ON SMOOTH DECOMPOSITIONS OF MATRICES*

LUCA DIECI† AND TIMO EIROLA‡

Abstract. In this paper we consider smooth orthonormal decompositions of smooth time varying matrices. Among others, we consider QR–, Schur–, and singular value decompositions, and their block-analogues. Sufficient conditions for existence of such decompositions are given and differential equations for the factors are derived. Also generic smoothness of these factors is discussed.

Key words. smooth orthonormal factorizations, QR factorization, Schur decomposition, singular values decomposition, polar decomposition

AMS subject classifications. 15A, 65F, 65L

1. Introduction. For very good reasons, orthogonal matrices (unitary in the complex case) are the backbone of modern matrix computation. They can be computed stably, and provide some of the most successful algorithmic procedures for a number of familiar tasks: finding orthonormal bases, solving least squares problems, eigenvalues and singular values computations, and so forth. The purpose of this work is to consider orthogonal decompositions for matrices depending on a real parameter. Thus we consider k times continuously differentiable matrix functions, i.e., $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$, $k \geq 0$.

We consider a number of basic tasks, such as the QR–, Schur–, and singular value decomposition (SVD) of A, and their block-analogues. Of course, in general, the matrix $A(t)$ will have, say, an SVD at each given t, but we are interested in conditions and procedures guaranteeing that the factors involved are smooth. This is desirable in several situations. For example, in updating techniques, perturbation theory, continuations processes, or to compute moving frames, e.g. to an invariant curve in space (see [4], [13], [9], [14]). A key motivation for us was provided by techniques for computing Lyapunov exponents (e.g., see [2] or [5]), in which case A is a solution of a linear system, in particular it is full rank for all t. (Which turns out to be a convenient sufficient condition for some of the factorizations considered below.)

Fundamental theoretical results on decompositions of parameter dependent matrices A are given in the book by Kato ([12]). There, the strongest results are obtained in case A is real analytic and Hermitian; then, for example, it has an analytic Schur decomposition. Similarly, Bunse-Gerstner et alia have shown that a real analytic A admits an analytic SVD ([3]). A different technique is used by Gingold and Hsieh in [6] to show that a real analytic (not necessarily Hermitian) matrix with only real eigenvalues admit an analytic Schur decomposition (triangularization). However, we do not want to require analyticity of A; many of the problems which motivated our work arise from linearization of a differential equation around a computed trajectory, and A may represent either the fundamental solution of the linearized problem, or the Jacobian matrix of the vector field. Thus we are interested in the case of A being a $\mathcal{C}^k$–matrix. Unfortunately, smoothness of A does not suffice for the existence of

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†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A. (dieci@math.gatech.edu).

‡Department of Mathematics, Joensuu University, FIN-80101, Joensuu, Finland (teirola@cc.joensuu.fi).

smooth decompositions for it. For example, in the case of a Hermitian matrix, it is well understood that a smooth Schur decomposition, in general, does not exist.

In this paper, we first give some (strong) sufficient conditions for smooth decompositions to exist. Differential equations for the factors will be given in such cases. Seemingly, these conditions appear very restrictive. For example, for the QR factorization of a matrix A one would need full rank of A; for the Schur factorization of a symmetric matrix or the SVD of a full rank matrix, one would need simple eigenvalues or singular values. We take three approaches to weaken our assumptions. First, we consider block-analogues of the standard decompositions. Also in this case differential equations are given. Second, we take a closer look at the type of singular behavior which can occur when, for example, A loses rank, or eigenvalues coalesce; this provides weaker sufficient conditions to guarantee existence of smooth decompositions, usually with some loss of smoothness. Third, we consider what can be expected in the generic case.

