SMOOTH SVD ON THE LORENTZ GROUP WITH APPLICATION TO COMPUTATION OF LYAPUNOV EXPONENTS

L. DIECI, C. ELIA AND L. LOPEZ

ABSTRACT. In this work we give a constructive argument to establish existence of a smooth singular value decomposition (SVD) for a C^k function X in the Lorentz group. We rely on the explicit structure of the polar factorization of X in order to justify the form of the SVD. Our construction gives a simple algorithm to find the SVD of X, which we have used to approximate the Lyapunov exponents of a differential system whose fundamental matrix solution evolves on the Lorentz group. Algorithmic details and examples are given.

1. INTRODUCTION

We consider a matrix valued function $X \in C^k(\mathbb{R}, \mathbb{R}^{(n+m)\times(n+m)})$, $k \geq 1$, belonging to the Lorentz group (the relativity group). That is, for all $t \in \mathbb{R}$, we have

$$X^T(t)DX(t) \; = \; D \; , \qquad \text{with} \qquad D = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix} \; , \tag{1}$$

where (without loss of generality) we will henceforth assume that $n \neq 0 \neq m$, and $n \geq m$. We will write $X \in \mathcal{L}(n,m)$ whenever X is in the Lorentz group. By differentiating $X^TDX = D$ and setting $A = \dot{X}X^{-1}$, it is simple to realize that $X \in \mathcal{L}(n,m)$ is a fundamental matrix solution for a linear system of differential equations of the form

$$\dot{X} = A(t)X \; , \qquad \text{where} \qquad A^T(t)D + DA(t) \; = \; 0 \, . \tag{2}$$

Finally, we will often write a matrix (be it a function or not) in block form, say $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where the partitioning is that inherited from D; i.e., $A_{11} \in \mathbb{R}^{n\times n}$, $A_{21} \in \mathbb{R}^{m\times n}$, $A_{12} \in \mathbb{R}^{n\times m}$, $A_{22} \in \mathbb{R}^{m\times m}$.

Example 1. *The following two facts are easy to verify directly and will be handy shortly.*

(i) *An orthogonal function $Q \in \mathcal{L}(n,m)$ must have the following structure: $Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix}$, where Q_{11} and Q_{22} are orthogonal (in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{m\times m}$, respectively).*

(ii) *A block triangular function $R \in \mathcal{L}(n,m)$ must be orthogonal.*

Our purpose in this work is to provide an explicit characterization of a singular value decomposition in $\mathcal{L}(n,m)$ of a generic function $X \in \mathcal{L}(n,m)$. In Section 2, we will rely on the polar factorization of X to characterize generic functions $X \in \mathcal{L}(n,m)$ and further arrive to an SVD-like in $\mathcal{L}(n,m)$ of X. In Section 3 we will then derive a set of differential equations satisfied by the SVD factors of X. Finally, in Section 4, we give an application:

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by using the explicit form of the SVD of X we will approximate the Lyapunov exponents of a dynamical system with fundamental matrix solution X satisfying (2).

2. Polar and Singular Value decompositions in the Lorentz group

Two of the most useful and commonly employed decompositions of a full rank function are its QR and SVD. It is well known (e.g., see [1]) that a C^k full rank function, for us $X \in \mathcal{L}(n,m)$ in (1), has C^k factors Q and R. However, see Example 1, in general Q and R will not be in $\mathcal{L}(n,m)$. Instead, here below we will show that –for a generic class of functions $X \in \mathcal{L}(n,m)$– X admits a C^k SVD with the factors in $\mathcal{L}(n,m)$. The stepping stone is given by the polar factorization of X, $X = QP$, with Q orthogonal and P symmetric positive definite. It is known that Q and P are C^k functions (see [1]), and that furthermore they can be taken in $\mathcal{L}(n,m)$, see [9]. The following result gives the structure of the polar factor P.

