LYAPUNOV EXPONENTS OF SYSTEMS EVOLVING ON QUADRATIC GROUPS

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Abstract. In this paper we show some symmetry properties of Lyapunov exponents of a dynamical system when the linearized problem evolves on a quadratic group, $X^THX = H$, with H orthogonal. It is well understood that in this case the exponents are symmetric with respect to the origin. Here, we give lower bounds on the number of Lyapunov exponents which are 0, and show that some Lyapunov exponents may have even multiplicity.

1. Introduction

Lyapunov exponents are a common tool to explore stability properties of dynamical systems; e.g., see the collection of works in [3, 4, 15] and the many references there. Given the n-dimensional system of differential equations defined for $t \geq 0$:

$$\dot{x} = f(x)\,, \quad x(0) = x_0\,, \tag{1.1}$$

the Lyapunov exponents are a characterization of the asymptotic properties of the solution $\phi^t x_0$ via analysis of the linearized problem: $dX/dt = f_x(\phi^t x_0)X$. More generally, we may consider the linear time varying system

$$\dot{x} = A(t)x\,, \quad A : \mathbb{R}^+ \to \mathbb{R}^{n\times n}\,. \tag{1.2}$$

With Φ we will indicate the *principal matrix solution* of (1.2), that is $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$, and with X any other fundamental matrix solution (that is, $X(t) = \Phi(t)X(0)$, $X(0)$ invertible). We assume that A is bounded and continuous.

Formally, the Lyapunov exponents associated to (1.2) may be defined as follows (e.g., see [1, 5] and cfr. with [14]). Let X be a fundamental matrix solution of (1.2), and let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$. Define the numbers $\lambda_i(X)$, $i = 1, \ldots, n$, as (in this paper, the norm is always the 2-norm)

$$\lambda_i(X) = \limsup_{t\to\infty} \frac{1}{t} \log \|X(t)e_i\|\,. \tag{1.3}$$

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When the sum of the $\lambda_i(X)$ is minimized over all initial conditions $X(0)$, the corresponding fundamental solution X is called *normal* and the numbers $\lambda_i(X)$, hereafter simply λ_i, $i = 1, \ldots, n$, are called (upper) *Lyapunov exponents* of the system. In general, see [1], the Lyapunov exponents satisfy

$$\sum_{i=1}^{n} \lambda_i \geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{trace}\big(A(s)\big) ds \,. \tag{1.4}$$

The normal fundamental matrix solution X, or just the system (1.2), is said to be *regular* if the time average of the trace in (1.4) has a finite limit and equality holds in (1.4). If X is regular, then the lim sups in (1.3) can be replaced by ordinary limits. Suppose that (1.2) is regular. Clearly there are at most n distinct Lyapunov exponents. We will call *Lyapunov spectrum* the collection of all Lyapunov exponents of the system, counted with their multiplicity, and indicate it with $\operatorname{Sp}(X)$.

It is well known that $\operatorname{Sp}(X)$ is unchanged under an orthogonal (time varying) transformation of X. That is, if $R = Q^T X$, Q an orthogonal function, then $\operatorname{Sp}(R) = \operatorname{Sp}(X)$. This fact is often used in computational works (e.g., see [5, 6]), whereby the orthogonal change of variable is used to triangularize X and thus one brings the coefficient matrix A in (1.2) to upper triangular form, say B. Then, see [1], regularity implies that the Lyapunov exponents are given by

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t B_{ii}(s) ds \,, \quad i = 1, \ldots, n \,. \tag{1.5}$$

To infer regularity of a given particular system is not easy. It is therefore important that regularity is a prevalent condition in a certain measure theoretic sense. Furthermore, since (1.2) typically arises from linearization of (1.1), the dependency of $\operatorname{Sp}(X)$ on the initial condition x_0 of (1.1) must also be assessed. These issues are at the heart of the theory of Oseledec. We refer to [5, 8, 12, 14] for details, here we highlight only some of the points from these works which we will use.

Suppose that ϕ^t, the flow of (1.1), is a flow on a smooth compact manifold M and let μ be an *invariant probability measure* on M (that is, $\mu(\phi^t A) = \mu(A)$ for all Borel sets A in M). The invariant measure μ is called *ergodic* if every set invariant under ϕ^t has measure 0 or 1. Let Φ_{x_0} be the principal matrix solution associated to the linearization of (1.1) along $\phi^t x_0$. We will write $\operatorname{Sp}\big(\Phi_{x_0}\big)$ for the Lyapunov spectrum (since it generally depends on x_0).

