). These tangential exits are a generic property of solution curves.

Remark 1.3. Observe that Filippov construction in the case of attractive sliding (1.11) requires non-vanishing vector fields. As such, it is a first order theory. Our generalizations below for the case $p \geq 2$ will also require non-vanishing vector fields.

Remark 1.4. A case not covered by Theorem 1.2 is the ill-posed case of *repulsive sliding*. This occurs when one has

$$[n^T(x)f_1(x)] < 0 \quad \text{and} \quad [n^T(x)f_2(x)] > 0, \qquad x \in \Sigma \ .$$

Though in principle one may enforce sliding by selecting α just as in (1.13), this case is ill-posed because at any instant of time we may leave Σ with either f_1 or f_2.

Reinterpretation, $p = 1$ **case**. We revisit in a new notation the case of attractive sliding motion. Let us begin by rewriting the problem in the form of a complementarity system as done by Acary and Brogliato in [1] and which is related to a rewriting first adopted by Stewart in [31].

We have the vector fields:

$$f_1 \quad \text{when} \quad h < 0 \ , \qquad \text{and} \qquad f_2 \quad \text{when} \quad h > 0 \ ,$$

and rewrite the Filippov differential inclusion (1.8) as

$$\dot{x} \in \frac{1 - \sigma(h(x))}{2} f_1(x) + \frac{1 + \sigma(h(x))}{2} f_2(x) \ , \tag{1.14}$$

where $h(x) = 0$ defines the discontinuity surface and where the function $\sigma(\cdot)$ is the multi-valued sign function

$$\sigma(x) = \begin{cases} 1 & x > 0 \\ [-1, 1] & x = 0 \\ -1 & x < 0 \end{cases} . \tag{1.15}$$

Note that this rewriting is exactly Filippov convexification formulation (1.8). Now, let us consider the case of x being on the surface, $x : \ h(x) = 0$. We seek a smooth selector (a function) $s(x)$, taking values in $[-1, 1]$, so that Filippov vector field becomes

$$\dot{x} = \frac{1 - s(x)}{2} f_1(x) + \frac{1 + s(x)}{2} f_2(x) \ , \quad \text{or} \quad \dot{x} = (1 - \alpha) f_1(x) + \alpha f_2(x) \ ,$$

where we have set $s(x) = 2\alpha(x) - 1$.

To obtain the Filippov sliding vector field (1.5) we need to find the function α so that $n^T(x)\dot{x} = 0$. We use the following notation (all quantities are understood to be evaluated at $x \in \Sigma$)

$$w_1 = n^T f_1 \ , \quad w_2 = n^T f_2 \ ,$$

so that the assumption of **attractivity** of the surface means the following insofar as the signs for w_1, w_2:

Table 1. Vector w: $p = 1$.

Component	$i = 1$	$i = 2$
w_i, $i = 1, 2$	> 0	< 0

The relation to be satisfied by α can be written as

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} (1 - \alpha) \\ \alpha \end{bmatrix} \ = \ 0 \, . \tag{1.16}$$

so that $\alpha \ = \frac{w_1}{w_1 - w_2}$, which is of course the same function of (1.13). [Of course, (1.16) has a unique solution, which can be trivially found from (1.16), as long as $w_1 \neq w_2$, but we are presently only concerned with the case of attractive sliding motion.]

We conclude this introduction with a simple result which will be useful in the next sections.

Lemma 1.5. *Let $s_1, s_2, \dots, s_p$ be real numbers. Consider the 2^p non repeated products of the form $(1 \pm s_1)(1 \pm s_2) \cdots (1 \pm s_p)$. Let S_p be the sum of all of these products. Then, $S_p = 2^p$.*

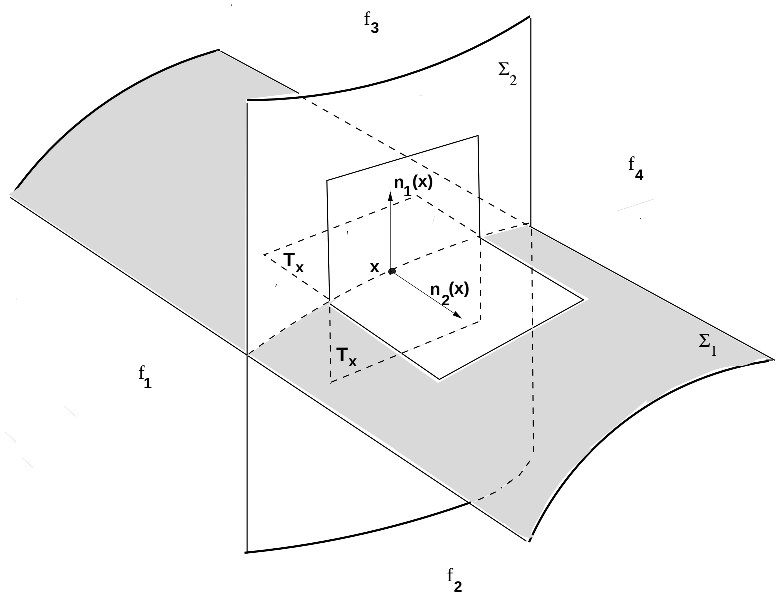

FIGURE 3. Intersection of discontinuity surfaces.

Proof. The proof is by induction on p. For $p = 1$, there are two terms to add: $(1-s_1)$ and $(1+s_1)$ which obviously add to 2. Now, suppose that the result is true for $(p-1)$ and let's show it for p. The observation is that there are two terms to add, the first is $(1-s_p)S_{p-1}$, the second is $(1+s_p)S_{p-1}$. Using the induction hypothesis, we have

$$(1-s_p)S_{p-1} + (1+s_p)S_{p-1} = 2S_{p-1} = 2^p\,.$$

□

Remark 1.6. For us, the values $s_1, s_2, \ldots, s_p$ in Lemma 1.5 will all be in $[-1, 1]$.

2. CO-DIMENSION 2 CASE

Here Σ is the intersection of two co-dimension 1 surfaces, $\Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_1 = \{\, x : h_1(x) = 0\ ,\ \ h_1 : \mathbb{R}^n \to \mathbb{R}\}$ and $\Sigma_2 = \{\, x :\ h_2(x) = 0\ ,\ \ h_2 : \mathbb{R}^n \to \mathbb{R}\}$, and they have smoothly varying unit normals n_1 and n_2, respectively, which we further assume to be linearly independent for $x \in \Sigma$. So, locally, the two surfaces split the phase space into four regions R_1, R_2, R_3 and R_4 and we have (1.1) with four smooth functions f_i, $i = 1, \ldots, 4$. We will use the labeling (see Figure 3):

$$(R_1)\ f_1 \quad \text{when} \quad h_1 < 0\ ,\ h_2 < 0\ , \qquad (R_2)\ f_2 \quad \text{when} \quad h_1 < 0\ ,\ h_2 > 0\ ,$$
$$(R_3)\ f_3 \quad \text{when} \quad h_1 > 0\ ,\ h_2 < 0\ , \qquad (R_4)\ f_4 \quad \text{when} \quad h_1 > 0\ ,\ h_2 > 0\ .$$

As we remarked in the introduction, it is well known that there is an ambiguity in defining the Filippov sliding vector field (1.5) on Σ, except in particular cases (e.g., see [18, 34] and Section 5). Recently, two different approaches have been proposed to eliminate this ambiguity when Σ satisfies appropriate attractivity assumptions (the nodal attractivity assumptions below). The first approach is due to Alexander and Seidman and is based on sigmoid blending techniques (see [2, 14]), the second one is based on geometric considerations and we proposed it in [16].

We now examing this case of $p = 2$ from a new viewpoint. First, rewrite the problem as (see [1])

$$\begin{aligned}\dot{x} \in & \frac{1-\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2}f_1(x) + \frac{1-\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2}f_2(x) \\ & + \frac{1+\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2}f_3(x) + \frac{1+\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2}f_4(x)\ ,\end{aligned} \tag{2.1}$$

where $\sigma(\cdot)$ is the sign function in (1.15). Off Σ, this rewriting is equivalent to Filippov convexification, in the sense that it represents the most general convex combination we can take (this is true also when x is on Σ_1 or Σ_2). Consider now $x \in \Sigma$. We want to replace the set valued sign functions $\sigma(h_1(x))$ and $\sigma(h_2(x))$ in (2.1) with smooth functions $s_1(x)$ and $s_2(x)$, defined for x in Σ and taking values in $[-1, 1]$, and consider the differential equation (see (2.1))

$$\begin{aligned}\dot{x} = & \frac{1-s_1(x)}{2}\frac{1-s_2(x)}{2}f_1(x) + \frac{1-s_1(x)}{2}\frac{1+s_2(x)}{2}f_2(x) \\ & + \frac{1+s_1(x)}{2}\frac{1-s_2(x)}{2}f_3(x) + \frac{1+s_1(x)}{2}\frac{1+s_2(x)}{2}f_4(x)\ ,\end{aligned} \tag{2.2}$$

and from Lemma 1.5 we have

$$\frac{1-s_1}{2}\frac{1-s_2}{2} + \frac{1-s_1}{2}\frac{1+s_2}{2} + \frac{1+s_1}{2}\frac{1-s_2}{2} + \frac{1+s_1}{2}\frac{1+s_2}{2} = 1\ .$$

Remark 2.1. In other words, for each $x \in \Sigma$, the functions $s_1(x)$ and $s_2(x)$ are specific selectors chosen from the set valued functions $\sigma(h_1(x))$ and $\sigma(h_2(x))$. By using scalar valued functions $s_1(x)$ and $s_2(x)$, with $s_{1,2}$ taking values in $[-1, 1]$, for all x in $\Sigma = \Sigma_1 \cap \Sigma_2$, we are taking a smooth convex combination of the vector fields, consistent with Filippov convexification approach.

To be consistent with previous notation, we set $s_1(x) = 2\alpha(x) - 1$ and $s_2(x) = 2\beta(x) - 1$, where $\alpha, \beta \in [0, 1]$, for all x of interest. [Unless truly necessary, we will omit highlighting the dependence on x, which is implicitly assumed.] Then, we look for a Filippov sliding vector field and differential equation on the intersection of the form

$$\dot{x} = (1-\alpha)(1-\beta)f_1(x) + (1-\alpha)\beta f_2(x) + \alpha(1-\beta)f_3(x) + \alpha\beta f_4(x) \tag{2.3}$$

and $\alpha, \beta \in [0, 1]$ must be found so that $n_1^T(x)\dot{x} = n_2^T(x)\dot{x} = 0$. The expression (2.3) is the same equation obtained in [2]. We now show that α and β can be uniquely found and are smooth functions if Σ is attractive.