Genericity considerations are common in dynamical systems' study. In our context, they are a way to rigorously classify which properties of matrices depending on a real parameter are typical. The starting point is to consider the space of one parameter functions of matrices. One then endows this space with a suitable topology, and calls a property of the topological space *generic* if it holds for a set which is of second Baire category. We refer to [1] and to [11] for the needed background on the topic. A generic property implies that we can perturb a given function not satisfying this property into one which does satisfy it. A generic statement will provide a convenient starting point for computational approaches. For example, we will show that, generically, Hermitian matrix functions of one variable have simple eigenvalues. Therefore, an appropriate starting point for computational procedures to find smooth eigendecompositions of Hermitian matrices is to assume simple eigenvalues: this gives the differential equations of §2. Likewise, the genericity results of §4 will legitimate the differential equations' models put forward in §2 also for QR factorization and SVD decompositions. We believe that a merit of this paper is to have a general framework for the relevant decompositions based on the differential equations models. However, it is premature to say whether or not numerical solution of these differential or differential-algebraic equations will lead to efficient algorithms of solution for the problems under study.

A plan of the paper is as follows. In §2 we consider QR–, Schur–, and block Schur factorizations, SVD and block SVD decompositions. In this section, differential equations are derived for the decompositions under some nondegenericity assumptions.

In §3 we extend existence results for smooth decompositions to the case in which some singular behavior is encountered. We consider QR factorizations for rank deficient matrices and Schur/SVD decompositions in the case of coalescing eigen/singular values.

In §4 we study how smooth the factors can be expected to be in generic cases.

2. Smooth decompositions. In this section we consider $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$, $k \geq 1$, and we give differential equations for the decompositions of interest. The differential equations for the QR factorization of a full rank matrix have been given before (e.g., see [5]); also the differential equations for the SVD have already appeared elsewhere (e.g., the earliest references of which we are aware are [8] and [16], with the former reference only for the case of a fundamental solution matrix).

Remark 2.1. In 1965, Sibuya [15, Theorem 3 and Remark 3] established that a $\mathcal{C}^k$ ($k \geq 0$) matrix with disjoint groups of eigenvalues admits a $\mathcal{C}^k$ block–diagonalization;

this result is the content of Remark 2.6 below. It seems possible to build on this result of Sibuya to establish some of the other results we give in this §2. However, the approach in [15] is not constructive and it seems hard to exploit it for devicing computational procedures. In contrast, the differential equations models are better exploitable to obtain computational procedures (but, of course, one needs $k \geq 1$).

We will use the following simple results

LEMMA 2.1. *Assume that in the linear system of equations $B(t)x(t) = b(t)$, $B(t)$ is invertible for all t and that B, $b \in \mathcal{C}^k$. Then, also $x \in \mathcal{C}^k$.*

Proof. This follows from repeated differentiation of $x(t) = B^{-1}(t)b(t)$, recalling that $\frac{d}{dt}B^{-1} = -B^{-1}\dot{B}B^{-1}$. ◻

LEMMA 2.2. *Let $Q_0 \in \mathbb{C}^{m\times n}$ be orthonormal* [1]*. Consider the differential equation*

$$\dot{Q} = S(Q,t)\ Q\,, \qquad Q(0) = Q_0\,, \tag{2.1}$$

where S is a $m \times m$–matrix function, locally Lipschitzian in Q and continuous in t. Then (2.1) *has a unique solution Q for all t. Moreover, if*

$$Q^*\big[S(Q,t) + S(Q,t)^*\big]Q \;=\; 0\ ,$$

for every t, then Q is orthonormal for all t.

Proof. Existence and uniqueness are standard results. (Note that the set of orthonormal matrices is compact.) If Q is the solution of (2.1), then $Q^*\dot{Q} + \dot{Q}^*Q = Q^*(S + S^*)Q = 0$, i.e., Q^*Q is constant. ◻

Remark 2.2. Quite often, we will use Lemma 2.2 in the case $m = n$ with Q satisfying $\dot{Q} = Q\ H(Q,t)$, where H is a skew-Hermitian $n \times n$ matrix function (i.e., $H^*(Q,t) = -H(Q,t)$).