Theorem 1.1. *Under the above assumptions, there is a subset M_0 of M, invariant under ϕ^t, and of measure 1, such that for any $x_0 \in M_0$ the following hold.*

(i) Φ_{x_0} *is regular.*

(ii) *The following limit exists*[1]

$$\lim_{t\to\infty} \frac{1}{t} \log\big(\Phi_{x_0}^T(t)\Phi_{x_0}(t)\big)^{1/2} . \tag{1.6}$$

(iii) $\mathrm{Sp}\big(\Phi_{x_0}\big)$ *is given by the eigenvalues of the symmetric matrix defined by* (1.6)*.*

(iv) *If* μ *is ergodic, then* $\mathrm{Sp}\big(\Phi_{x_0}\big)$ *is independent of* $x_0 \in M_0$*.*

From (1.6), we see that $\mathrm{Sp}\big(\Phi_{x_0}\big)$ is given by the limits of the time averages of the logarithms of the singular values of the principal matrix solution $\Phi_{x_0}(t)$. We refer to [9, 10] for numerical approximation of $\mathrm{Sp}\big(\Phi_{x_0}\big)$ exploiting this point of view. But, regardless of whether one adopts (1.5) or (1.6) as basis of an algorithm to approximate the Lyapunov exponents, it must be appreciated that either one of (1.5) or (1.6) can be specialized to target the p most dominant Lyapunov exponents, for example all the positive Lyapunov exponents of a system[2]. This is convenient, since one may know before hand that the Lyapunov spectrum enjoys some symmetries. Unarguably, the most important symmetry of the spectrum is the one with respect to the origin. This property is well known in the symplectic case (see [5, 8, 13]). In this work, we give some results on symmetries of Lyapunov exponents associated to fundamental matrix solutions evolving on other quadratic groups, namely for which $X^T(t)HX(t) = H$, with $H^TH = I$, for all t. To be precise, in this case, we will be able to give lower bounds on the number of singular values of X which are identically 1 for all t, by looking at the distribution of eigenvalues of the matrix H defining the quadratic group. We will further give some bounds on the number of singular values of X which have even multiplicity. These facts, coupled with Theorem 1.1, will translate into bounds on the Lyapunov exponents of $\mathrm{Sp}\big(\Phi_{x_0}\big)$. As a result, one may end up having to approximate only a few Lyapunov exponents in order to recover the entire Lyapunov spectrum. In particular, our results will apply to the case of the Lorentz and Minkowski groups. Maxwell's equations are the most famous example of a system satisfying invariance under the Lorentz group, and in this case only one Lyapunov exponent will need to be approximated; for this, and other examples of systems invariant under the Lorentz and Minkowski groups, see [2].

2. How many Lyapunov exponents are zero?

The following result is essentially given by Gupalo et al. in [11].

Theorem 2.1. *Let* X *be a fundamental matrix solution of* (1.2)*, and suppose that, for all* t*,* $X(t)$ *verifies*

$$(a) \quad X^T(t)HX(t) = H \ , \qquad and \qquad (b) \quad X(t)HX^T(t) = H \ , \tag{2.1}$$

[1]for all t, $\log\big(\Phi_{x_0}^T(t)\Phi_{x_0}(t)\big)^{1/2}$ is the unique symmetric logarithm of the unique symmetric positive definite square root

[2]of relevance to approximate the *entropy*, see [8]

where $H \in \mathbb{R}^{n\times n}$ is nonsingular. Then the function A in (1.2) satisfies for all t

$$(2.2) \qquad (a) \quad A^T(t)H + HA(t) = 0\ , \qquad and \qquad (b) \quad A(t)H + HA^T(t) = 0\,.$$

Further, the logarithms of the singular values of $X(t)$ are symmetric with respect to the origin, for all t. Finally, under the assumptions and with the notation of Theorem 1.1, i.e., if Φ_{x_0}, $x_0 \in M_0$, satisfies (2.1), then

$$(2.3) \qquad \mathrm{Sp}\big(\Phi_{x_0}\big) \qquad \text{is symmetric with respect to the origin.}$$

In this paper, we are interested in exploring further symmetries of Lyapunov exponents. From (2.3) in Theorem 2.1, if the dimension n is an odd number, then obviously there must be at least one Lyapunov exponent equal to 0. But, in general, can we anticipate how many Lyapunov exponents are guaranteed to be 0?