Attractive Σ. We use the following notation (all quantities to be evaluated at x)

$$\begin{aligned} w_1^1 = n_1^T f_1\ ,\ w_2^1 = n_1^T f_2\ ,\ w_3^1 = n_1^T f_3\ ,\ w_4^1 = n_1^T f_4\ , \\ w_1^2 = n_2^T f_1\ ,\ w_2^2 = n_2^T f_2\ ,\ w_3^2 = n_2^T f_3\ ,\ w_4^2 = n_2^T f_4\ , \end{aligned}$$

and set $W = \begin{bmatrix} w^1 & w^2 \end{bmatrix} \in \mathbb{R}^{4\times 2}$, so that the system to be satisfied by α, β can be written as

$$W^T a = 0\ , \quad a = \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix}\ .$$

For each given x, this is a nonlinear system in α, β, which we rewrite as

$$\begin{aligned} (1-\alpha)\big[(1-\beta)w_1^1(x) + \beta w_2^1(x)\big] + \alpha\big[(1-\beta)w_3^1(x) + \beta w_4^1(x)\big] = 0 \\ (1-\beta)\big[(1-\alpha)w_1^2(x) + \alpha w_3^2(x)\big] + \beta\big[(1-\alpha)w_2^2(x) + \alpha w_4^2(x)\big] = 0\ , \end{aligned} \tag{2.4}$$

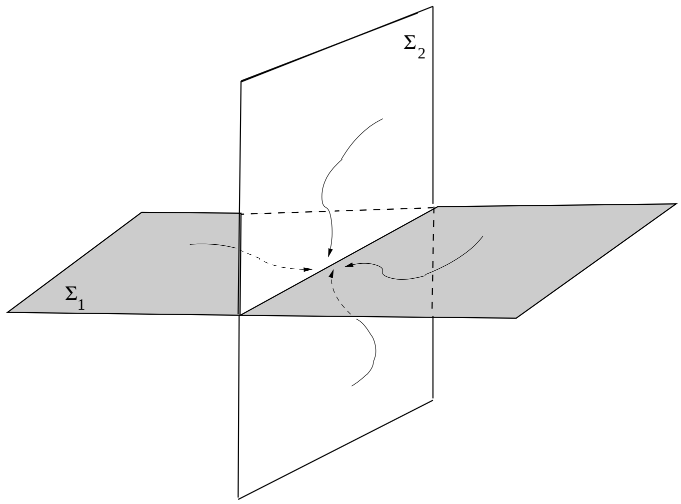

FIGURE 4. Attractive Σ: Nodal case.

or more compactly as

$$\begin{aligned} &\alpha\big[A_1^1(x,\beta) - A_2^1(x,\beta)\big] + A_2^1(x,\beta) = 0 \\ &\beta\big[A_1^2(x,\alpha) - A_1^2(x,\alpha)\big] + A_2^2(x,\alpha) = 0 \ , \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} A_1^1 &= (1-\beta)w_3^1(x) + \beta w_4^1(x) \ , \ \ A_2^1 = (1-\beta)w_1^1(x) + \beta w_2^1(x) \ , \\ A_1^2 &= (1-\alpha)w_2^2(x) + \alpha w_4^2(x) \ , \ \ A_2^2 = (1-\alpha)w_1^2(x) + \alpha w_3^2(x) \ . \end{aligned}$$

We now make the assumption of **attractivity** of the intersection, and further differentiate between three different cases of attractivity.
(a): **Nodal Attractivity**. This is the attractivity case considered in [2] and [16]. This case is analogous to the case of a stable node in a planar system, and it is characterized by the constraints on the signs of w^1 and w^2 expressed in Table 2.

TABLE 2. Matrix W: $p = 2$. Nodal Attractivity.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1, $i=1:4$	>0	>0	<0	<0
w_i^2, $i=1:4$	>0	<0	>0	<0

Remark 2.2. What does the assumption of nodal attractivity mean? The geometrical meaning of it is that if a trajectory is near Σ, it will be attracted towards Σ and if the trajectory is on either one of the two co-dimension 1 surfaces Σ_1 or Σ_2, then it will slide on this lower co-dimension surface while approaching the intersection Σ. See Figure 4.

Existence[1] Because of the signs in Table 2, we have that for all $(\alpha,\beta) \in [0,1]\times[0,1]$:

$$\begin{aligned} &A_1^1(x,\beta) < 0 \ , \ \ A_2^1(x,\beta) > 0 \ , \\ &A_1^2(x,\alpha) < 0 \ , \ \ A_2^2(x,\alpha) > 0 \ . \end{aligned}$$

[1]In this case, unique solvability for for α and β in (2.3) was also obtained in [2] by somewhat different means.

So, formally expressing α and β from the first, respectively the second, relation in (2.5), we can formulate the problem as a nonlinear fixed point problem, $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = T(\alpha,\beta)$:

$$\begin{aligned} \alpha &= \frac{A_2^1(x,\beta)}{A_2^1(x,\beta) - A_1^1(x,\beta)}\ , \\ \beta &= \frac{A_2^2(x,\alpha)}{A_2^2(x,\alpha) - A_1^2(x,\alpha)}\ , \end{aligned} \tag{2.6}$$

and by virtue of the signs of the coefficients we have that the map T is a continuous map from $[0,1]\times[0,1]$ into itself. Therefore, there is a fixed point in the unit square and a solution α,β of (2.5).

Uniqueness. The issue of uniqueness is more complex, because it is not easy to verify if the map T is a contraction.

Remark 2.3. The reason why is not easy to verify (a priori) that T is a contraction can be seen as follows. We have the map $T : \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \to \begin{bmatrix}T_1(\beta)\\ T_2(\alpha)\end{bmatrix}$, and its derivative is $DT = \begin{bmatrix}0 & T_1'(\beta)\\ T_2'(\alpha) & 0\end{bmatrix}$, so that DT cannot have norm less than 1 if either $T_1'(\beta)$ or $T_2'(\alpha)$ are greater than 1 (in magnitude). Explicit computation gives

$$T_1'(\beta) = \frac{1}{A_2^1 - A_1^1}\left[(1-\alpha)(w_2^1 - w_1^1) - \alpha(w_3^1 - w_4^1)\right]$$

and it is not possible to decide priori if $|T_1'(\beta)| < 1$ for $\alpha,\beta \in [0,1]^2$, without putting extra requirements on the w_j^i's besides those of Table 2 (and we do not want to do this). Similarly for $T_2'(\alpha)$. Notice that we are not saying that T is not a contraction at its fixed point, we do not know this for sure, but simply that it is not possible to decide a priori that DT has norm bounded by 1 for all $\alpha,\beta \in [0,1]$.

Nevertheless, it is possible to decide that the fixed point is unique, that is that there is a unique solution $(\alpha,\beta)\in[0,1]^2$, by reasoning in geometrical terms.

First, from (2.6), we substitute the relation satisfied by α in the formula expressing β and obtain a quadratic equation for β, $P(\beta)=0$, where $P(\beta) = c_2\beta^2 + c_1\beta + c_0$, and

$$\begin{aligned} c_2 &= (w_2^1 w_4^2 + w_1^1 w_3^2) - (w_1^1 w_4^2 + w_2^1 w_3^2) - (w_2^2 w_4^1 + w_1^2 w_3^1) + (w_1^2 w_4^1 + w_2^2 w_3^1) \\ c_1 &= w_1^1 w_4^2 + w_2^1 w_3^2 - 2w_1^1 w_3^2 - w_1^2 w_4^1 - w_2^2 w_3^1 + 2w_1^2 w_3^1\ , \quad c_0 = w_1^1 w_3^2 - w_1^2 w_3^1\ . \end{aligned}$$

We seek $\beta \in [0,1]$ satisfying this equation. Because of the signs in Table 2 above, we have

$$P(0) = c_0 = w_1^1 w_3^2 - w_1^2 w_3^1 > 0 \quad P(1) = c_2 + c_1 + c_0 = w_2^1 w_4^2 - w_2^2 w_4^1 < 0\ ,$$

and therefore there is a root β^* in $[0,1]$ and since the function is a parabola the root in $[0,1]$ is unique. Using this for α in (2.6) we have that also α varies smoothly (in x).

Again, there is no guarantee that $|P'(\beta)|<1$ nor that $P'(\beta)\neq 0$, for $\beta\in[0,1]$, though clearly $P'(\beta^*)\neq 0$ (see Figure 2).

Example 2.4. As an **alternative** to directly solving the nonlinear system $n_1^T\dot{x} = n_2^T\dot{x} = 0$, to find α and β for each $x\in\Sigma$, it is possible to find the solution by solving an eigenvalue problem. We have to solve

$$W^T a = 0\ , \quad a = \begin{bmatrix} (1-\alpha)\begin{bmatrix}1-\beta\\ \beta\end{bmatrix} \\ \alpha\begin{bmatrix}1-\beta\\ \beta\end{bmatrix}\end{bmatrix}\ .$$

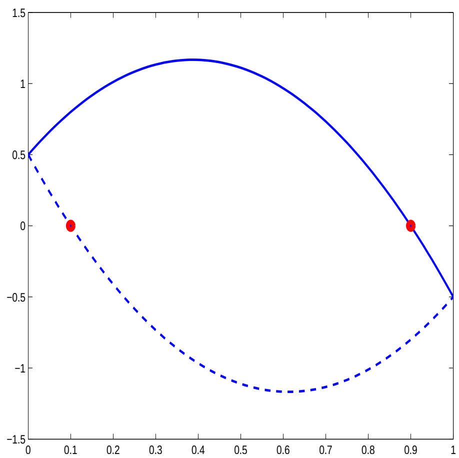

FIGURE 5. $p = 2$: Possible intersections.

By virtue of the signs in Table 2 above, the leading (2×2) matrix in W^T is invertible, which means that we need to solve a problem of the form $\begin{bmatrix} I & B \end{bmatrix} a = 0$, which is the eigenvalue problem

$$\begin{bmatrix} \alpha(I - B) - I \end{bmatrix} \begin{bmatrix} 1 - \beta \\ \beta \end{bmatrix} = 0 \, .$$

This gives at most two solutions, and we want the one with $\alpha \in [0, 1]$ and the eigenvector normalized with positive entries and 1-norm 1.

The other cases of attractive Σ we consider is when the intersection is attractive in a spiral-like manner. We distinguish between two such cases.

(b-1): **Spiral-like Attractivity: Case 1**. This is analogous to the case of a stable spiral in a planar system. Recent work by Brogliato, [8], seems concerned with a similar situation. The clockwise situation is characterized by the constraints on the signs of w^1 and w^2 and by the additional well-posedness conditions on the relative magnitudes of w_i^1 and w_i^2, $i = 1, \dots, 4$, expressed in Table 3. These are required to hold on –and in a sufficiently small neighborhood of– Σ. There is of course an obvious counter-clockwise situation as well.

TABLE 3. Matrix W: $p = 2$. Spiral Attractivity: Case 1.