2.1. QR factorization.

We begin by giving differential equations for the QR factorization of a full rank matrix $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$, $m \geq n$. Suppose that $A(0) = Q(0)R(0)$ is a given QR factorization (i.e., $Q(0) \in \mathbb{C}^{m\times n}$ is orthonormal, and $R(0) \in \mathbb{C}^{n\times n}$ is upper triangular). We want to find, if possible, orthonormal Q and upper triangular R, both in $\mathcal{C}^k$, such that $A(t) = Q(t)R(t)$. Therefore, if feasible, differentiating $A = QR$ and $Q^*Q = I$ we get

$$\dot{A} = \dot{Q}R + Q\dot{R} \quad \text{and} \quad \dot{Q}^*Q + Q^*\dot{Q} = 0\ . \tag{2.2}$$

Hence $H := Q^*\dot{Q}$ is skew–Hermitian and from the first of (2.2) we get

$$\begin{aligned} \dot{R} &= Q^*\dot{A} - Q^*\dot{Q}R = Q^*\dot{A} - HR\,, \\ \dot{Q} &= \dot{A}R^{-1} - Q\dot{R}R^{-1} = (I - QQ^*)\dot{A}R^{-1} + QH\,. \end{aligned} \tag{2.3}$$

Since $\dot{R}$ has to be upper triangular, we require

$$\begin{aligned} (Q^*\dot{A})_{i,1} &= H_{i,1}R_{1,1}\ , && i \geq 2 \\ (Q^*\dot{A})_{i,2} &= H_{i,1}R_{1,2} + H_{i,2}R_{2,2}\ , && i \geq 3 \\ &\ \vdots \\ (Q^*\dot{A})_{n,n-1} &= H_{n,1}R_{1,n-1} + H_{n,2}R_{2,n-1} + \ldots + H_{n,n-1}R_{n-1,n-1}\ . \end{aligned} \tag{2.4}$$

[1] We call $Q \in \mathbb{C}^{m\times n}$ *orthonormal*, if $Q^*Q = I$ and *unitary*, if further $m = n$.

These are clearly solvable for $H_{i,j}$, $i > j$, assuming that $R_{1,1}, \ldots, R_{n-1,n-1}$ are nonzero. Then the skew–Hermitian property defines the part above the diagonal. Diagonal entries of H we set to purely imaginary values in such a way that the diagonal of $Q^*\dot{A} - HR$ becomes real[2]. This defines H as a smooth function of Q and R. Rewriting the differential equation for Q as

$$\dot{Q} = [(I - QQ^*)\dot{A}R^{-1}Q^* + QHQ^*]Q := S(Q,t)Q\,,$$

since $Q^*(S + S^*)Q = 0$, (2.3) and Lemmata 2.1 and 2.2 give the desired result as long as R stays invertible. On the other hand, if matrices Q and R satisfy the differential equations (2.3), and the initial condition $Q(0)R(0) = A(0)$, then they provide a QR factorization of A. Thus we get:

PROPOSITION 2.3. *Any full column rank $\mathcal{C}^k$–matrix has a $\mathcal{C}^k$ QR–decomposition.*

Remark 2.3. The case in which A is a solution of a linear system, $\dot{A}(t) = B(t)A(t)$, leads to some simplifications in formulas (2.3). In fact, $\dot{A} = BQR$ gives

$$\begin{aligned} \dot{R} &= (Q^*BQ - H)R\,, \\ \dot{Q} &= (I - QQ^*)BQ + QH\,. \end{aligned} \tag{2.5}$$

2.2. Schur decompositions. Next, we consider differential equations for Schur decomposition. We begin with the case of a Hermitian matrix.