To make some progress, we will assume that H in Theorem 2.1 is orthogonal:

$$(2.4) \qquad X^T(t)HX(t) \ = \ H\ , \quad \text{for all } t\ , \quad H^TH = HH^T = I\,,$$

that is a fundamental matrix solution X evolves on the quadratic group defined by the orthogonal matrix H. In the case of (2.4), either one of (a) or (b) in (2.1) and (2.2) is redundant. To witness, from (2.1)-(a) we have $X^T(t)HX(t) = H \Leftrightarrow H^TX^T(t)H = X^{-1}(t) \Leftrightarrow X(t)H^TX^T(t) = H^T \Leftrightarrow X(t)HX^T(t) = H$, and similarly for (2.2).

With this special choice of H orthogonal, we will next show some properties of the singular values of X. These properties, coupled with (1.6), will then be used to obtain bounds on the number of Lyapunov exponents which are zero, and will further tell if some of them have even multiplicity.

Example 2.2. Naturally, the orthogonal group is included in (2.4) if $H = I_n$; in this case, all Lyapunov exponents are 0. Further, the symplectic group is also included if $H = J$ with

$$(2.5) \qquad J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

In this case, a priori one should not expect any of the Lyapunov exponents to be 0. Included in (2.4) is also the Minkowski group (i.e., the *"relativity"* group) where $H = D$ with

$$(2.6) \qquad D = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}\ ,$$

and $n_1 + n_2 = n$. The particular case $n_1 = 3$ and $n_2 = 1$ is the Lorentz group.

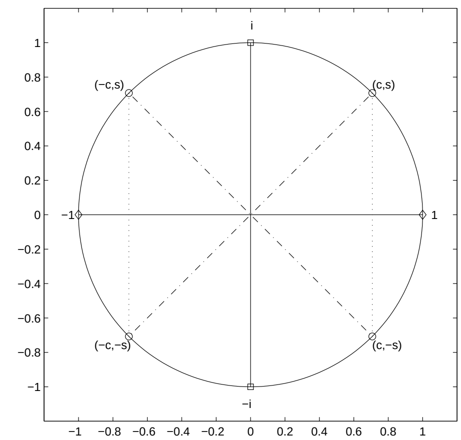
Before proceeding, let us simplify the problem. Let U be an orthogonal matrix giving the real Schur form of H, grouping the eigenvalues of H on the unit circle as follows:

$$K := U^T H U = \begin{bmatrix} D & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & J \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix} & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \end{bmatrix}, \tag{2.7}$$

where C comprises the eigenvalues of H different from ± 1 and $\pm i$:

$$\begin{aligned} & C = \operatorname{diag}(C_1, \dots, C_p) \ , \ \ C_j = \begin{bmatrix} Q_j \otimes I_{n_1(j)} & 0 \\ 0 & -Q_j \otimes I_{n_2(j)} \end{bmatrix} , \\ & Q_j = \begin{bmatrix} c_j & s_j \\ -s_j & c_j \end{bmatrix} \ , \ \ c_j^2 + s_j^2 = 1 \ , \ c_j \neq 0 \ , \ s_j \neq 0 \ , \ j = 1, \dots, p \, . \end{aligned} \tag{2.8}$$

In other words, we have blocked the eigenvalues of H grouping together the eigenvalues $\cos(\phi_j) \pm i \sin(\phi_j)$ and those out of phase by π: $\cos(\phi_j + \pi) \pm i \sin(\phi_j + \pi)$, and we have ordered them so that the angles are increasing from 0 to $\pi/2$; see the figure on the right. Naturally, not for every complex conjugate pair of eigenvalues, $e^{\pm i\phi}$, there need to be a complex conjugate pair out of phase by π with it, or viceversa. That is, in (2.8), $n_1(j)$ or $n_2(j)$ may be 0.