Lemma 2. *Let $P \in \mathcal{L}(n,m)$ be symmetric positive definite. Then P has the following block structure:*

$$P = \begin{pmatrix} (I_n + P_{21}^T P_{21})^{1/2} & P_{21}^T \\ P_{21} & (I_m + P_{21}P_{21}^T)^{1/2} \end{pmatrix}, \qquad P_{21} \in \mathbb{R}^{m\times n}\,. \tag{3}$$

If $P \in C^k(\mathbb{R}, \mathbb{R}^{(n+m)\times(n+m)})$, then $P_{21} \in C^k(\mathbb{R}, \mathbb{R}^{m\times n})$.

Proof. We write P in block form: $P = \begin{pmatrix} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix}$, and use the relation $P^TDP = D$ to obtain the following conditions:

$$\text{(a) } P_{11}^2 - P_{21}^T P_{21} = I_n\ , \quad \text{(b) } P_{21}P_{11} - P_{22}P_{21} = 0\ , \quad \text{(c) } P_{21}P_{21}^T - P_{22}^2 = -I_m\ .$$

From (a) and (c) it follows that $P_{11} = (I_n + P_{21}^T P_{21})^{1/2}$ and $P_{22} = (I_m + P_{21}P_{21}^T)^{1/2}$. Given these forms for P_{11} and P_{22}, condition (b) is automatically satisfied. To see this, consider (at any given t) the singular value decomposition of the block P_{21}: $P_{21} = U\Sigma V^T$. Then $P_{11}^2 = V(I_n + \Sigma^T\Sigma)V^T$ and $P_{11} = V(I_n + \Sigma^T\Sigma)^{1/2}V^T$. Similarly, we have $P_{22} = U(I_m + \Sigma\Sigma^T)^{1/2}U^T$. Using these, we see that (b) is identically satisfied. Thus (3) follows. ☐

Consider now the function $P_{21} \in C^k(\mathbb{R}, \mathbb{R}^{m\times n})$ of Lemma 2. Endow the space $C^k(\mathbb{R}, \mathbb{R}^{m\times n})$ with the Witney (or fine) topology; see [5] for definition of this topology. From [1, Theorem 4.3], we know that a *generic* function $A \in C^k(\mathbb{R}, \mathbb{R}^{m\times n})$ has distinct singular values for all t. [Recall that a property is generic if the set of elements that do not have this property is a countable union of nowhere dense sets]. With these preparations, the following definition is justified.

Definition 3. *A generic function $X \in \mathcal{L}(n,m)$ is one for which the singular values of the function P_{21} in* (3) *are distinct for all t.*

As it is well understood, it is a trivial matter to recover an SVD of X from a Schur decomposition of the symmetric positive definite function P. Indeed, if we have $P = ZEZ^T$, with $Z^TZ = I$ and E diagonal, then we immediately have the SVD of X: $X = (QZ)EZ^T$. We will now see that a simple expression exists for the eigendecomposition of a generic $P \in \mathcal{L}(n,m)$, and hence for the SVD of $X \in \mathcal{L}(n,m)$.

Lemma 4. *For a generic symmetric positive definite C^k function $P \in \mathcal{L}(n,m)$, we have*

$$W^TPW = \begin{pmatrix} (I_n + \Sigma^T\Sigma)^{1/2} & \Sigma^T \\ \Sigma & (I_m + \Sigma\Sigma^T)^{1/2} \end{pmatrix}, \tag{4}$$

where $W = \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}$ *and* $P_{21} = U\Sigma V^T$ *is a* C^k *SVD of the function* P_{21}*. If we further let* $\Sigma = (\Sigma_1 \ \ 0)$*,* $\Sigma_1 = \mathrm{diag}(\alpha_1, \dots, \alpha_m)$*, and* $S = \begin{pmatrix} \frac{1}{\sqrt{2}} I_m & 0 & -\frac{1}{\sqrt{2}} I_m \\ 0 & I_{n-m} & 0 \\ \frac{1}{\sqrt{2}} I_m & 0 & \frac{1}{\sqrt{2}} I_m \end{pmatrix}$*, then from* (4) *we have the following eigendecomposition of* P*:*

$$(WS)^T P (WS) = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & \Lambda^{-1} \end{pmatrix}, \quad \Lambda = (I_m + \Sigma_1^2)^{1/2} + \Sigma_1 \,. \tag{5}$$