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
w_i^1, $i = 1:4$	> 0	< 0	> 0	< 0
w_i^2, $i = 1:4$	< 0	< 0	> 0	> 0
Well posedness conditions: $w_1^1 > -w_1^2$, $w_2^1 > w_2^2$, $w_3^2 > w_3^1$, $-w_4^1 > w_4^2$				

Remark 2.5. Near the intersection, the surfaces $\Sigma_{1,2}$ are crossed, but the distance between successive crossings decreases. The well posedness condition is necessary to guarantee that the surface Σ is attractive, see Figure 6 for a sketch of the situation. We emphasize that with the conditions expressed in Table 3, the function $V(x(t)) = |h_1(x(t))| + |h_2(x(t))|$ decreases along solution trajectories.

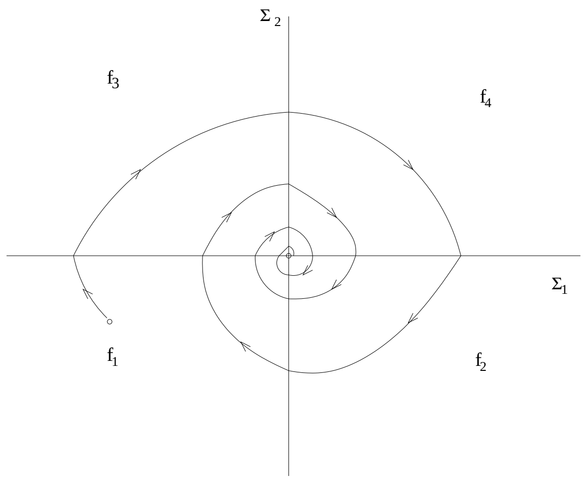

FIGURE 6. Attractive Σ: Spiral case 1.

Verifying existence and uniqueness of the solution of (2.4) goes in a similar way to the case of nodal attractivity. Letting $\widehat{\alpha} = \beta$ and $\widehat{\beta} = 1 - \alpha$ in (2.4), the system to be solved is rewritten as

$$(2.7)\qquad \begin{aligned} &(1-\widehat{\alpha})\big[(1-\widehat{\beta})w_3^1(x)+\widehat{\beta}w_1^1(x)\big]+\widehat{\alpha}\big[(1-\widehat{\beta})w_4^1(x)+\widehat{\beta}w_2^1(x)\big]=0\\ &(1-\widehat{\beta})\big[(1-\widehat{\alpha})w_3^2(x)+\widehat{\alpha}w_4^2(x)\big]+\widehat{\beta}\big[(1-\widehat{\alpha})w_1^2(x)+\widehat{\alpha}w_2^2(x)\big]=0\ . \end{aligned}$$

At this point, formally solving for $\widehat{\alpha}$ from the first equation in (2.7) and substituting in the second equation, we end up with the following quadratic equation for $\widehat{\beta}$:

$$P(\widehat{\beta}) = 0\ , \quad P(\widehat{\beta}) = c_0 + c_1\widehat{\beta} + c_2\widehat{\beta}^2\ ,$$

where

$$c_0 = w_3^1w_4^2 - w_4^1w_3^2\ , \quad c_1 = 2w_4^1w_3^2 - w_4^1w_1^2 - w_2^1w_3^2 - 2w_3^1w_4^2 + w_1^1w_4^2 + w_3^1w_2^2\ ,$$
$$c_2 = w_1^1w_2^2 - w_2^1w_1^2 - w_4^1w_3^2 + w_4^1w_1^2 + w_2^1w_3^2 + w_3qw_4^2 - w_1^1w_4^2 - w_3^1w_2^2\ .$$

Now, using the signs of Table 3, this gives $P(0) = c_0 > 0$ and $P(1) = w_1^1w_2^2 - w_2^1w_1^2 < 0$ and again there is a unique solution β in $(0, 1)$.

We note that the well-posedness condition of Table 3 is not needed to obtain a unique solution to (2.7); the signs of w_i^1, w_i^2, $i = 1, \ldots, 4$, in Table 3 are sufficient for this. The well-posedness condition renders Σ attractive. Without this well-posedness condition, one may have ill-posed repulsive sliding motion, similarly to the situation of Remark 1.4,

(b-2): **Spiral-like Attractivity: Case 2**. This case has no immediate similarity with a smooth planar dynamical system. One of the surfaces is crossed on both sides of the intersection, while the other surface is crossed on one side, but sliding motion towards Σ occurs on the other side. The clockwise situation when we slide on the portion of Σ_1 to the right of Σ is characterized by the constraints on the signs of the entries of w^1 and w^2 expressed in Table 4, and by the additional well-posedness condition expressed in Table 4. These conditions must hold on –and in a sufficiently small neighborhood of– Σ. Naturally, there are similar counter-clockwise situations, as well as when we slide on the other side of Σ_1 or on Σ_2.

Table 4. Matrix W: $p = 2$. Spiral Attractivity: Case 2.

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
w_i^1, $i = 1:4$	> 0	> 0	> 0	< 0
w_i^2, $i = 1:4$	< 0	< 0	> 0	> 0
Well posedness condition: $-w_4^1 w_2^2 + w_2^1 w_4^2 < 0$				

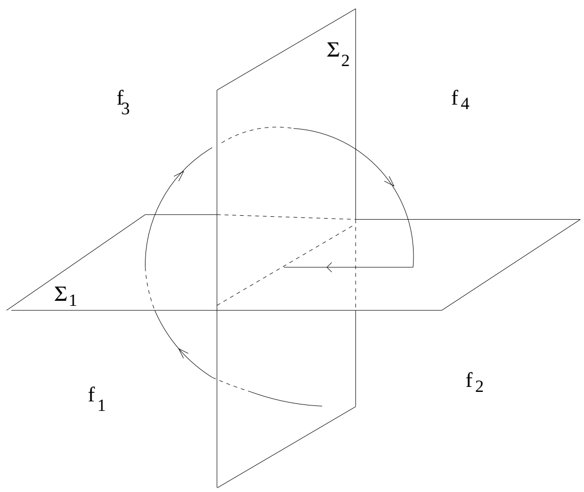

Figure 7. Attractive Σ: Spiral case 2.

Remark 2.6. The conditions expressed in Table 4, including the well-posedness condition, guarantee that near the intersection the surface Σ_2 is crossed, as it is the portion of Σ_1 on the left of the intersection. However, upon reaching Σ_1 on the right, the motion becomes a sliding motion towards Σ. The well posedness condition is crucial to guarantee that the surface Σ is reached, see Figure 7. Indeed, on the portion of Σ_1 to the right of Σ, we will have attractive sliding motion with vector field $f_{\rm F} = (1-\alpha)f_2 + \alpha f_4$ and $\alpha = \frac{w_2^1}{w_2^1 - w_4^1}$; to reach the intersection Σ, we are asking to have $n_2^T f_{\rm F} < 0$, which leads to the well-posedness condition $-w_4^1 w_2^2 + w_2^1 w_4^2 < 0$.

To show existence and uniqueness of the solution to (2.4) in this case, we proceed as follows. Letting $\widehat{\alpha} = 1 - \alpha$ and $\widehat{\beta} = \beta$ in (2.4), the system to be solved is rewritten as

$$
\begin{aligned}
&(1-\widehat{\alpha})\big[(1-\widehat{\beta})w_3^1 + \widehat{\beta} w_4^1\big] + \widehat{\alpha}\big[(1-\widehat{\beta})w_1^1(x) + \widehat{\beta} w_2^1(x)\big] = 0 \\
&(1-\widehat{\alpha})\big[(1-\widehat{\beta})w_3^2 + \widehat{\beta} w_4^2\big] + \widehat{\alpha}\big[(1-\widehat{\beta})w_1^2(x) + \widehat{\beta} w_2^2(x)\big] = 0\,.
\end{aligned} \tag{2.8}
$$

Solving for $\widehat{\alpha}$ from the second equation in (2.8), and substituting in the first equation, we end up with the following quadratic equation for $\widehat{\beta}$:

$$
P(\widehat{\beta}) = 0\ , \quad P(\widehat{\beta}) = c_0 + c_1\widehat{\beta} + c_2\widehat{\beta}^2\ ,
$$

where

$$
c_0 = w_1^1 w_3^3 - w_3^1 w_1^2\ , \quad c_1 = 2w_3^1 w_1^2 - w_4^1 w_1^2 - w_3^1 w_2^2 - 2w_1^1 w_3^2 + w_1^1 w_4^2 + w_2^1 w_3^2\ ,
$$
$$
c_2 = w_4^1 w_1^2 - w_3^1 w_1^2 + w_3^1 w_2^2 - w_4^1 w_2^2 + w_1^1 w_3^2 - w_1^1 w_4^2 - w_2^1 w_3^2 + w_2^1 w_4^2\ .
$$

Using Table 4, this gives $P(0) = w_3^1 w_4^2 - w_4^1 w_3^2 > 0$ and $P(1) = -w_3^1 w_1^2 + w_2^1 w_4^2$ which is negative because of the well-posedness condition of Table 4. So, again, there is a unique solution $\beta \in (0,1)$, smoothly varying in $x \in \Sigma$. We stress that in this situation, the well-posedness condition of Table 4 is needed to obtain a unique solution to (2.8).

3. Co-dimension $p = 3$

Again, we will consider the problem by taking into account the structure of the data which we have on the intersection. We only consider the case of nodal attractivity, though appropriate generalizations of the case of spiral attractivity also should be possible.