Now, if X is a fundamental matrix solution of (1.2) satisfying (2.4), then the matrix function $R = U^T X U$ satisfies

$$R^T(t) K R(t) \ = \ K \ , \quad \text{for all } t \, . \tag{2.9}$$

with K as in (2.7). Since the singular values of X and R are the same, we can assume to have the simplified form of orthogonal matrices as in (2.7). In this case, we can simplify the form of R satisfying (2.9).

Lemma 2.3. *Let $R \in \mathbb{R}^{n \times n}$ be any matrix satisfying $R^T K R = K$, with K given in* (2.7)*. Then, R has the block structure*

$$R = \begin{bmatrix} W & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & S \end{bmatrix} , \tag{2.10}$$

where the partitioning is that inherited by the form of K.

Proof. Write R in block form: $R = \begin{bmatrix} R_{11} & [R_{12} \ R_{13}] \\ \begin{bmatrix} R_{21} \\ R_{31} \end{bmatrix} & \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix} \end{bmatrix}$. Now, use the relations $R^T K R = K$ and $R^T K^T R = K^T$. In particular, from the respective $(2,2)$ blocks, we have

$$\begin{bmatrix} R_{12}^T \\ R_{13}^T \end{bmatrix} D \begin{bmatrix} R_{12} & R_{13} \end{bmatrix} + \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} C & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & J \end{bmatrix}$$

$$\begin{bmatrix} R_{12}^T \\ R_{13}^T \end{bmatrix} D \begin{bmatrix} R_{12} & R_{13} \end{bmatrix} + \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} C^T & 0 \\ 0 & -J \end{bmatrix} \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} C^T & 0 \\ 0 & -J \end{bmatrix},$$

from which

$$\begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} (C-C^T)/2 & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} (C-C^T)/2 & 0 \\ 0 & J \end{bmatrix}.$$

Therefore, $\begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}$ must be invertible. Now, from the $(2,1)$ blocks, we get

$$\begin{aligned} &\begin{bmatrix} R_{12}^T \\ R_{13}^T \end{bmatrix} DR_{11} + \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} C & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} R_{21} \\ R_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\begin{bmatrix} R_{12}^T \\ R_{13}^T \end{bmatrix} DR_{11} + \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} C^T & 0 \\ 0 & -J \end{bmatrix} \begin{bmatrix} R_{21} \\ R_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{2.11}$$

from which it follows that

$$\begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} (C-C^T)/2 & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} R_{21} \\ R_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and hence $R_{21} = 0$ and $R_{31} = 0$. At this point, the relation for the $(1,1)$ block gives $R_{11}^T DR_{11} = D$, from which it follows that R_{11} must be invertible. Writing out the relations for the $(1,2)$ blocks, in a similar way to the above, it follows that $R_{12} = 0$ and $R_{13} = 0$. With this, adding the two relations satisfied by the $(2,2)$ blocks, one gets

$$\begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix}^T \begin{bmatrix} (C+C^T)/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} (C+C^T)/2 & 0 \\ 0 & 0 \end{bmatrix}.$$

From this, it follows that R_{22} is invertible and $R_{23} = 0$, $R_{32} = 0$, and hence necessarily that R_{33} is invertible. □

Because of Lemma 2.3, we can restrict attention to simpler cases of fundamental matrix solutions W, Z, and S, where for all t:

$$W^T(t)DW(t) = D, \quad D \text{ in (2.6)}, \tag{2.12}$$

$$Z^T(t)CZ(t) = C, \quad C \text{ in (2.8)}, \tag{2.13}$$

$$S^T(t)JS(t) = J, \quad J \text{ in (2.5)}. \tag{2.14}$$

A goal of ours is to give lower bounds on the number of singular values of fundamental matrix solutions X satisfying (2.4) that are 1 for all t. Our arguments will use the difference in multiplicities of the eigenvalues of H which are out of phase by

π with one another. If this difference is 0, the lower bound is 0. For this reason, we will focus attention on (2.12) and (2.13) only, that is on the W-part and Z-part of the system.