Finally, letting $C = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & \Lambda^{-1} \end{pmatrix}$ *and* $Y = QW$*, we obtain the following form for a* C^k *SVD of* X*:*

$$X = YSCS^T W^T \ . \tag{6}$$

Proof. We only need to recall (see again [1, Theorem 4.3]) that if P_{21} has distinct singular values for all t, then we have $P_{21}(t) = U(t)\Sigma(t)V(t)^T$, for all t, where $U \in C^k(\mathbb{R}, \mathbb{R}^{m\times m})$, $\Sigma \in C^k(\mathbb{R}, \mathbb{R}^{m\times n})$, $V \in C^k(\mathbb{R}, \mathbb{R}^{n\times n})$, and $\Sigma = (\Sigma_1 \ \ 0)$, $\Sigma_1 = \mathrm{diag}(\alpha_1, \dots, \alpha_m)$, with $\alpha_i \neq \alpha_j$, $i \neq j$, for all t. The rest follows by straightforward manipulation. □

Remark 5. *Obviously, the function* P (*respectively,* X) *has eigenvalues* (*respectively, singular values*) $\sqrt{1+\alpha_i^2} + \alpha_i$*,* $\sqrt{1+\alpha_i^2} - \alpha_i$*,* $i = 1, \dots, m$*, and* $(n-m)$ *eigenvalues* (*singular values*) *identically equal to* 1 *for all* t*. Without loss of generality, we can also assume that the eigenvalues* (*singular values*) *are ordered from largest to smallest.*

Remark 6. *Clearly, the factors* W *and* Y *in* (5) *and* (6) *are in* $\mathcal{L}(n,m)$*. The role of the constant matrix* S (*not in* $\mathcal{L}(n,m)$) *is simply to explicitly diagonalize* $W^T P W$*. The form* $X = Y(SCS^T)W$ *is the SVD-like in* $\mathcal{L}(n,m)$ *of* $X \in \mathcal{L}(n,m)$*.*

3. Smooth SVD of X

The standard general construction of differential equations for a smooth SVD of the function $X \in \mathcal{L}(n,m)$ (e.g., see [8] or [1]) in general will fail, because the multiple singular values which are identically 1 lead to singularities in the differential equations derived in these cited works. However, relying on Lemma 4, we now derive differential equations for the factors Y, C, and W, in the SVD (6) of X, without encountering singularities.

Differentiating (6), we have

$$\dot{X} = A(YS)C(WS)^T = \dot{Y}SCS^T W^T + YS\dot{C}S^T W^T + YSCS^T \dot{W}^T .$$

Let $H = Y^T\dot{Y}$ and $K = W^T\dot{W}$, and note that H and K must be skew-symmetric, so that we must have

$$(S^T H S)C + \dot{C} - C(S^T K S) = (S^T G S)C, \tag{7}$$

$$\dot{Y} = YH, \tag{8}$$

$$\dot{W} = WK, \tag{9}$$

where we have set $G = Y^T AY$. Next, notice that $Y = \begin{pmatrix} Q_{11}V & 0 \\ 0 & Q_{22}U \end{pmatrix}$, and that H and K have this same block structure as well. Write

$$H = \begin{pmatrix} H_{11} & H_{12} & 0 \\ -H_{12}^T & H_{22} & 0 \\ 0 & 0 & H_{33} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} & 0 \\ -K_{12}^T & K_{22} & 0 \\ 0 & 0 & K_{33} \end{pmatrix},$$

where H_{11}, H_{33}, K_{11}, K_{33} are $C^{k-1}(\mathbb{R}, \mathbb{R}^{m\times m})$ and skew-symmetric, H_{22}, K_{22} are $C^{k-1}(\mathbb{R}, \mathbb{R}^{(n-m)\times(n-m)})$ skew-symmetric, and H_{12}, K_{12} are $C^{k-1}(\mathbb{R}, \mathbb{R}^{m\times(n-m)})$. Then (7) can be rewritten as