We have $\Sigma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$, and the three co-dimension 1 surfaces $\Sigma_1 = \{ x : \ h_1(x) = 0\}$, $\Sigma_2 = \{ x : \ h_2(x) = 0\}$ and $\Sigma_3 = \{ x : \ h_3(x) = 0\}$, have smoothly varying and well defined unit normals n_1, n_2 and n_3, respectively, which are linearly independent on Σ. Locally, Σ separates $\mathbb{R}^n$ in eight regions R_i inside which we have eight smooth functions f_i, $i = 1, \dots, 8$. We assume that they are labeled as follows:

$$\begin{aligned}
&f_1 \ \text{ when } h_1 < 0 \ , \ h_2 < 0 \ , \ h_3 < 0 \ , \quad && f_2 \ \text{ when } h_1 < 0 \ , \ h_2 < 0 \ , \ h_3 > 0 \ , \\
&f_3 \ \text{ when } h_1 < 0 \ , \ h_2 > 0 \ , \ h_3 < 0 \ , \quad && f_4 \ \text{ when } h_1 < 0 \ , \ h_2 > 0 \ , \ h_3 > 0 \ , \\
&f_5 \ \text{ when } h_1 > 0 \ , \ h_2 < 0 \ , \ h_3 < 0 \ , \quad && f_6 \ \text{ when } h_1 > 0 \ , \ h_2 < 0 \ , \ h_3 > 0 \ , \\
&f_7 \ \text{ when } h_1 > 0 \ , \ h_2 > 0 \ , \ h_3 < 0 \ , \quad && f_8 \ \text{ when } h_1 > 0 \ , \ h_2 > 0 \ , \ h_3 > 0 \ .
\end{aligned}$$

Similarly to before, near Σ, we rewrite the problem (1.1) as

(3.1)
$$\begin{aligned}
\dot{x} \in & \frac{1-\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2}\frac{1-\sigma(h_3(x))}{2} f_1(x) + \frac{1-\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2}\frac{1+\sigma(h_3(x))}{2} f_2(x) \\
& + \frac{1-\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2}\frac{1-\sigma(h_3(x))}{2} f_3(x) + \frac{1-\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2}\frac{1+\sigma(h_3(x))}{2} f_4(x) \\
& + \frac{1+\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2}\frac{1-\sigma(h_3(x))}{2} f_5(x) + \frac{1+\sigma(h_1(x))}{2}\frac{1-\sigma(h_2(x))}{2}\frac{1+\sigma(h_3(x))}{2} f_6(x) \\
& + \frac{1+\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2}\frac{1-\sigma(h_3(x))}{2} f_7(x) + \frac{1+\sigma(h_1(x))}{2}\frac{1+\sigma(h_2(x))}{2}\frac{1+\sigma(h_3(x))}{2} f_8(x) \ ,
\end{aligned}$$

where of course the function $\sigma(\cdot)$ is the multivalued sign function of (1.15).

In our search for a specific Filippov vector field on Σ, similarly to what we did before, we seek selectors to replace the set valued functions $\sigma(h_1(\cdot)), \sigma(h_2(\cdot))$ and $\sigma(h_3(\cdot))$ with some smooth (single valued) functions s_1, s_2 and s_3. That is, for each $x \in \Sigma$, we seek $s_1(x)$, $s_2(x)$, $s_3(x)$, and replace the inclusion (3.1) by the differential equation given by the following special convex combination (see Lemma 1.5 and (1.5))

$$\begin{aligned}
\dot{x} = & \frac{1-s_1(x)}{2}\frac{1-s_2(x)}{2}\frac{1-s_3(x)}{2} f_1(x) + \frac{1-s_1(x)}{2}\frac{1-s_2(x)}{2}\frac{1+s_3(x)}{2} f_2(x) \\
& + \frac{1-s_1(x)}{2}\frac{1+s_2(x)}{2}\frac{1-s_3(x)}{2} f_3(x) + \frac{1-s_1(x)}{2}\frac{1+s_2(x)}{2}\frac{1+s_3(x)}{2} f_4(x) \\
& + \frac{1+s_1(x)}{2}\frac{1-s_2(x)}{2}\frac{1-s_3(x)}{2} f_5(x) + \frac{1+s_1(x)}{2}\frac{1-s_2(x)}{2}\frac{1+s_3(x)}{2} f_6(x) \\
& + \frac{1+s_1(x)}{2}\frac{1+s_2(x)}{2}\frac{1-s_3(x)}{2} f_7(x) + \frac{1+s_1(x)}{2}\frac{1+s_2(x)}{2}\frac{1+s_3(x)}{2} f_8(x) \ .
\end{aligned} \tag{3.2}$$

To be consistent with previous notation, for all $x \in \Sigma$, we set $s_1(x) = 2\alpha(x)-1$, $s_2(x) = 2\beta(x)-1$ and $s_3(x) = 2\gamma(x)-1$, where the functions α, β, γ map $x \in \Sigma$ into $[0,1]$ and must be found. Then, the differential equation on the intersection becomes

$$\begin{aligned}\dot{x} =&(1-\alpha)(1-\beta)(1-\gamma)f_1(x) + (1-\alpha)(1-\beta)\gamma f_2(x) + (1-\alpha)\beta(1-\gamma)f_3(x)+\\ &(1-\alpha)\beta\gamma f_4(x) + \alpha(1-\beta)(1-\gamma)f_5(x) + \alpha(1-\beta)\gamma f_6(x)\\ &+ \alpha\beta(1-\gamma)f_7(x) + \alpha\beta\gamma f_8(x)\end{aligned}$$

and $\alpha, \beta, \gamma \in [0,1]$ must be found so that $n_1^T(x)\dot{x} = n_2^T(x)\dot{x} = n_3^T(x)\dot{x} = 0$. We use the following notation (all quantities to be evaluated at x)

$$\begin{aligned} w^1 &= (w_i^1)_{i=1}^8 \ : \ w_i^1 = n_1^T f_i \ , \ i = 1, \ldots, 8 \ , \\ w^2 &= (w_i^2)_{i=1}^8 \ : \ w_i^2 = n_2^T f_i \ , \ i = 1, \ldots, 8 \ , \\ w^3 &= (w_i^3)_{i=1}^8 \ : \ w_i^3 = n_3^T f_i \ , \ i = 1, \ldots, 8 \ , \end{aligned}$$

and form the matrix $W = \begin{bmatrix} w^1 & w^2 & w^3 \end{bmatrix} \in \mathbb{R}^{8\times 3}$, so that the system to be satisfied by α, β, γ can be written as

$$W^T a = 0 \ , \quad a = \begin{bmatrix} (1-\alpha)(1-\beta)(1-\gamma) \\ (1-\alpha)(1-\beta)\gamma \\ (1-\alpha)\beta(1-\gamma) \\ (1-\alpha)\beta\gamma \\ \alpha(1-\beta)(1-\gamma) \\ \alpha(1-\beta)\gamma \\ \alpha\beta(1-\gamma) \\ \alpha\beta\gamma \end{bmatrix} .$$

For each given x, this is a nonlinear system in α, β, γ, which we rewrite compactly as

$$\begin{aligned} &\alpha\big[A_1^1(x,\beta,\gamma) - A_2^1(x,\beta,\gamma)\big] + A_2^1(x,\beta,\gamma) = 0 \\ &\beta\big[A_1^2(x,\alpha,\gamma) - A_2^2(x,\alpha,\gamma)\big] + A_2^2(x,\alpha,\gamma) = 0 \\ &\gamma\big[A_1^3(x,\alpha,\beta) - A_2^3(x,\alpha,\beta)\big] + A_2^3(x,\alpha,\beta) = 0 \ , \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A_1^1 &= (1-\beta)(1-\gamma)w_5^1(x) + (1-\beta)\gamma w_6^1(x) + \beta(1-\gamma)w_7^1 + \beta\gamma w_8^1 \ , \\ A_2^1 &= (1-\beta)(1-\gamma)w_1^1(x) + (1-\beta)\gamma w_2^1(x) + \beta(1-\gamma)w_3^1 + \beta\gamma w_4^1 \ , \\ A_1^2 &= (1-\alpha)(1-\gamma)w_3^2(x) + (1-\alpha)\gamma w_4^2(x) + \alpha(1-\gamma)w_7^2 + \alpha\gamma w_8^2 \ , \\ A_2^2 &= (1-\alpha)(1-\gamma)w_1^2(x) + (1-\alpha)\gamma w_2^2(x) + \alpha(1-\gamma)w_5^2 + \alpha\gamma w_6^2 \ , \\ A_1^3 &= (1-\alpha)(1-\beta)w_2^3(x) + (1-\alpha)\beta w_4^3(x) + \alpha(1-\beta)w_6^3 + \alpha\beta w_8^3 \ , \\ A_2^3 &= (1-\alpha)(1-\beta)w_1^3(x) + (1-\alpha)\beta w_3^3(x) + \alpha(1-\beta)w_5^3 + \alpha\beta w_7^3 \ . \end{aligned}$$

As before, we make the assumption of **nodal attractivity** of the intersection. In the present context, this means the assumption on the signs of w^1, w^2, w^3, summarized in Table 5. Again, we

TABLE 5. Matrix W: $p = 3$.

Component	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
w_i^1, $i=1:8$	>0	>0	>0	>0	<0	<0	<0	<0
w_i^2, $i=1:8$	>0	>0	<0	<0	>0	>0	<0	<0
w_i^3, $i=1:8$	>0	<0	>0	<0	>0	<0	>0	<0

observe that our assumption of attractivity means that (locally, near Σ), trajectories approach Σ, and further that we have attracting sliding motions on the lower co-dimensions (1 and 2) surfaces intersecting on Σ, with all such sliding vector fields leading toward Σ.

Using Table 5, we observe that for all $(\alpha,\beta,\gamma)\in[0,1]\times[0,1]\times[0,1]$:

$$\begin{aligned}
&A_1^1(x,\beta,\gamma)<0\ ,\ \ A_2^1(x,\beta,\gamma)>0\ ,\\
&A_1^2(x,\alpha,\gamma)<0\ ,\ \ A_2^2(x,\alpha,\gamma)>0\ ,\\
&A_1^3(x,\alpha,\beta)<0\ ,\ \ A_2^3(x,\alpha,\beta)>0\ .
\end{aligned}$$

So, solving for α, β and γ, from the first, second, and third, respectively, equation in (3.3), we can formulate the problem as the nonlinear fixed point problem, $\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=T(\alpha,\beta,\gamma)$:

$$\begin{aligned}
\alpha=&\frac{A_2^1(x,\beta,\gamma)}{A_2^1(x,\beta,\gamma)-A_1^1(x,\beta,\gamma)}\ ,\\
\beta=&\frac{A_2^2(x,\alpha,\gamma)}{A_2^2(x,\alpha,\gamma)-A_1^2(x,\alpha,\gamma)}\ ,\\
\gamma=&\frac{A_2^3(x,\alpha,\beta)}{A_2^3(x,\alpha,\beta)-A_1^3(x,\alpha,\beta)}\ ,
\end{aligned}\tag{3.4}$$

and by virtue of the signs of Table 5, the map T is a continuous map from $[0,1]\times[0,1]\times[0,1]$ into itself. Therefore, there is a fixed point in the unit cube and a solution α,β,γ of (3.3).

Remark 3.1. Proving uniqueness appears to be considerably more complicated. Based upon genericity arguments, we do expect the solution to be isolated. This statement is easy to justify since the expressions in (3.4) represent three surfaces in the unit cube in $\mathbb{R}^3$ and, generically, we expect that if they intersect (and we know they do) they do so at isolated points.

4. Case of general co-dimension $p>3$

In this section we show how to generalize the previous construction of a Filippov vector field. That is, we obtain a solution to the nonlinear system arising from the specific form of the convex combination we seek on the intersection of p surfaces.

To be specific, the general form of the problem we seek to solve is the following.