Lemma 2.4. *Let $W \in \mathbb{R}^{n\times n}$ be any matrix satisfying $W^TDW = D$, with D given in* (2.6)*, and $n = n_1 + n_2$. Let $\nu_0(W)$ be the number of singular values of W which are equal to* 1*. Then, we have*

$$\nu_0(W) \geq |n_1 - n_2|\,.$$

Proof. Let W be partitioned similarly to D, that is $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$, where $W_{ii} \in \mathbb{R}^{n_i\times n_i}$, $i = 1, 2$, and $W_{12} \in \mathbb{R}^{n_1\times n_2}$, $W_{21} \in \mathbb{R}^{n_2\times n_1}$. Let $Y = \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix}$. Then, since Y has at most $2\min(n_1, n_2)$ linearly independent columns, we see that $\dim\big(\ker(Y)\big) \geq n-2\min(n_1, n_2) = |n_1 - n_2|$. Next, observe that, since $W^TDW = D$ and $D^2 = I$, we have

$$\begin{aligned} \ker(W^TW - I) &= \ker(W^TW - DW^TDW) = \ker\big((W^T - DW^TD)W\big) \\ &= \ker\Big(\begin{bmatrix} 0 & 2W_{21}^T \\ 2W_{12}^T & 0 \end{bmatrix} W\Big)\,. \end{aligned}$$

Thus, we have $\dim\big(\ker(W^TW - I)\big) \geq |n_1 - n_2|$. □

Next, we show some results concerning the Z-part of the system.

Lemma 2.5. *Let $Z \in \mathbb{R}^{n\times n}$ be any matrix satisfying Z^TCZ, where C is given in* (2.8) *with $n = 2\sum_{j=1}^{p} [n_1(j) + n_2(j)]$. Then, Z is a block diagonal matrix:*

$$Z \;=\; \operatorname{diag}(Z_1, \dots, Z_p)\ ,\ \ Z_j^TC_jZ_j = C_j\ ,\ \ j = 1, \dots, p\,. \tag{2.15}$$

Moreover, for $j = 1, \dots, p$, Z_j satisfy

$$Z_j^TD_jZ_j \;=\; D_j\ ,\ \ D_j = \begin{bmatrix} I_2\otimes I_{n_1(j)} & 0 \\ 0 & -I_2\otimes I_{n_2(j)} \end{bmatrix}\ ,\ \ \textit{and} \tag{2.16}$$

$$Z_j^T\widehat{J}_jZ_j = \widehat{J}_j,\ \widehat{J}_j = \begin{bmatrix} J_{n_1(j)} & 0 \\ 0 & -J_{n_2(j)} \end{bmatrix},\ J_{n_k(j)} = \begin{bmatrix} 0 & I_{n_k(j)} \\ -I_{n_k(j)} & 0 \end{bmatrix},\ k = 1, 2. \tag{2.17}$$

Proof. Since $Z^TCZ = C$ and $ZCZ^T = C$, one also has $Z^TC^TZ = C^T$ and $ZC^TZ^T = C^T$. Adding these relations pairwise, we obtain

$$Z^TNZ = N\ ,\ \ ZNZ^T = N\ ,\qquad \text{where}\qquad N = (C + C^T)/2\,. \tag{2.18}$$

Given the form of C in (2.8), the matrix N has the form

$$N \;=\; \operatorname{diag}(c_1D_1, \dots, c_pD_p)\ ,\ D_j = \begin{bmatrix} I_2\otimes I_{n_1(j)} & 0 \\ 0 & -I_2\otimes I_{n_2(j)} \end{bmatrix}\ ,\ \ j = 1, \dots, p\,,$$

and $c_j = \cos(\phi_j)$, $0 < \phi_1 < \cdots < \phi_p < \pi/2$. Now, from (2.18), one has

$$\text{(a)}\quad Z^{-T} = NZN^{-1}\qquad \text{and}\qquad \text{(b)}\quad Z^{-T} = N^{-1}ZN\,.$$

Write Z in block form, and equate the (i,j)-th blocks of (a) and (b):

$$\frac{c_i}{c_j} D_i Z_{ij} D_j \;=\; \frac{c_j}{c_i} D_i Z_{ij} D_j \;;$$

thus, we must have $(c_i^2 - c_j^2) Z_{ij} = 0$. For $i \neq j$, this implies $Z_{ij} = 0$. Hence, Z must be block diagonal and (2.15) holds. The form (2.16) is obtained at once from $Z_j^T (C_j + C_j^T) Z_j = (C_j + C_j^T)$, while (2.17) is obtained from $Z_j^T (C_j - C_j^T) Z_j = (C_j - C_j^T)$.[3] □