(a) Diagonalization of a Hermitian matrix. Denote by $\mathbb{C}_H^{n\times n}$ the set of Hermitian $n\times n$ matrices. Let $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}_H^{n\times n})$. Suppose that a Schur decomposition $A(0) = Q(0)D(0)Q(0)^*$ is given, i.e., $Q(0)$ is unitary, and $D(0)$ is diagonal. We want to find –if possible– $Q, D \in \mathcal{C}^k$, unitary and diagonal, respectively, such that $Q^*(t)A(t)Q(t) = D(t)$ for all t. Here we write $D = \mathrm{diag}(d_1, \ldots, d_n)$. If feasible, we must have

$$\dot{D} = Q^*\dot{A}Q + DQ^*\dot{Q} + \dot{Q}^*QD\,,$$

or by letting $H = Q^*\dot{Q}$, and noticing that H is skew-Hermitian, $H^* = -H$, we get the following system for D and Q:

$$\begin{aligned} \dot{D} &= Q^*\dot{A}Q + DH - HD\,, \\ \dot{Q} &= Q\,H\,. \end{aligned} \tag{2.6}$$

From this it is immediate to obtain $\dot{d}_i = (Q^*\dot{A}Q)_{ii}$. Notice that (2.6) is really a system of differential-algebraic equations (DAEs), and we wish to use the algebraic equations (the part relative to the off diagonal entries in $\dot{D}$) to determine H. Because of skewness, H_{ii}'s have to be purely imaginary, otherwise arbitrary, e.g. zero. If the eigenvalues of A are distinct, then we get

$$H_{ij} = \frac{(Q^*\dot{A}Q)_{ij}}{d_j - d_i}\,,\ i \neq j\,. \tag{2.7}$$

So, from (2.6) and (2.7) we get the smooth Schur factorization of A if all eigenvalues of $A(t)$ are distinct for all t.

[2]With this choice of the diagonal of H we get the diagonal elements of R real if they are such for $R(0)$.

PROPOSITION 2.4. *Any Hermitian $\mathcal{C}^k$–matrix with simple eigenvalues is diagonalizable with a unitary $\mathcal{C}^k$–matrix .*

Under the previous assumptions, an obvious consequence of the Schur decomposition of a positive definite matrix is that we can very simply obtain smooth square roots of the matrix, once the decomposition $A = QDQ^*$ is at hand. It suffices to consider square roots of the eigenvalues d_i.

Remark 2.4. A construction similar to the one above can be done for the eigendecomposition of a general matrix $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n\times n})$ with distinct eigenvalues. Now we seek (given appropriate initial conditions $V(0)$ and $D(0)$) a smooth decomposition $A(t) = V(t)D(t)V(t)^{-1}$. By letting $P = V^{-1}\dot{V}$, we have

$$\begin{aligned} \dot{d}_i &= (V^{-1}\dot{A}V)_{ii}\,, && i = 1, \ldots, n, \\ \dot{V} &= V\,P\,, \quad P_{ij} = \frac{(V^{-1}\dot{A}V)_{ij}}{d_j - d_i}\,, && i \neq j\,. \end{aligned} \tag{2.8}$$

The diagonal entries P_{ii} are not uniquely determined, and we may set them to 0. Another choice is to require the columns of V to have constant norms, which means $\mathrm{Re}(v_i^* \dot{v}_i) = 0$ giving the condition

$$\mathrm{Re}(v_i^* v_i\, P_{ii}) = -\sum_{j\neq i} \mathrm{Re}(v_i^* v_j\, P_{ji}).$$

Again, the factors V and D are as smooth as A.

(b) Schur decomposition of a general matrix. Let $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n\times n})$. For given $U(0)$ and $R(0)$, respectively unitary and upper triangular, such that $R(0) = U^*(0)A(0)U(0)$, we seek unitary U and triangular R, as smooth as A, such that $R(t) = U^*(t)A(t)U(t)$, for all t. If feasible, we then must have

$$\dot{R} = U^*\dot{A}U + \dot{U}^*AU + U^*A\dot{U} = U^*\dot{A}U + (\dot{U}^*U)U^*AU + U^*AU(U^*\dot{U}).$$

Again we set $H := U^*\dot{U}$ and obtain the system

$$\begin{aligned} \dot{R} &= U^*\dot{A}U + RH - HR\,, \\ \dot{U} &= U\,H\,. \end{aligned} \tag{2.9}$$

Conditions that R is upper triangular and $H^* = -H$ bring this to a DAE system. We use the strictly lower triangular part in the first equation of (2.9) to determine H (except for its diagonal). This can be done by first finding the vector $(H_{21}, \ldots, H_{n1})^*$, then $(H_{32}, \ldots, H_{n2})^*$, etc. up to $H_{n,n-1}$. To find these vectors one needs to solve triangular systems which are easily seen to be nonsingular if $R_{ii}(t) \neq R_{jj}(t)$, $i \neq j$. Then, by solving (2.9), we get R and U as smooth as A. The entries $H_{ii}(t)$ are undetermined; we may set them to 0 (in any case, they must be purely imaginary).