$$\begin{pmatrix} \frac{H_{11}+H_{33}}{2} & \frac{H_{12}}{\sqrt{2}} & \frac{-H_{11}+H_{33}}{2} \\ \frac{-H_{12}^T}{\sqrt{2}} & H_{22} & \frac{H_{12}^T}{\sqrt{2}} \\ \frac{-H_{11}+H_{33}}{2} & \frac{-H_{12}}{\sqrt{2}} & \frac{H_{11}+H_{33}}{2} \end{pmatrix} \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & \Lambda^{-1} \end{pmatrix} + \begin{pmatrix} \dot{\Lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{d}{dt}\Lambda^{-1} \end{pmatrix} -$$

$$(10) \qquad \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} \frac{K_{11}+K_{33}}{2} & \frac{K_{12}}{\sqrt{2}} & \frac{-K_{11}+K_{33}}{2} \\ \frac{-K_{12}^T}{\sqrt{2}} & K_{22} & \frac{K_{12}^T}{\sqrt{2}} \\ \frac{-K_{11}+K_{33}}{2} & \frac{-K_{12}}{\sqrt{2}} & \frac{K_{11}+K_{33}}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{G_{11}+G_{13}+G_{13}^T+G_{33}}{2} & \frac{G_{12}+G_{23}^T}{\sqrt{2}} & \frac{-G_{11}+G_{13}-G_{13}^T+G_{33}}{2} \\ \frac{-G_{12}^T+G_{23}}{\sqrt{2}} & G_{22} & \frac{G_{12}^T+G_{23}}{\sqrt{2}} \\ \frac{-G_{11}-G_{13}+G_{13}^T+G_{33}}{2} & \frac{-G_{12}+G_{23}^T}{\sqrt{2}} & \frac{G_{11}-G_{13}-G_{13}^T+G_{33}}{2} \end{pmatrix} \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & \Lambda^{-1} \end{pmatrix}.$$

Let σ_i, $i = 1, \dots, m$, be the diagonal elements of Λ^{-1}. From the differential part in (10), using the fact that the relations from the $(1,1)$ and $(3,3)$ blocks imply that $\text{diag}(G_{11} + G_{33}) = 0$, we have

$$(11) \qquad \dot{\sigma_i} = -(G_{13})_{ii}\sigma_i, \quad i = 1, \dots, m.$$

From the algebraic part in (10) we obtain

$$(12) \qquad H_{22} - K_{22} = G_{22}$$

$$(13) \qquad \begin{pmatrix} \Lambda^{-1} & -I_m \\ -I_m & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} H_{12} \\ K_{12} \end{pmatrix} = \begin{pmatrix} \Lambda^{-1}(G_{23}^T + G_{12}) \\ -G_{12} + G_{23}^T \end{pmatrix}.$$

From (13) we can uniquely determine H_{12}, K_{12}. We further set K_{22} to zero and thus H_{22} is uniquely determined from (12). [This choice for K_{22} is for convenience only. Any other choice by which we recover smooth H_{22} and K_{22} in (12) is possible].

To find the blocks H_{11}, K_{11}, H_{33} and K_{33}, we reason as follows. We look at the $(1,3)$, $(3,1)$ and $(3,3)$ blocks of (10) (note that the $(1,1)$ block is the same as the $(3,3)$ block). From the $(1,3)$ and $(3,1)$ blocks we get

$$\begin{pmatrix} 0 & -\Lambda \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} -H_{11} + H_{33} \\ -K_{11} + K_{33} \end{pmatrix} + \begin{pmatrix} -H_{11} + H_{33} \\ -K_{11} + K_{33} \end{pmatrix} \Lambda^{-1} = \begin{pmatrix} (-G_{11} + G_{13} - G_{13}^T + G_{33})\Lambda^{-1} \\ \Lambda(G_{11} + G_{13} - G_{13}^T - G_{33}) \end{pmatrix},$$

so that, for $i, j = 1, \dots, m$, $i \neq j$, we get

$$(14) \qquad \begin{aligned} (-H_{11} + H_{33})_{ij} &= (-G_{11} + G_{33})_{ij} + (G_{13} - G_{13}^T)_{ij} \frac{\sigma_i^2\sigma_j^2 + 1}{\sigma_i^2\sigma_j^2 - 1}, \quad \text{and} \\ (-K_{11} + K_{33})_{ij} &= 2(G_{13} - G_{13}^T)_{ij} \frac{\sigma_j\sigma_i}{\sigma_i^2\sigma_j^2 - 1}. \end{aligned}$$