Lemma 2.6. *With the notation of Lemma 2.5, we have*

$$\nu_0(Z) = \sum_{j=1}^{p} \nu_0(Z_j)\,,$$

where $\nu_0(Z)$ *and* $\nu_0(Z_j)$ *denote the number of singular values of* Z *and* Z_j *that are* 1*, and*

$$\nu_0(Z_j) \geq 2|n_1(j) - n_2(j)|\ ,\ j = 1, \ldots, p\ .$$

Further, the singular values of each Z_j *have even multiplicity.*

Proof. The statement on $\nu_0(Z) = \sum_{j=1}^{p} \nu_0(Z_j)$ is clear from (2.15). The fact that $\nu_0(Z_j) \geq 2|n_1(j) - n_2(j)|$ is now a consequence of Lemma 2.4 and of (2.16).

Now, for given j, suppose that $Z_j^T Z_j x = \frac{1}{\lambda} x$, $\|x\| = 1$. Then, we have at once

$$D_j Z_j^T Z_j x \;=\; \frac{1}{\lambda} D_j x \qquad \text{and} \qquad \widehat{J}_j Z_j^T Z_j x \;=\; \frac{1}{\lambda} \widehat{J}_j x\,.$$

Now, since $Z_j^T D_j Z_j = D_j$ and $Z_j^T \widehat{J}_j Z_j = \widehat{J}_j$, one also has $Z_j D_j Z_j^T = D_j$ and $Z_j \widehat{J}_j Z_j^T = \widehat{J}_j$. Thus, we get

$$(Z_j^T Z_j)^{-1} (D_j x) = \frac{1}{\lambda} (D_j x) \qquad \text{and} \qquad (Z_j^T Z_j)^{-1} (\widehat{J}_j x) = \frac{1}{\lambda} (\widehat{J}_j x)\,.$$

Therefore, since the eigenvalues of $Z_j^T Z_j$ arise as $\{\lambda, 1/\lambda\}$, and the eigenvectors $D_j x$ and $\widehat{J}_j x$ are orthogonal unit vectors, we conclude that each eigenvalue of $Z_j^T Z_j$ has even multiplicity. □

Remark 2.7. Suppose that W is a fundamental matrix solution of (1.2) satisfying (2.12) for all t. Since the eigenvalues of the continuous function $W^T W$ can be labelled so to be continuous functions of t, then we can label the singular values of W in such a way that they are continuous functions of t and at least $|n_1 - n_2|$ of them are identically 1 for all t. Likewise, let Z be a fundamental matrix solution of (1.2) satisfying (2.13) for all t. Since the eigenvalues of the functions $Z_j^T Z_j$, $j = 1, \ldots, p$, can be labelled so to be continuous functions of t, then the singular values of Z can be labelled so that they are continuous functions of t, at least $2 \sum_{j=1}^{p} |n_1(j) - n_2(j)|$ are identically 1 for all t, and they have even multiplicity for any t.

[3](2.16) and (2.17) are equivalent to $Z_j^T C_j Z_j = C_j$

Remark 2.8. As far as the $S-$part of the differential system is concerned, that is when S satisfies (2.14), then a priori we cannot be certain that any of its singular values will be identically 1 nor that they will posses even multiplicity.

Finally, let $\nu_0(X)$ be the number of singular values of a fundamental matrix solution X satisfying (2.4) which are identically 1 for all t. By putting together the results obtained in this section, we see that a lower bound on $\nu_0(X)$ can be obtained by looking at the distribution of eigenvalues of H on the unit circle. In case in which the assumptions leading to (1.6) hold, then this will give us a lower bound on how many Lyapunov exponents will be 0. We summarize these considerations in the following theorem, which holds as a consequence of the previous results.