We have p surfaces in $\mathbb{R}^n$, $\Sigma_1,\Sigma_2,\dots,\Sigma_p$, with $n\geq p+1$, and Σ is their intersection: $\Sigma=\bigcap_{i=1}^p\Sigma_i$. Each surface Σ_i is characterized as the zero set of a scalar function $h_i(x)$, $i=1,\dots,p$, and we assume to have well defined, smooth, and linearly independent gradients $\nabla h_i(x)\neq 0$, for $x\in\Sigma$. Let $n_1(x),n_2(x),\dots,n_p(x)$ be the unit normals. Locally, the intersection of these p surfaces divides the space $\mathbb{R}^n$ in 2^p regions with respective vector fields f_i, $i=1,\dots,2^p$, and Filippov's convexification is given in (1.4). To select a specific Filippov vector field, we reason as follows.

First, we label the 2^p vector fields f_i's in a similar way to what we did in Sections 2 and 3. Specifically, we will label f_1 the function in the region $R_1:=\{x:\ \ h_i(x)<0\ ,\ \ i=1,\dots,p\}$, f_2 the function in the region $R_2:=\{x:\ \ h_i(x)<0\ ,\ \ i=1,\dots,p-1,\ \ h_p(x)>0\}$, etc.. To have a systematic description, we adopt the following notation. Let $\mathbf{1}_k$ be the row vector of all 1's, of size 2^k $(k=1,\dots,p-1)$. For given value of p, define the $(p,2^p)$ sign matrix B inductively as:

$$B^{(1)}=\begin{bmatrix}-1 & 1\end{bmatrix}\ ,\quad B^{(k)}=\begin{bmatrix}-\mathbf{1}_{k-1} & \mathbf{1}_{k-1}\\ B^{(k-1)} & B^{(k-1)}\end{bmatrix}\ ,\ k=2,\dots,p\,,\quad B=B^{(p)}\,.\tag{4.1}$$

For $i = 1, \dots, 2^p$, each function f_i corresponds to the region $R_i := \{x : \ \mathrm{sign}(h_j(x)) = B_{j,i}, \ j = 1, \dots, p\}$. With this, we rewrite Filippov's convexification (1.4) in the form

$$x'(t) \in \sum_{i=1}^{2^p} \prod_{k=1}^{p} \frac{1 + B_{k,i}\sigma(h_k(x))}{2} f_i(x)\,, \tag{4.2}$$

where $\sigma(\cdot)$ is the usual multi-valued sign function of (1.15).

Similarly to what we have done in the previous sections, we seek single valued functions $s_k(x)$, $k = 1, \dots, p$, to replace the set valued functions $\sigma(h_k(x))$ for x on the intersection Σ, and the differential inclusion (4.2) with the differential equation on Σ:

$$x'(t) = \sum_{i=1}^{2^p} \prod_{k=1}^{p} \frac{1 + B_{k,i}s_k(x)}{2} f_i(x)\,. \tag{4.3}$$

In order to have a sliding solution, as usual we ask that the vector field in (4.3) be in the tangent plane. This requires solving the nonlinear system

$$\begin{cases} \displaystyle\sum_{i=1}^{2^p} \prod_{k=1}^{p} \frac{1 + B_{k,i}s_k(x)}{2} \big(n_1^T(x) f_i(x)\big) \ = \ 0 \ , \\ \qquad\qquad \cdots\cdots\cdots\cdots\cdots \\ \displaystyle\sum_{i=1}^{2^p} \prod_{k=1}^{p} \frac{1 + B_{k,i}s_k(x)}{2} \big(n_p^T(x) f_i(x)\big) \ = \ 0 \ , \end{cases} \tag{4.4}$$

for the p unknowns $s_1(x), \dots, s_p(x)$, at a given $x \in \Sigma$.

The next task is to show that this system is solvable, which we will do under conditions of **nodal attractivity** of Σ. To state these conditions, we use a notation similar to what we did in Sections 2 and 3. First, let

$$s_k(x) = 2\alpha_k(x) - 1 \ , \ \ \alpha_k \ : x \in \Sigma \ \to \ [0,1] \ , \quad k = 1, \dots, p\,,$$

and let

$$w^k = (w_i^k)_{i=1}^{2^p} \ , \ \ w_i^k = n_k^T f_i\,, \quad i = 1, \dots, 2^p\,, \ \ k = 1, \dots, p\,.$$

Thus, the nonlinear system (4.4) is rewritten as (omitting x for simplicity)

$$\sum_{i=1}^{2^p} \Big(\prod_{k=1}^{p} \frac{1 - B_{k,i} + 2B_{k,i}\alpha_k}{2}\Big) w_i^j \ = \ 0 \ , \quad j = 1, \dots, p\,, \tag{4.5}$$

or

$$\sum_{i=1}^{2^p} \Big[\Big(\frac{1 - B_{j,i}}{2} + B_{j,i}\alpha_j\Big) \prod_{k=1, k\neq j}^{p} \Big(\frac{1 - B_{k,i}}{2} + B_{k,i}\alpha_k\Big)\Big] w_i^j \ = \ 0 \ , \quad j = 1, \dots, p\,.$$

Now, for each $j = 1, \dots, p$, among the 2^p terms $B_{j,i}$ there are 2^{p-1} terms which are equal to -1, and 2^{p-1} terms which are equal to 1. Accordingly, we break the sum in (4.5) as

$$\begin{aligned} &\sum_{i:\ B_{j,i}=-1} \Big[(1 - \alpha_j) \prod_{k=1, k\neq j}^{p} \Big(\frac{1 - B_{k,i}}{2} + B_{k,i}\alpha_k\Big)\Big] w_i^j + \\ &\sum_{i:\ B_{j,i}=1} \Big[\alpha_j \prod_{k=1, k\neq j}^{p} \Big(\frac{1 - B_{k,i}}{2} + B_{k,i}\alpha_k\Big)\Big] w_i^j \ = \ 0 \ , \quad j = 1, \dots, p\,, \end{aligned} \tag{4.6}$$

and there are 2^{p-1} indices i in each sum.

With these preparations, we rewrite the nonlinear system (4.6) by explicitly expressing α_j from the j-th equation. For all $j = 1, \dots, p$, we can write:

$$\alpha_j = \frac{A_2^j(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_p)}{A_2^j(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_p) - A_1^j(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_p)} , \tag{4.7}$$

where

$$A_1^j = \sum_{i:\ B_{j,i}=1} \Big[\prod_{k=1, k\neq j}^{p} \big(\frac{1 - B_{k,i}}{2} + B_{k,i}\alpha_k\big) \Big] w_i^j$$

and

$$A_2^j = \sum_{i:\ B_{j,i}=-1} \Big[\prod_{k=1, k\neq j}^{p} \big(\frac{1 - B_{k,i}}{2} + B_{k,i}\alpha_k\big) \Big] w_i^j .$$

So, altogether, we have rewritten the nonlinear system (4.7) as the map $\alpha = T(\alpha)$, for $\alpha = (\alpha_1 \dots, \alpha_p)^T$.

At this point, we make the key observation that, in agreement with the choice we adopted for the sign matrix B, our assumption of nodal attractivity of the (intersection) surface Σ means the following assumption on the signs of the w^k 's:

$$\text{sign}(W^T) \ = \ -B , \qquad \text{where} \qquad W = \begin{bmatrix} w^1 & w^2 & \cdots & w^p \end{bmatrix} ;$$

in particular: $w_i^j > 0$ when $B_{j,i} = -1$ and $w_i^j < 0$ when $B_{j,i} = 1$. We further notice that the term $\prod_{k=1,k\neq j}^{p} (\frac{1 - B_{k,i}}{2} + B_{k,i}\alpha_k)$ is always the product of terms like α_k or $(1 - \alpha_k)$, and therefore it is always in $[0, 1]$ if each $\alpha_k \in [0, 1]$. As a consequence, we have that

$$A_1^j < 0 \qquad \text{and} \qquad A_2^j > 0 ,$$

for all $j = 1, \dots, p$, and all $\alpha_j \in [0, 1]$, $j = 1, \dots, p$. Therefore, the map $\alpha = T(\alpha)$, maps the hypercube $[0, 1]^p$ into itself (strictly), and thus it has a fixed point there. That is, there is a solution to the nonlinear problem (4.4) as we wanted to show.

Uniqueness is more elusive. Although extensive computational experiments, and several partial results, lead us to suspect that –under our nodal attractivity assumptions– the solution in the unit cube is unique, a complete proof is still lacking.

Remark 4.1. As already remarked in Section 1, an interesting task is to explore the connection of the Filippov vector field we selected in this work to the limiting behavior of some regularized vector field. For co-dimension $p > 1$, this study is being done in [15].

5. Some examples

Here we consider several examples, to highlight certain specific features of our construction and contrast it to the choice of vector field we made in [16]; henceforth, we will call f_{DL} the vector field choice we made in [16], to which we refer for details.

First, we consider a mechanical system where the physics of the problem itself renders a unique Filippov solution, regardless of the nominal co-dimension of the sliding surface. Then, we will consider examples in the style of the control problems from [33], to highlight that there may be ambiguities in the constructions of either Filippov and Utkin vector fields.

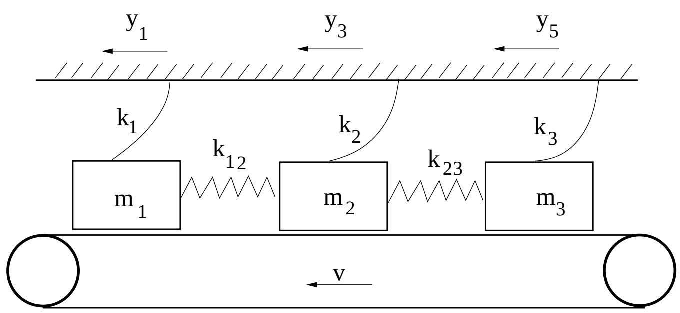

FIGURE 8. The stick-slip 3 blocks system.

Example 5.1. (The stick-slip 3 blocks system, see Galvanetto, [21, 22]). This is an example of the classic stick–slip class, in which all the known techniques give the same sliding vector field.

The general mechanical model under investigation is depicted in Figure 8. There are three blocks arranged along a line. Each block is connected to the adjacent blocks and to a fixed body by linear springs and is supported by a moving belt. The velocity of the belt is constant and is called the driving velocity $v > 0$. See [21, 22] for a detailed description of the model.