Moreover, from the $(3,3)$ block of (10) we also get, for $i,j=1,\dots,m$, $i\neq j$,

$$(15)\qquad \begin{aligned}(H_{11}+H_{33})_{ij} &= (G_{11}+G_{33})_{ij}-(G_{13}+G_{13}^T)_{ij}\frac{\sigma_j^2+\sigma_i^2}{\sigma_j^2-\sigma_i^2}, \quad \text{and}\\ (K_{11}+K_{33})_{ij} &= -2(G_{13}+G_{13}^T)_{ij}\frac{\sigma_j\sigma_i}{\sigma_j^2-\sigma_i^2}.\end{aligned}$$

Using (14) and (15), we can determine H_{11}, H_{33}, K_{11}, K_{33} and hence H and K.

To summarize, to find the SVD of $X\in\mathcal{L}(n,m)$, we will proceed as follows:

1. Given initial conditions in $\mathcal{L}(n,m)$ (i.e., $X(0)$ or its polar factors in $\mathcal{L}(n,m)$).
2. Integrate (8), (9) and (11) making use of the algebraic relations (12) with $K_{22}=0$, (13), (14), (15) to obtain K and H.
3. Form the SVD (6) if desired.

Remark 7. *It must be stressed that it is because of the assumption of genericity on the class of functions considered (see Definition (3)), that we are able to write down the differential equations for the evolutions of the factors in the SVD. In particular, we have been able to restrict to singular values which are distinct (hence (15) is legitimate) and less than* 1 *in magnitude (hence (14) is legitimate).*

4. Approximation of the Lyapunov exponents

An application of the SVD of a fundamental matrix solution X of a linear system is in the approximation of its Lyapunov exponents. This requires some technical conditions on the linear system, and for the sake of brevity we refer to [6, 7] for formal definitions and justifications. Presently, we recall (e.g., see [4]) that if σ_i, $i=1,\dots,(n+m)$, are the continuous singular values of X, then the Lyapunov exponents can be obtained as

$$(16)\qquad \lambda_i \;=\; \lim_{t\to\infty}\frac{1}{t}\ln(\sigma_i(t))\,, \quad i=1\,\dots,n+m\,.$$

Of course, in practice we will approximate the Lyapunov exponents on a finite interval $[0,\,T]$ as $\lambda_i\approx\frac{1}{T}\ln\sigma_i(T)$, $i=1,\dots,n+m$.

In our case of $X\in\mathcal{L}(n,m)$, we will make use of the structure of a generic function $X\in\mathcal{L}(n,m)$ to compute only singular values which are distinct for all t. The argument is as follows. In [2], it was already observed that the Lyapunov exponents of $X\in\mathcal{L}(n,m)$ are symmetric with respect to the origin and that $(n-m)$ of them are equal to 0 (a consequence of having $(n-m)$ singular values identically 1). Thus, we can compute just the m Lyapunov exponents that are less than 0. [Exploiting this fact, we avoid working with quantities that blow up as time increases.] These are associated to singular values which are less than 1 for all t. By virtue of Definition (3), these singular values are also distinct for all t. Now, in order to compute these m singular values, at every step we will solve the differential equations (11) for the σ_i and (8) for Y; there is no need to find W in (9). Also, instead of solving (11), we rather solve

$$(17)\qquad \frac{d}{dt}(\ln(\sigma_i)) = -(G_{13})_{ii}, \quad i=1,\dots,m.$$

Then, to avoid reforming explicitly the σ_i's, we modified (14)-(15) to obtain the entries of H_{11}, H_{33} (and K_{11}, K_{33}) as follows. For $i = 2, \dots, m, \quad j = 1, \dots, i-1$,