Theorem 2.9. *Let X be a fundamental matrix solution of* (1.2) *satisfying* (2.4). *Let orthogonal U give the ordered Schur form of H as in* (2.7) *and* (2.8), *with $n = n_1 + n_2 + 2m + 2\sum_{j=1}^{p}[n_1(j) + n_2(j)]$. With the understanding that some of the indices below may be* 0, *H has*

(1) *n_1 eigenvalues equal to* 1, *and n_2 eigenvalues equal to* -1;
(2) *$2n_1(j)$ eigenvalues equal to $e^{\pm i\phi_j}$, and $2n_2(j)$ eigenvalues equal to $e^{\pm i(\phi_j+\pi)}$, for $j = 1, \dots, p$, and $0 < \phi_1 < \dots < \phi_p < \pi/2$.*
(3) *$2m$ eigenvalues equal to $\pm i$.*

Then, for $\nu_0(X)$, we have

$$\nu_0(X) \geq |n_1 - n_2| + 2\sum_{j=1}^{p} |n_1(j) - n_2(j)| \,. \tag{2.19}$$

Moreover, consider the subproblem associated to the eigenvalues $e^{\pm i\phi_j}$, $e^{\pm i(\phi_j+\pi)}$ of (2), *that is consider Z in* (2.15). *Then, X has at least as many non-simple singular values as Z does.*

Finally, under the assumptions and with the notation of Theorem 1.1, for $x_0 \in M_0$, $\mathrm{Sp}\big(\Phi_{x_0}\big)$ is symmetric with respect to the origin, and has at least $\big[|n_1 - n_2| + 2\sum_{j=1}^{p}|n_1(j) - n_2(j)|\big]$ Lyapunov exponents equal to 0. *Also, $\mathrm{Sp}\big(\Phi_{x_0}\big)$ contains at least as many repeated Lyapunov exponents as the number of distinct singular values of the Z-part of $U^T\Phi_{x_0}U$, all of which have even multiplicity.*

Proof. The only things to justify are the statements about $\mathrm{Sp}\big(\Phi_{x_0}\big)$. With previous notation, for all t we must have (see (2.10) and (2.12), (2.13), (2.14)) $U^T\Phi_{x_0}(t)U =$

$\begin{bmatrix} W_{x_0}(t) & 0 & 0 \\ 0 & Z_{x_0}(t) & 0 \\ 0 & 0 & S_{x_0}(t) \end{bmatrix}$, so that in particular

$$U^T \lim_{t\to\infty} \frac{1}{t} \log\big(\Phi_{x_0}^T(t)\Phi_{x_0}(t)\big)^{1/2}\; U \;=$$
$$\lim_{t\to\infty} \frac{1}{t} \begin{bmatrix} \log\big(W_{x_0}^T(t)W_{x_0}(t)\big)^{1/2} & 0 & 0 \\ 0 & \log\big(Z_{x_0}^T(t)Z_{x_0}(t)\big)^{1/2} & 0 \\ 0 & 0 & \log\big(S_{x_0}^T(t)S_{x_0}(t)\big)^{1/2} \end{bmatrix}.$$

Thus, the symmetry with respect to the origin and the bound on the number of 0 Lyapunov exponents are consequences of the fact that the singular values of the function Φ_{x_0} can be chosen continuous functions of t, and of previous results. The statement on the multiplicity relatively to the Lyapunov exponents associated to Z_{x_0} is also consequence of continuity of the singular values and of the fact that the limit matrix above is symmetric, hence diagonalizable. In fact, if we let $\Psi_{x_0} = \lim_{t\to\infty} \frac{1}{t}\log\big(Z_{x_0}^T(t)Z_{x_0}(t)\big)^{1/2}$, then continuity of the eigenvalues of $Z_{x_0}^T(t)Z_{x_0}(t)$ precludes from having any of the eigenvalues of Ψ_{x_0} with odd multiplicity. □

Remark 2.10. As we remarked in point (i) of Theorem 1.1, in general $\mathrm{Sp}\big(\Phi_{x_0}\big)$ depends on $x_0 \in M_0$, and on the invariant measure μ (M_0 does). The lower bounds given in Theorem 2.9, instead, hold for all x_0 (and μ). The situation is similar to (2.3) in Theorem 2.1, whereby the symmetry of the Lyapunov spectrum with respect to the origin holds regardless of x_0. In order to further infer that $\mathrm{Sp}\big(\Phi_{x_0}\big)$ does not depend on $x_0 \in M_0$, we would need condition (iv) in Theorem 1.1 to hold.