Together with Remark 2.4 we get:

PROPOSITION 2.5. *Any $\mathcal{C}^k$–matrix with simple eigenvalues has a $\mathcal{C}^k$–Schur decomposition and is diagonalizable with a $\mathcal{C}^k$–matrix .*

Clearly, the differential equation model above breaks down if some of the eigenvalues of $A(t)$ coalesce. Besides, also for simple eigenvalues, numerical difficulties can be expected in case two or more eigenvalues become close. In such cases, it may be better to compute the invariant subspaces relative to a cluster of eigenvalues.

(c) Block-Schur decomposition. Consider $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n\times n})$. We want to find unitary Q and block upper triangular S, as smooth as A, such that $Q^*(t)A(t)Q(t) = S(t)$, where S is partitioned as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} .$$

We start with a given decomposition $A(0) = Q(0)S(0)Q(0)^*$. Assume that $\lambda_1(t), \dots, \lambda_n(t) \in \Lambda(A(t))$ are continuous such that $\Lambda_1(t) = \{\lambda_1(t), \dots, \lambda_m(t)\}$ and $\Lambda_2(t) = \{\lambda_{m+1}(t), \dots, \lambda_n(t)\}$ are disjoint for all t and $\Lambda(S_{jj}(0)) = \Lambda_j(0)$, $j = 1, 2$.

Differentiating the relation $S = Q^*AQ$ and letting $H := Q^*\dot{Q}$ we obtain the system of DAE's

$$\begin{aligned} \dot{S} &= Q^*\dot{A}Q + SH - HS \ , \\ \dot{Q} &= Q\,H \ , \\ S_{21} &= 0\,, \quad H^* = -H \ . \end{aligned} \tag{2.10}$$

This can be reduced to a system of DEs by eliminating the algebraic part. By rewriting the first equation of (2.10) in block form

$$\begin{bmatrix} \dot{S}_{11} & \dot{S}_{12} \\ 0 & \dot{S}_{22} \end{bmatrix} = \begin{bmatrix} (Q^*\dot{A}Q)_{11} & (Q^*\dot{A}Q)_{12} \\ (Q^*\dot{A}Q)_{21} & (Q^*\dot{A}Q)_{22} \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ -H_{12}^* & H_{22} \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ -H_{12}^* & H_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} ,$$

we realize that we must have

$$S_{22}H_{12}^* - H_{12}^*S_{11} = (Q^*\dot{A}Q)_{21}\,. \tag{2.11}$$

Thus, $H_{12} \in \mathbb{C}^{m\times(n-m)}$ is the unique solution of (2.11), since S_{11} and S_{22} have no common eigenvalue (e.g., see [7]). The blocks H_{11} and H_{22} are not uniquely determined, and we may set them both to 0 (in any case, they must be skew-Hermitian). Thus, taking $H = \begin{bmatrix} 0 & -H_{12}^* \\ -H_{12} & 0 \end{bmatrix}$, and the differential equations (2.10) for Q, S_{11}, S_{12}, S_{22} we obtain the desired result.