The model may be described by the following discontinuous differential system in $\mathbb{R}^6$

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= -\tfrac{1}{m_1}[k_1 y_1 + k_{12}(y_1 - y_3) + k_{13}(y_1 - y_5)] + \begin{cases} \frac{F_{s,1}}{1-\gamma(y_2-v)}, & \text{when} \quad y_2 < v \ , \\ -\frac{F_{s,1}}{1+\gamma(y_2-v)}, & \text{when} \quad y_2 > v \ , \end{cases} \\
y_3' &= y_4 \\
y_4' &= -\tfrac{1}{m_2}[k_2 y_3 + k_{12}(y_3 - y_1) + k_{23}(y_3 - y_5)] + \begin{cases} \frac{F_{s,2}}{1-\gamma(y_4-v)}, & \text{when} \quad y_4 < v \ , \\ -\frac{F_{s,2}}{1+\gamma(y_4-v)}, & \text{when} \quad y_4 > v \ , \end{cases} \\
y_5' &= y_6 \\
y_6' &= -\tfrac{1}{m_3}[k_3 y_5 + k_{13}(y_5 - y_1) + k_{23}(y_5 - y_3)] + \begin{cases} \frac{F_{s,3}}{1-\gamma(y_6-v)}, & \text{when} \quad y_6 < v \ , \\ -\frac{F_{s,3}}{1+\gamma(y_6-v)}, & \text{when} \quad y_6 > v \ . \end{cases}
\end{aligned}
\tag{5.1}
$$

where k_1, k_2, k_3 k_{12}, k_{13}, k_{23}, $F_{s,1}, F_{s,2}, F_{s,3}$, are suitable constants. Hence, by setting $h_1(y) = y_2 - v$, $h_2(y) = y_4 - v$, $h_3(y) = y_6 - v$, these three functions define three planes $\Sigma_i = \{y \in \mathbb{R}^6 | h_i(y) = 0\}$ for $i = 1, 2, 3$, which divide the space into 8 regions R_j with respective vector fields f_j, $j, \ldots, 8$.

Let us denote:

$$
\begin{aligned}
A(y_1, y_3, y_5) &= -\frac{1}{m_1}[k_1 y_1 + k_{12}(y_1 - y_3) + k_{13}(y_1 - y_5)] \ , \\
B(y_1, y_3, y_5) &= -\frac{1}{m_2}[k_2 y_3 + k_{12}(y_3 - y_1) + k_{23}(y_3 - y_5)] \ , \\
C(y_1, y_3, y_5) &= -\frac{1}{m_3}[k_3 y_5 + k_{13}(y_5 - y_1) + k_{23}(y_5 - y_3)] \ ,
\end{aligned}
$$

then the vector fields $f_i(y)$, for $i = 1, \dots, 8$, on the intersection $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$, (where $y_2 = y_4 = y_6 = v$) are given by:

(5.2)

$$f_1(y) = \begin{bmatrix} v \\ A + F_{s,1} \\ v \\ B + F_{s,2} \\ v \\ C + F_{s,3} \end{bmatrix}, \quad f_2(y) = \begin{bmatrix} v \\ A + F_{s,1} \\ v \\ B + F_{s,2} \\ v \\ C - F_{s,3} \end{bmatrix}, \quad f_3(y) = \begin{bmatrix} v \\ A + F_{s,1} \\ v \\ B - F_{s,2} \\ v \\ C + F_{s,3} \end{bmatrix}, \quad f_4(y) = \begin{bmatrix} v \\ A + F_{s,1} \\ v \\ B - F_{s,2} \\ v \\ C - F_{s,3} \end{bmatrix},$$

(5.3)

$$f_5(y) = \begin{bmatrix} v \\ A - F_{s,1} \\ v \\ B + F_{s,2} \\ v \\ C + F_{s,3} \end{bmatrix}, \quad f_6(y) = \begin{bmatrix} v \\ A - F_{s,1} \\ v \\ B + F_{s,2} \\ v \\ C - F_{s,3} \end{bmatrix}, \quad f_7(y) = \begin{bmatrix} v \\ A - F_{s,1} \\ v \\ B - F_{s,2} \\ v \\ C + F_{s,3} \end{bmatrix}, \quad f_8(y) = \begin{bmatrix} v \\ A - F_{s,1} \\ v \\ B - F_{s,2} \\ v \\ C - F_{s,3} \end{bmatrix}.$$

The unit normal vectors to the surfaces Σ_i for $i = 1, 2, 3$, are respectively $n_1 = [0, 1, 0, 0, 0, 0]^T$, $n_2 = [0, 0, 0, 1, 0, 0]^T$, $n_3 = [0, 0, 0, 0, 0, 1]^T$. Hence, it follows that:

$$\text{(5.4)} \quad \begin{cases} n_1^T f_i = [f_i]_2 = A + F_{s,1} & i \in \{1, 2, 3, 4\}\,;\; n_1^T f_i = [f_i]_2 = A - F_{s,1} \quad i \in \{5, 6, 7, 8\} \\ n_2^T f_i = [f_i]_4 = B + F_{s,2} & i \in \{1, 2, 5, 6\}\,;\; n_2^T f_i = [f_i]_4 = B - F_{s,2} \quad i \in \{3, 4, 7, 8\} \\ n_3^T f_i = [f_i]_6 = C + F_{s,3} & i \in \{1, 3, 5, 7\}\,;\; n_3^T f_i = [f_i]_6 = C - F_{s,3} \quad i \in \{2, 4, 8, 8\} \end{cases}$$

where $[f_i]_j$, for $i = 1, \dots, 8$, and $j = 1, \dots, 6$, denotes the $j-$th component of f_i.

Observe that on the intersection we must have $y_2 = y_4 = y_6 = v$, and therefore we anticipate that we must have $y_2' = y_4' = y_6' = 0$. Further, since $y_1' = y_2, y_3' = y_4, y_5' = y_6$, we know that –on physical grounds– the vector field on the intersection will be $[v, 0, v, 0, v, 0]^T$. For didactic purposes, we verify algebraically that this is indeed the case for all methods.

The sliding Filippov's vector field is given by $f_{\mathrm{F}} = \sum_{i=1}^{8} \lambda_i f_i$ where $\lambda_i \geq 0$, for $i = 1, \dots, 8$, and $\sum_{i=1}^{8} \lambda_i = 1$. The coefficients λ_i must be found by imposing the following orthogonality conditions:

$$\text{(5.5)} \quad \sum_{i=1}^{8} \lambda_i n_j^T f_i = 0, \quad j = 1, 2, 3.$$

By setting $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 - \alpha$, $\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 = 1 - \beta$, $\lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 = 1 - \gamma$, and by considering (5.4), then the linear system (5.5) becomes:

$$\text{(5.6)} \quad \begin{cases} A + F_{s,1}[(1-\alpha) - \alpha] = 0 & \Leftrightarrow \quad \alpha = \frac{F_{s,1} + A}{2F_{s,1}} \\ B + F_{s,2}[(1-\beta) - \beta] = 0 & \Leftrightarrow \quad \beta = \frac{F_{s,2} + B}{2F_{s,2}} \\ C + F_{s,3}[(1-\gamma) - \gamma] = 0 & \Leftrightarrow \quad \gamma = \frac{F_{s,3} + C}{2F_{s,3}}, \end{cases}$$

from which it follows that the sliding Filippov vector is given by $f_{\mathrm{F}} = [v, 0, v, 0, v, 0]^T$. In other words, there is no ambiguity in selecting a unique Filippov vector field. As a consequence, also the approach proposed in our present work must give the same sliding vector field. Let us verify this fact formally. We have the following vector field on the intersection:

$$f_S = (1-\alpha)(1-\beta)(1-\gamma)f_1 + (1-\alpha)(1-\beta)\gamma f_2 + (1-\alpha)\beta(1-\gamma)f_3 + (1-\alpha)\beta\gamma f_4 +$$

$$\text{(5.7)} \quad +\alpha(1-\beta)(1-\gamma)f_5 + \alpha(1-\beta)\gamma f_6 + \alpha\beta(1-\gamma)f_7 + \alpha\beta\gamma f_8\ .$$

Since this is a convex combination, from (5.2) it follows that $[f_S]_1 = [f_S]_3 = [f_S]_5 = v$. The coefficients α, β, γ are determined by imposing:

$$n_j^T f_S = 0, \quad j = 1, 2, 3, \tag{5.8}$$

which, by using (5.4), is just the linear system in (5.6). Then the sliding vector field (on the intersection $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$) obtained by this approach is indeed $f_S = [v, 0, v, 0, v, 0]^T$.

We next determine the sliding vector on the intersection $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ by employing the approach used in [16]. Set $N = [n_1, n_2, n_3]$, and form the projector on the intersection $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$:

$$\Pi = I - N(N^T N)^{-1} N^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The projections of the 8 vector fields in (5.2) on $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ are all the same, that is $\Pi f_i = [v, 0, v, 0, v, 0]^T$, for $i = 1, \dots, 8$. Since the sliding vector field $f_{\rm DL}$ on $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ is given by $f_{\rm DL} = \sum_{i=1}^8 \alpha_i \, \Pi f_i$, with coefficients $\alpha_i \geq 0$, $i = 1, \dots, 8$, and $\sum_{i=1}^8 \alpha_i = 1$, again we must have $f_{\rm DL} = [v, 0, v, 0, v, 0]^T$.

To recap, this generalized stick-slip problem leads to a unique vector on the tangent plane, and therefore a unique Filippov vector field, which coincides also with $f_{\rm DL}$. We further remark that the same situation will arise if we considered an arbitrary number of connected blocks, not just three.

The next three examples are variations on an example of Utkin from [33]. Before presenting the examples, we recall that Utkin considers discontinuous PWS systems of the form $x' = f(x, u)$, where u are control variables which change discontinuously as a solution trajectory reaches one or more discontinuity surfaces. By letting N to be the matrix collecting the normal(s) to the surface, Utkin searches for the sliding vector field in the form of $f_{\rm U} = f(x, u_{\rm eq})$, where $u_{\rm eq}$ must be found from solving the nonlinear system $N^T f(x, u_{\rm eq}) = 0$. If the controls appear nonlinearly, in general the classical constructions of Utkin and Filippov do not uniquely select a sliding vector field. To exemplify, we now present three examples with different features: In Example 5.2, from [33], Filippov construction is ambiguous, but Utkin's is not, in Example 5.3, Filippov construction is not ambiguous, but Utkin's is, and in Example 5.4 both Filippov and Utkin's constructions are ambiguous. In all cases, the vector field we set forth in this paper is well defined, and it further coincides with the one we introduced in [16] (but see Remark 5.5). The aforementioned ambiguities are reflected in different dynamical behaviors on the sliding surfaces. In particular, in the simple examples considered below, our selection choice always gives a (stable) equilibrium solution, whereas the other choices in general do not.

Example 5.2. [33, p. 64] This is an example where Filippov approach gives an ambiguous vector field, while Utkin's approach does not. Both the approach considered in this paper and the approach we introduced in [16] give the same vector field as with Utkin's approach.