$$\begin{aligned}(H_{11})_{ij} = (G_{11})_{ij} - &\frac{1}{2}(G_{13})_{ij}(\coth(\ln(\sigma_j/\sigma_i)) + \coth(\ln(\sigma_j\sigma_i))) - \\ &\frac{1}{2}(G_{13})_{ji}(\coth(\ln(\sigma_j/\sigma_i)) - \coth(\ln(\sigma_j\sigma_i)))\ , \\ (H_{33})_{ij} = (G_{33})_{ij} - &\frac{1}{2}(G_{13})_{ij}(\coth(\ln(\sigma_j/\sigma_i)) - \coth(\ln(\sigma_j\sigma_i))) - \\ &\frac{1}{2}(G_{13})_{ji}(\coth(\ln(\sigma_j/\sigma_i)) + \coth(\ln(\sigma_j\sigma_i)))\ ,\end{aligned}$$

and similarly

$$(H_{12})_{ij} = (G_{12})_{ij} + (G_{23})_{ji}\coth(\ln\sigma_i), \quad i = 1, \dots, m,\ j = 1, \dots, n-m.$$

Finally, when we integrate (8), we must maintain the approximation orthogonal at grid points, which we have enforced by using a simple projected integrator, whereby at each step we orthogonalize the solution obtained with an explicit Runge-Kutta integrator.

5. Numerical examples

In this section we apply the algorithm to approximate numerically the Lyapunov exponents, and we compare the results obtained using this technique with those obtained with the tried-and-true continuous QR-method (see [3]). We take two systems for which we are able to compute exactly the Lyapunov exponents by other means, so that we can test the accuracy of the methods. Integration of the relevant differential equations is done with explicit Runge Kutta schemes of order 2 and 4 and constant stepsize $h = 0.1$, implemented in `Matlab5`. Initial conditions are chosen at random (using the `rand` command in `Matlab5`). Finally, we notice that with the SVD approach we approximate the negative Lyapunov exponents, while with the QR method we approximate the positive ones.

Example 1. Consider the linear system (2) with $n = m = 2$ given by

$$A(t) = \begin{pmatrix} 0 & \cos(t) & -1 & \frac{1}{1+t} \\ -\cos(t) & 0 & \frac{3}{1+t^2} & 5 \\ 1 & -\frac{3}{1+t^2} & 0 & -\sin(t) \\ \frac{1}{1+t} & 5 & -\sin(t) & 0 \end{pmatrix}.$$

The Lyapunov exponents are $\{-5, 0, 0, 5\}$. Numerical results are summarized in Table (1), where we show:

- The endpoint of integration T.
- The `Method` used. Namely, SVD2 and SVD4 are the SVD methods, QR2 and QR4 are the QR methods, with integrators of order 2 or 4, respectively.
- λ are the approximations of the Lyapunov exponents at T.

Example 2. Here we have (2) with $n = 4$, $m = 2$, given by

$$A(t) = \begin{pmatrix} 0 & \sin(t) & \frac{1}{1+t} & -1 & \frac{2}{1+t^2} & \cos(t) \\ -\sin(t) & 0 & \frac{1}{2+t} & 0 & 4 & \frac{1}{1+t^2} \\ -\frac{1}{1+t} & -\frac{1}{t+2} & 0 & 0 & \cos(t) & 2+\sin(t) \\ 1 & 0 & 0 & 0 & \frac{1}{1+t} & 0 \\ \frac{2}{1+t^2} & 4 & \cos(t) & \frac{1}{1+t} & 0 & -\sin(t) \\ \cos(t) & \frac{1}{1+t^2} & 2+\sin(t) & 0 & \sin(t) & 0 \end{pmatrix}.$$

Table 1. Example 1: QR and SVD methods

	Method	λ
$T = 10^2$	SVD2	-5.0114
	SVD4	-5.0101
	QR2	4.9940
	QR4	4.9928
$T = 10^3$	SVD2	-5.0018
	SVD4	-5.0009
	QR2	4.9997
	QR4	4.9992
$T = 10^4$	SVD2	-5.0009
	SVD4	-5.00009
	QR2	4.9999
	QR4	4.9999

The two positive LEs for this system are $\lambda_1 = 3.85140$, $\lambda_2 = 2.20210$. Numerical results are summarized in Table (2).