The above procedure immediately generalizes to a block-Schur factorization in p blocks. That is, one seeks the factorization where S is partitioned as

$$S = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1p} \\ 0 & S_{22} & \dots & S_{2p} \\ & & \ddots & \vdots \\ 0 & 0 & & S_{pp} \end{bmatrix} , \tag{2.12}$$

and each diagonal block is a square matrix. With obvious notation, block-partitioning $H = Q^*\dot{Q}$ conformally to S, and assuming that $\Lambda(S_{ii}(t)) \cap \Lambda(S_{jj}(t)) = \emptyset$ for all t and $i \neq j$, we obtain H by solving the system for the H_{ji}:

$$\begin{aligned} &\sum_{k=i}^{p} S_{ik}H_{jk}^* - \sum_{k=1}^{j} H_{ki}^*S_{kj} = (Q^*\dot{A}Q)_{ij} \\ &\qquad\qquad j = 1, \dots, p-1, \quad i = j+1, \dots, p\,. \end{aligned} \tag{2.13}$$

The blocks H_{ii} are not uniquely determined and we may set them to 0. In summary, we have:

PROPOSITION 2.6. *Any $\mathcal{C}^k$–matrix with disjoint groups of eigenvalues has a $\mathcal{C}^k$–block Schur decomposition, the blocks corresponding to these groups.*

Remark 2.5. From the practical point of view, the strongest assumption we made is that the initial decomposition determines a correct blocking for all t.

In case A is Hermitian, the above block-Schur decomposition gives a block diagonal form. Then (2.13) becomes $S_{ii}H_{ji}^* - H_{ji}^*S_{jj} = (Q^*\dot{A}Q)_{ij}$, $i \neq j$. Here, since the initial conditions for S_{ii} are Hermitian, we will have $S_{ii}(t)$ Hermitian for all t, for every skew–Hermitian H_{ii}, because $\dot{S}_{ii} = (Q^*\dot{A}Q)_{ii} + S_{ii}H_{ii} - H_{ii}S_{ii}$.

Remark 2.6. Of course, a block diagonalization of general matrices can also be envisioned similarly to our previous Remark 2.4, and see also [15]. That is, if feasible, we now seek block-diagonal $D = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_p \end{bmatrix}$ and invertible V, both as smooth as A, such that $A(t) = V(t)D(t)V^{-1}(t)$. Differentiating this, letting $P = V^{-1}\dot{V}$, and partitioning P conformally to D, we get

$$\begin{aligned} \dot{D}_i &= (V^{-1}\dot{A}V)_{ii} + D_iP_{ii} - P_{ii}D_i\,, \quad i = 1, \dots, p\,, \\ 0 &= (V^{-1}\dot{A}V)_{ij} + D_iP_{ij} - P_{ij}D_j\,, \quad i \neq j\,, \\ \dot{V} &= VP\;. \end{aligned} \tag{2.14}$$

If $\Lambda(D_i(t)) \cap \Lambda(D_j(t)) = \emptyset$, $i \neq j$, the second of these equations can be solved for the P_{ij}. P_{ii} are not uniquely determined, and we may set them to 0, or to require $V_i^*V_i$ to be constant, where $[V_1 \dots V_p]$ is the partitioning of V in column blocks. This amounts to

$$P_{ii}V_i^*V_i + V_i^*V_iP_{ii} = -\sum_{i\neq j}(P_{ji}^*V_j^*V_i + V_i^*V_jP_{ji})\;,$$

which is uniquely solvable for P_{ii}, since V_i has full rank.

2.3. Singular value decompositions. Next we consider the SVD of a matrix. Again, we also consider block analogs.

(a) Smooth SVD. This has been recently considered in [3] and [16] in the analytic case. We have a matrix $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$, $m \geq n$, and look for $\mathcal{C}^k$ unitary U and V, and real "diagonal" Σ such that $A(t) = U(t)\Sigma(t)V^*(t)$. Here $\Sigma = \begin{bmatrix} S \\ 0 \end{bmatrix}$, $S = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$. We assume that $0 \neq \sigma_i(t) \neq \sigma_j(t)$, $i \neq j$, for all t. Let an initial SVD: $A(0) = U(0)\Sigma(0)V^*(0)$ be given. Differential equations for the factors are derived as follows. From $A = U\Sigma V^*$, we get

$$U^*\dot{A}V \;=\; (U^*\dot{U})\,\Sigma + \dot{\Sigma} + \Sigma\,(\dot{V}^*V)\,.$$