Consider the system:

$$\begin{cases} x_1' = u_1 \\ x_2' = u_2 \\ x_3' = u_1 u_2 \end{cases} , \tag{5.9}$$

with the two discontinuous controls u_1 and u_2:

$$u_1 = \begin{cases} +1 & \text{when } x_1 < 0 \\ -1 & \text{when } x_1 > 0 \end{cases} , \qquad u_2 = \begin{cases} +1 & \text{when } x_2 < 0 \\ -1 & \text{when } x_2 > 0 \end{cases} .$$

So, we have two event functions $h_1(x) = x_1$, $h_2(x) = x_2$, two discontinuity planes $\Sigma_1 = \{(x_1, x_2, x_3) : x_1 = 0\}$ and $\Sigma_2 = \{(x_1, x_2, x_3) : x_2 = 0\}$ and four vector fields:

$$f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} . \tag{5.10}$$

The normal vectors to Σ_1 and Σ_2 are simply $n_1^T = [1, 0, 0]$ and $n_2^T = [0, 1, 0]$ from which it is trivial to verify that the attractivity conditions of Table 2 are satisfied.

Filippov sliding vector field $f_{\rm F} = \sum_{i=1}^4 \lambda_i f_i$ in (1.5) requires us to impose $n_1^T f_{\rm F} = 0$, $n_2^T f_{\rm F} = 0$, and this leads to the underdetermined system

$$\begin{cases} \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \ , \\ \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0 \ , \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \ . \end{cases} \tag{5.11}$$

The solution can be written as $\lambda_1 = \lambda_4$, $\lambda_2 = \lambda_3 = 1/2 - \lambda_1$ and λ_1 is an undefined value in $[0, 1/2]$, which leads to a family of vector fields on the intersection $\Sigma_1 \cap \Sigma_2$: $f_{\rm F} = [0, 0, -1 + 4\lambda_1]^T$.

Utkin's equivalent control approach requires $x_1' = u_{1,\rm eq} = 0$, $x_2' = u_{2,\rm eq} = 0$, and thus Utkin's sliding vector is $f_{\rm U} = [0, 0, 0]^T$.

Consider now the approach we used in [16]. Let $N = [n_1, n_2]$, and form the projector on the intersection $\Sigma_1 \cap \Sigma_2$:

$$\Pi = I - N(N^T N)^{-1} N^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Take the projections of the four vector fields in (5.10) on $\Sigma_1 \cap \Sigma_2$:

$$\mathtt{v}_1 = \Pi f_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathtt{v}_2 = \Pi f_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathtt{v}_3 = \Pi f_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathtt{v}_4 = \Pi f_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{5.12}$$

The approach in [16] suggests to take the sliding vector on the intersection as a convex linear combination of these four projected vectors, that is

$$f_{\rm DL} = \sum_{i=1}^4 \alpha_i \ \mathtt{v}_i = \begin{bmatrix} 0 \\ 0 \\ \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 \end{bmatrix} .$$

To find the coefficients α_i's, in [16] we form the vectors w_i's of Section 2:
(5.13)

$$w_1 = \begin{bmatrix} n_1^T f_1 \\ n_2^T f_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} n_1^T f_2 \\ n_2^T f_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} n_1^T f_3 \\ n_2^T f_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad w_4 = \begin{bmatrix} n_1^T f_4 \\ n_2^T f_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} ,$$

and select

$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{5.14}$$

Then we take

$$\mu_i = \frac{\left[\prod_{j=1, j\neq i}^{4} a_j^T w_j\right]}{\left[\prod_{j=1, j\neq i}^{4} a_j^T w_j\right] - a_i^T w_i}\ , \qquad \alpha_i = \frac{\mu_i}{\sum_{i=1}^{4} \mu_i}, \quad \text{for } i = 1, \dots, 4\ . \tag{5.15}$$

From (5.13) and (5.14) it follows that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 4/5$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/4$, and the sliding vector is given by $f_{\mathrm{DL}} = \sum_{i=1}^{4} \alpha_i\ \mathtt{v}_i = [0,0,0]^T$.

The choice we presented in the present paper, instead, seeks the sliding vector field in the form

$$f_{\mathrm{S}} = (1-\alpha)(1-\beta)f_1 + (1-\alpha)\beta f_2 + \alpha(1-\beta)f_3 + \alpha\beta f_4\ , \tag{5.16}$$

where α and β are determined from solving $n_1^T f_{\mathrm{S}} = 0$, $n_2^T f_{\mathrm{S}} = 0$. In this case, this gives

$$\begin{cases} (1-\alpha)(1-\beta) + (1-\alpha)\beta - \alpha(1-\beta) - \alpha\beta \ = 0 \\ (1-\alpha)(1-\beta) - (1-\alpha)\beta + \alpha(1-\beta) - \alpha\beta \ = 0\ , \end{cases} \tag{5.17}$$

the solution of which is given by $\alpha = \beta = 1/2$ and so once more $f_{\mathrm{S}} = [0,0,0]^T$.

Example 5.3. This is an example where Filippov approach gives a well defined vector field, which is also recovered by the approach we considered in this paper and the approach we introduced in [16], while Utkin's approach gives an ambiguous vector field.

Let

$$\begin{cases} x_1' = u_1 \\ x_2' = u_2 \\ x_3' = (u_1)^2 u_2 \end{cases}\ , \tag{5.18}$$

with the two discontinuous controls u_1 and u_2:

$$u_1 = \begin{cases} +1 & \text{when } x_1 < 0 \\ -1 & \text{when } x_1 > 0 \end{cases}\ , \qquad u_2 = \begin{cases} +1 & \text{when } x_3 < 0 \\ -1 & \text{when } x_3 > 0 \end{cases}\ ,$$

which define two event functions $h_1(x) = x_1$, $h_2(x) = x_3$, two discontinuity planes $\Sigma_1 = \{(x_1, x_2, x_3) :\ x_1 = 0\}$ and $\Sigma_2 = \{(x_1, x_2, x_3) :\ x_3 = 0\}$ and four vector fields:

$$f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\ , \tag{5.19}$$

respectively in the regions R_1, R_2, R_3 and R_4.

The normal vectors to Σ_1 and Σ_2 are $n_1^T = [1,0,0]$ and $n_2^T = [0,0,1]$ from which it is easy to verify that the attractivity conditions of Table 2 are satisfied. Just like in Example 5.2, Filippov's approach leads to the ambiguous choice $\lambda_1 = \lambda_4$, $\lambda_2 = \lambda_3 = 1/2 - \lambda_1$. Nevertheless, this now gives a unique Filippov sliding vector: $f_{\mathrm{F}} = [0,0,0]^T$.

On the other hand, Utkin's equivalent control approach requires $x_1' = u_{1,\mathrm{eq}} = 0$, $x_3' = (u_{1,\mathrm{eq}})^2 u_{2,\mathrm{eq}} = 0$, so x_3' could be zero for any value of $u_{2,\mathrm{eq}}$ and Utkin's vector is ambiguous: $f_{\mathrm{U}} = [0, u_2, 0]$.

If we consider the approach of [16], similar computations to those we performed in Example 5.2 give the sliding vector $f_{\mathrm{DL}} = \sum_{i=1}^{4} \alpha_i\ \mathtt{v}_i = [0,0,0]^T$.

Finally, the approach considered in the present paper –selecting a Filippov vector field– also renders $f_{\mathrm{S}} = [0,0,0]^T$.

Example 5.4. In this last example, both Filippov and Utkin approaches give an ambiguous vector field, while the approach considered in this paper and the approach introduced in [16] give a well defined vector field. Consider the following systems in $\mathbb{R}^4$:

$$\begin{cases} x_1' = u_1 \\ x_2' = (u_1)^2 u_2 \\ x_3' = (u_1)^2 u_3 \\ x_4' = u_2 u_3 \end{cases} , \tag{5.20}$$

with the three discontinuous controls u_1, u_2 and u_3:

$$u_1 = \begin{cases} +1 & \text{when } x_1 < 0 \\ -1 & \text{when } x_1 > 0 \end{cases} , \qquad u_2 = \begin{cases} +1 & \text{when } x_2 < 0 \\ -1 & \text{when } x_2 > 0 \end{cases} , \qquad u_3 = \begin{cases} +1 & \text{when } x_3 < 0 \\ -1 & \text{when } x_3 > 0 \end{cases} ,$$

which define three discontinuity planes $\Sigma_1 = \{(x_1, x_2, x_3) : x_1 = 0\}$ and $\Sigma_2 = \{(x_1, x_2, x_3) : x_2 = 0\}$, $\Sigma_3 = \{(x_1, x_2, x_3) : x_3 = 0\}$ and eight vector fields:

$$f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} , \tag{5.21}$$

$$f_5 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad f_6 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad f_7 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad f_8 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} , \tag{5.22}$$

respectively in the regions R_i, for $i = 1 \dots, 8$. On Σ, the normal vectors n_i, $i = 1, 2, 3$, are just the first three unit vectors, from which it is easy to verify that the attractivity conditions of Table 5 are satisfied.

Filippov sliding vector field (1.5) has the form $f_{\mathrm{F}} = \sum_{i=1}^{8} \lambda_i \, f_i$, where the coefficients $\lambda_i \geq 0$, for $i = 1, \dots, 8$, and $\sum_{i=1}^{8} \lambda_i = 1$. By imposing the orthogonality conditions ($n_1^T f_{\mathrm{F}} = 0$, $n_2^T f_{\mathrm{F}} = 0$, $n_3^T f_{\mathrm{F}} = 0$), Filippov's approach leads to solve the following linear system

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - (\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) = 0 \ , \\ \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 - \lambda_7 - \lambda_8 = 0 \ , \\ \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8 = 0 \ , \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1 \ , \end{cases} \tag{5.23}$$

which implies that the Filippov sliding vector is $f_{\mathrm{F}} = [0, 0, 0, 4(\lambda_4 + \lambda_8) - 1]^T$, and thus f_{F} is undetermined.

Utkin's approach requires $x_1' = u_{1,\mathrm{eq}} = 0$, $x_2' = (u_{1,\mathrm{eq}})^2 u_{2,\mathrm{eq}} = 0$, $x_3' = (u_{1,\mathrm{eq}})^2 u_{3,\mathrm{eq}} = 0$, thus x_2' or x_3' could be zero for any value of $u_{2,\mathrm{eq}}$, $u_{3,\mathrm{eq}}$ and Utkin's sliding vector field is $f_{\mathrm{U}} = [0, 0, 0, u_{2,\mathrm{eq}} \, u_{3,\mathrm{eq}}]$ and again the last component is undetermined.

Next, consider our approach in [16]. This requires us to seek the vector on Σ as convex combination of the projections of the vector fields f_i's on the tangent plane. In this case, this means that we have

$$f_{\mathrm{DL}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 \end{bmatrix} ,$$

where the nonnegative coefficients α_i, for $i = 1, \dots, 8$, have to be found in the following way. Consider the vectors w_i, $i = 1, \dots, 8$, from Section 3, that is:

$$w_i = \begin{bmatrix} n_1^T f_i \\ n_2^T f_i \\ n_3^T f_i \end{bmatrix} \ , \ \ i = 1, \dots, 8 \ ,$$

and select

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad a_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \ ,$$

$$a_5 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad a_6 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad a_7 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad a_8 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ ,$$

such that $a_1^T w_1 > 0$, and $a_i^T w_i < 0$, for $i = 2, \dots, 8$. Then we will take:

$$\mu_i = \frac{\left[\prod_{j=1, j\neq i}^8 a_j^T w_j\right]}{\left[\prod_{j=1, j\neq i}^8 a_j^T w_j\right] - a_i^T w_i} \ , \qquad \alpha_i = \frac{\mu_i}{\sum_{i=1}^4 \mu_i}, \quad \text{for } i = 1, \dots, 8 \ . \tag{5.24}$$

In this specific case, we get $\mu_i = 7/8$, and $\alpha_i = 1/8$, for $i = 1, \dots, 8$, and the sliding vector will be given by $f_{\mathrm{DL}} = [0, 0, 0, 0]^T$.