Table 2. Example 2: QR and SVD methods

	Method	λ_1	λ_2
$T = 10^2$	SVD2	-3.8799	-2.2065
	SVD4	-3.8718	-2.1988
	QR2	3.8691	2.1851
	QR4	3.8669	2.1832
$T = 10^3$	SVD2	-3.8566	-2.2042
	SVD4	-3.8542	-2.2017
	QR2	3.8557	2.2020
	QR4	3.8538	2.2001
$T = 10^4$	SVD2	-3.8535	-2.2043
	SVD4	-3.8517	-2.2022
	QR2	3.8534	2.2039
	QR4	3.8517	2.2021

From Tables (1) and (2), we observe that the QR and SVD methods give numerical approximations of the Lyapunov exponents with the same degree of accuracy. Also, upon using the `flops` count provided by `Matlab5`, we found that the SVD methods are about 32% more expensive than the QR methods. Furthermore, in Figure 1 we show that the rate of approach to the Lyapunov exponents of the two methods is essentially identical. Therefore, and in spite of the fact that in the QR method the factors Q and R do not belong to $\mathcal{L}(n, m)$, we conclude that, if only the Lyapunov exponents are desired, the QR method is preferable. However, by also integrating for W in (9), the SVD method allows us to rebuild the entire function X without explicitly resolving growth behaviors, while this does not appear possible with the QR method.

6. Conclusions

Using the particular structure of the polar factor in the polar factorization of a smooth matrix function X evolving on the Lorentz group $\mathcal{L}(n, m)$, we gave a simple formula for a

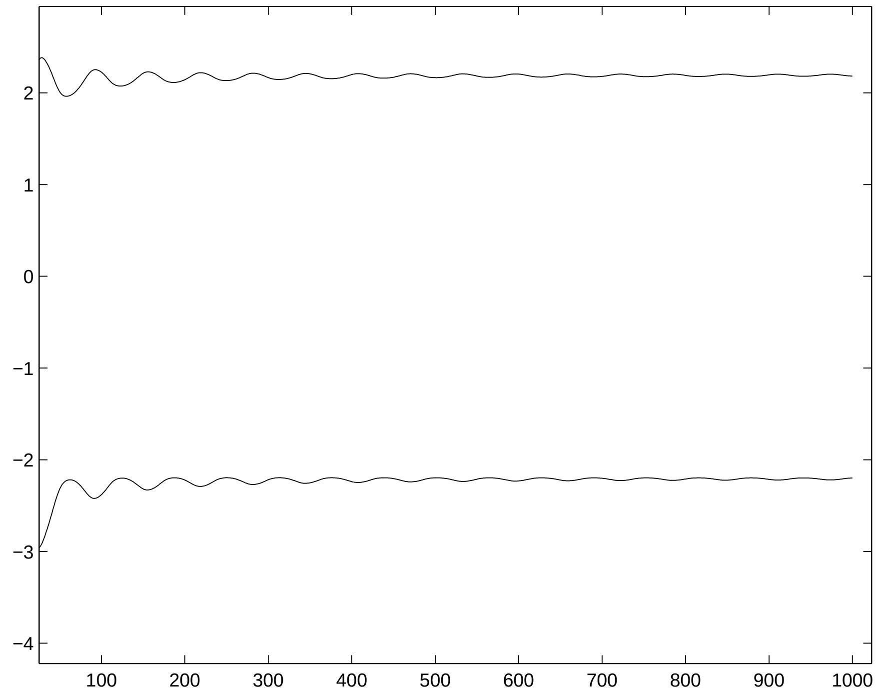

FIGURE 1. Example 2: QR and SVD methods approaching λ_2

smooth singular value decomposition of X with all factors in $\mathcal{L}(n,m)$. We were also able to derive differential equations for the factors in the SVD of X, in spite of the fact that X in general has multiple singular values (identically 1). As an application, we have used the SVD of X to approximate its Lyapunov exponents.

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SCHOOL OF MATHEMATICS, GEORGIA TECH, ATLANTA, GA 30332 U.S.A.

E-mail address: `dieci@math.gatech.edu, elia@math.gatech.edu`

DIPARTIMENTO DI MATEMATICA, UNIV. OF BARI, I-70100, BARI, ITALY

E-mail address: `lopez@pascal.dm.uniba.it`