Remark 2.11. An extension of our results (cfr. [7, 11]) is obtained by replacing (2.4) with

$$X^T(t)HX(t) = e^{at}H\,,\ \forall t\,,\ H^TH = I\,. \tag{2.20}$$

It is a simple verification that one arrives at (2.20) upon considering the shifted system $\dot{x} = (A(t)+a/2\; I)x$, instead of (1.2). In this case, one has $A^T(t)H + HA(t) = aH$, instead of (2.2)-(a). Now $\mathrm{Sp}\big(\Phi_{x_0}\big)$ will be shifted by $a/2$.

3. Examples

The numerical results below have been obtained using the so-called "continuous QR method" (see [6]). That is, we use the technique leading to (1.5) as follows:

- Q is approximated by the classic Runge-Kutta scheme of order 4 to integrate the equation for Q and the solution is orthogonalized after each step;
- the Lyapunov exponents are approximated from (1.5) using the composite trapezoidal rule.

For the problems below, we fix the interval of integration to $[0, 10^4]$, take initial condition to the identity, and perform integration with a constant stepsize $h = 1/10$. These examples are purposely built starting from a periodic coefficient matrix, to which we add a term which goes to 0 as $t \to \infty$, so that $\mathrm{Sp}(\Phi)$ reduces to the set of Floquet exponents of the periodic problem. On one hand, this allows us to compute $\mathrm{Sp}(\Phi)$ by other means and to check the accuracy of the obtained answers. On the other hand, we remark that when we attempted a direct time integration for the full monodromy matrix on these problems we obtained very inaccurate approximations of the Floquet exponents (only the largest one was accurate).

Example 3.1. This is a system evolving on the Lorentz group. We have

$$A(t) = \begin{bmatrix} 0 & \cos(t) & -1 & \frac{1}{1+t} \\ -\cos(t) & 0 & \frac{3}{1+t^2} & 5 \\ 1 & -\frac{3}{1+t^2} & 0 & -\sin(t) \\ \frac{1}{1+t} & 5 & -\sin(t) & 0 \end{bmatrix}, \qquad t \geq 0\,.$$

The Lyapunov exponents are $\{5, 0, 0, -5\}$. Approximating all four Lyapunov exponents, we obtain (at six digits):

$$\lambda_1 = 4.99959\ ,\ \lambda_2 = 0.000353\ ,\ \lambda_3 = 0.00000277230\ ,\ \lambda_4 = -4.99994\ .$$

Approximating only the dominant Lyapunov exponent, by integrating just for the first column of Q, we get $\lambda_1 = 4.99958$. This second computation takes 20% of the time required by the first one.

Example 3.2. Here we consider a problem whose fundamental matrix solution $Z(t)$ satisfies $Z^T(t)CZ(t) = C$ with $C = \left[\begin{smallmatrix} Q\otimes I_2 & 0 \\ 0 & -Q \end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix} c & s \\ -s & c \end{smallmatrix}\right]$, and $c = \cos(\phi)$, $s = \sin(\phi)$, $0 < \phi < \frac{\pi}{2}$. We take the following coefficient matrix:

$$A(t) = \begin{bmatrix} 0 & 2 & -1 & \frac{1}{1+t} & 1 & 2 \\ -2 & 0 & \frac{1}{1+t} & 5 & \cos(t) & 4 \\ 1 & -\frac{1}{1+t} & 0 & 2 & -2 & 1 \\ -\frac{1}{1+t} & -5 & -2 & 0 & -4 & \cos(t) \\ 1 & \cos(t) & -2 & -4 & 0 & \sin(t) \\ 2 & 4 & 1 & \cos(t) & -\sin(t) & 0 \end{bmatrix}, \qquad t \geq 0\ .$$

We expect two zero and two possibly nonzero Lyapunov exponents, symmetric with respect to the origin, each of multiplicity 2. In other words, only one Lyapunov exponent really needs to be computed. In fact, the two nonzero Lyapunov exponents for this problem are (at four digits) $\{\pm 3.027\}$. Approximating all six Lyapunov exponents, we get

$$\lambda_1 = 3.028\ ,\ \lambda_2 = 3.028\ ,\ \lambda_3 = 0.5347 \times 10^{-4}\ ,$$
$$\lambda_4 = 0.4374 \times 10^{-3}\ ,\ \lambda_5 = -3.028\ ,\ \lambda_6 = -3.028\,.$$

Directly approximating only the dominant Lyapunov exponent, we get $\lambda_1 = 3.027$ and this second computation takes 12.5% of the time required to approximate all Lyapunov exponents.

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