Finally, consider the approach presented in the present paper. The sliding vector is of the form:

$$f_{\mathrm{S}} = (1-\alpha)(1-\beta)(1-\gamma) f_1 + (1-\alpha)(1-\beta)\gamma f_2 + (1-\alpha)\beta(1-\gamma) f_3 + (1-\alpha)\beta\gamma f_4 +$$
$$+ \alpha(1-\beta)(1-\gamma) f_5 + \alpha(1-\beta)\gamma f_6 + \alpha\beta(1-\gamma) f_7 + \alpha\beta\gamma f_8 \ ,$$

where α, β and γ are determined by imposing the orthogonality conditions $n_1^T f_{\mathrm{S}} = 0$, $n_2^T f_{\mathrm{S}} = 0$, $n_3^T f_{\mathrm{S}} = 0$, which gives:

$$\begin{cases} (1-\alpha)(1-\beta)(1-\gamma) + (1-\alpha)(1-\beta)\gamma + (1-\alpha)\beta(1-\gamma) + (1-\alpha)\beta\gamma - \\ \quad -\alpha(1-\beta)(1-\gamma) - \alpha(1-\beta)\gamma - \alpha\beta(1-\gamma) - \alpha\beta\gamma = 0 \\ \\ (1-\alpha)(1-\beta)(1-\gamma) + (1-\alpha)(1-\beta)\gamma - (1-\alpha)\beta(1-\gamma) - (1-\alpha)\beta\gamma + \\ \quad +\alpha(1-\beta)(1-\gamma) + \alpha(1-\beta)\gamma - \alpha\beta(1-\gamma) - \alpha\beta\gamma = 0 \\ \\ (1-\alpha)(1-\beta)(1-\gamma) - (1-\alpha)(1-\beta)\gamma + (1-\alpha)\beta(1-\gamma) - (1-\alpha)\beta\gamma + \\ \quad +\alpha(1-\beta)(1-\gamma) - \alpha(1-\beta)\gamma + \alpha\beta(1-\gamma) - \alpha\beta\gamma = 0 \ . \end{cases}$$

This is a trivial linear system in disguise, with the unique solution $\alpha = \beta = \gamma = 1/2$. Therefore, once more we have $f_{\mathrm{S}} = [0, 0, 0, 0]^T$.

Remark 5.5. The fact that for Examples 5.2, 5.3 and 5.4 the vector fields f_{S} and f_{DL} coincide is due to the symmetries present in the problems. Likewise, also the fact that the system to be solved for f_{S} (to find α, β, γ) is linear is a consequence of the symmetries in the problem. In general, neither of these facts is true: f_{S} and f_{DL} are generally different, and to find $\alpha, \beta, \gamma, \dots$ to form f_{S} one generally needs to solve a nonlinear system.

6. Conclusions

In this paper we have considered piecewise smooth dynamical systems, and in particular how to define a Filippov sliding vector field on a surface of co-dimension $p \geq 2$. In this case, it is well understood that –in general– there is no uniquely defined Filippov vector field. Yet, it is desirable to arrive at a non ambiguous definition of sliding motion, in order to perform numerical simulation of the system and to obtain information on the system's dynamics.

For this reason, we proposed a selection of a Filippov sliding vector field in the case of co-dimension $p \geq 2$, for the case when the surface is attractive. In the co-dimension $p = 2$ case, we distinguished between three different types of attractive surface (nodal, and spiral-like) and in all cases proved existence, uniqueness, and smoothness of the resulting Filippov sliding field. In the case of co-dimension $p \geq 3$, and nodal attractivity assumption, we proved existence. We further illustrated our construction, and compared it with alternatives, on several examples.

References

[1] V. Acary and B. Brogliato. *Numerical Methods for Nonsmooth Dynamical Systems. Applications in Mechanics and Electronics*. Lecture Notes In Applied and Computational Mechanics. Springer-Verlag, Berlin, 2008.
[2] J.C. Alexander and T. Seidman. Sliding modes in intersecting switching surfaces, I: Blending. *Houston J. Math.*, 24:545–569, 1998.
[3] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, Berlin, 1984.
[4] G. Bartolini and A. Ferrara. A Switching Controller for Systems with Hard Uncertainties. *IEEE Transactions on Circuits and Systems-I:Fundamental Theory and Applications*, 50:984–990, 2003.
[5] G. Bartolini, A. Pisano, E. Punta, and E. Usai. A survey of applications of second-order sliding mode control mechanical systems. *Int. J. Control*, 76:875–892, 2003.
[6] G. Batt, D. Rogers, H. de Jong, and J. Geiselmann. Qualitative analysis and verification of hybrid models of genetics regulatory networks: Nutritional stress response in escherichia coli. In M. Morari and L. Thiele, editors, *HSCC, LNCS*, pages 134–150. Springer-Verlag, 2005.
[7] I. Boiko. *Discontinuous Control Systems*. Birkhauser, Boston, 2009.
[8] B. Brogliato. *Implicit Euler Numerical Simulation of Sliding Mode Systems*. Conference SDS 2010, `http://www.dm.uniba.it/ delbuono/sds10/LectureBrogliatoSDS2010.pdf`.
[9] R. Casey, H. de Jong, and Gouze J.-L. Piecewise-linear models of genetics regulatory networks: Equilibria and their stability. *J.Math. Biol.*, 52:27–56, 2006.
[10] M.L. Corradini and G. Orlando. Linear unstable plants with saturating actuators: robust stabilization by a time varying sliding surface. *Automatica*, 43:88–94, 2007.
[11] J. Davila, L. Fridman, and Levant A. Second order sliding mode observers for mechanical systems. *IEEE Transactions on Automatic Control*, 50:1785–1789, 2005.
[12] H. de Jong, J.-L. Gouze, C. Hernandez, M. Page, T. Sari, and J. Geiselmann. Qualitative simulation of genetic regulatory networks using piecewise-linear models. *Bulletin of Mathematical Biology*, 66:301–340, 2004.
[13] H. de Jong, Gouze J.-L., C. Hernandez, T. Sari, and J. Geiselmann. Dealing with discontinuities in the qualitative simulation of genetic regulatory network. In F. van Harmelen, editor, *Proceedings of the 15th European Conference on Artificial Intelligence*, pages 725–730. IOS Press, Amsterdam, 2002.
[14] M. di Bernardo, C.J. Budd, A.R. Champneys, and P. Kowalczyk. *Piecewise-smooth Dynamical Systems. Theory and Applications*. Applied Mathematical Sciences 163. Springer-Verlag, Berlin, 2008.
[15] L. Dieci and N. Guglielmi. Regularization methods for discontinuous differential equations. *In preparation*, 2009.
[16] L. Dieci and L. Lopez. Sliding motion in Filippov differential systems: Theoretical results and a computational approach. *SIAM J. Numer. Anal.*, 47-3:2023–2051, 2009.
[17] C. Edwards and S.K. Spurgeron. *Sliding Mode Control: Theory and Applications*. Taylor and Francis, New York, 1998.
[18] A.F. Filippov. *Differential Equations with Discontinuous Right-Hand Sides*. Mathematics and Its Applications, Kluwer Academic, Dordrecht, 1988.
[19] L. Fridman and A. Levant. Higher order sliding modes. In J.P. Barbot and W. Perruguetti, editors, *Sliding Mode Control in Engineering*, pages 263–293. Marcel Dekker, 2002.
[20] G. Fusco and N. Guglielmi. A regularization for discontinuous differential equations with application to state-dependent delay differential equations of neutral type. *Preprint*, 2008.

[21] U. Galvanetto. Computational techniques for nonlinear dynamics in multiple frictions oscillators. *Comput. Methods Appl. Mech. Engrg.*, 163:373–382, 1998.
[22] U. Galvanetto. Non-linear dynamics of multiple frictions oscillators. *Comput. Methods Appl. Mech. Eng.*, 178:291–306, 1999.
[23] J.-L. Gouze and T. Sari. A class of piecewise linear differential equations arsing in biological models. *Dynamical Systems*, 17:299–319, 2002.
[24] K.H. Johansson, A. Rantzer, and K.J. Astrom. Fast swiyches in relay feedback systems. *Automatica*, 35:539–552, 1999.
[25] R.I. Leine. Bifurcations of equilibria in mechanical systems. *Physica D*, 223:121–137, 2006.
[26] R.I Leine and D.H. van Campen. Discontinuous fold bifurcations in non-smooth dynamical systems. *Archive of Applied Mechanics*, 72:138–146, 2002.
[27] R.I Leine, D.H. van Campen, and B.L. van de Vrande. Bifurcations in nonlinear discontinuous systems. *Nonlinear Dynamics*, 23:105–164, 2000.
[28] J. Llibre, P.R. da Silva, and M.A. Teixeira. Study of singularities in nonsmooth dynamical systems via singular perturbation. *SIAM J. Appl. Dyn. Syst.*, 8:508–526, 2009.
[29] J. Llibre, P. R. Silva, and M. A. Teixeira. Regularization of discontinuous vector fields via singular perturbation. *J. Dynam. Differential Equations*, 19:309–331, 2007.
[30] J. Sotomayor and M.A. Teixeira. Regularization of discontinuous vector fields. In *International Conference on Differential Equations*, pages 207–223, 1996.
[31] D.E. Stewart. A high accuracy method for solving ODEs with discontinuity right-hand side. *Numer. Math.*, 58:299–328, 1990.
[32] D.E. Stewart. Rigid-body dynamics with friction and impact. *SIAM Review*, 42:3–39, 2000.
[33] V.I. Utkin. *Sliding Modes and Their Application in Variable Structure Systems.* MIR Publisher, Moskow, 1978.
[34] V.I. Utkin. *Sliding Mode in Control and Optimization*. Springer, Berlin, 1992.
[35] V.I. Utkin and H.-C. Chang. Sliding mode control on electro-mechanical systems. *Mathematical Problems in Engineering*, 8:451–473, 2002.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 U.S.A.
E-mail address: dieci@math.gatech.edu

Dip. di Matematica, Universitá di Bari, Via Orabona 4, 70125 Italy
E-mail address: lopezl@dm.uniba.it