Let us set

$$H := U^*\dot{U} \in \mathbb{C}^{m\times m}\,, \quad K := V^*\dot{V} \in \mathbb{C}^{n\times n}\,,$$

both of which are skew–Hermitian. Therefore, we obtain the system of DAEs

$$\begin{aligned} \dot{\Sigma} &= U^*\dot{A}V - H\Sigma + \Sigma K\,, \\ \dot{U} &= UH\,, \\ \dot{V} &= VK\,, \end{aligned} \tag{2.15}$$

with the requirement that Σ is real and diagonal. We proceed to eliminate the algebraic part, thereby reducing the above to a system of DEs. First we notice that the equations for the singular values are all decoupled:

$$\dot\sigma_i = (U^*\dot AV)_{ii} - H_{ii}\sigma_i + \sigma_i K_{ii}\,, \quad i = 1,\dots,n\,.$$

Next, consider the algebraic part of the first equation in (2.15). For $i \neq j$, $i,j = 1,\dots,n$ we must have

$$0 = (U^*\dot AV)_{ij} - H_{ij}\sigma_j + \sigma_i K_{ij}$$
$$0 = (U^*\dot AV)_{ji} - H_{ji}\sigma_i + \sigma_j K_{ji}\,,$$

from which – using skewness of H and K – we get

$$H_{ij} = \frac{\sigma_j(U^*\dot AV)_{ij} + \sigma_i(U^*\dot AV)_{ji}}{\sigma_j^2 - \sigma_i^2}$$
$$K_{ij} = \frac{\sigma_j(U^*\dot AV)_{ji} + \sigma_i(U^*\dot AV)_{ij}}{\sigma_j^2 - \sigma_i^2}$$

$i,j = 1,\dots,n,\ i \neq j$. Also, it is easy to obtain

$$H_{ij} = -\bar H_{ji} = \frac{(U^*\dot AV)_{ij}}{\sigma_j}\,, \quad i = n+1,\dots,m,\ j = 1,\dots,n\,. \tag{2.16}$$

For $\dot\sigma_i$ to be real the diagonal elements of H and K need to satisfy:

$$(H_{ii} - K_{ii}) = \mathrm{Im}((U^*\dot AV)_{ii})/\sigma_i\ .$$

We can choose, e.g., $K_{ii} = 0$, which then determines H_{ii} .

Finally, the bottom right $(m-n)\times(m-n)$ block of H is not determined; one may thus set it to 0 (or any other skew matrix). These give the desired result:

PROPOSITION 2.7. *Any full column rank $\mathcal{C}^k$–matrix with distinct singular values has a $\mathcal{C}^k$ singular value decomposition.*

In case A is the solution of a linear system: $\dot A(t) = B(t)A(t)$ with $A(0)$ full rank, the above formulas simplify by $\dot A = BU\Sigma V^*$. The details are easy and one obtains

$$\dot\sigma_i = (U^*BU)_{ii}\sigma_i - H_{ii}\sigma_i + \sigma_i K_{ii}\ , \qquad i = 1,\dots,n$$

$$H_{ij} = \frac{\sigma_j^2(U^*BU)_{ij} + \sigma_i^2(U^*BU)_{ji}}{\sigma_j^2 - \sigma_i^2}\ , \qquad i,j = 1,\dots,n,\ i \neq j$$

$$H_{ij} = (U^*BU)_{ij}\ , \qquad i = n+1,\dots,m,\ j = 1,\dots,n$$

$$K_{ij} = \sigma_i\sigma_j\frac{(U^*BU)_{ji} + (U^*BU)_{ij}}{\sigma_j^2 - \sigma_i^2}\ , \qquad i,j = 1,\dots,n,\ i \neq j\,.$$

It should be noticed that the matrix V does not need to be computed if only information on the singular values is desired.

Remark 2.7. It should be clear that the U and V factors of the SVD of A are nothing but unitary Schur factors of the positive semidefinite matrices AA^* and A^*A, respectively. As usual, we can think of the square part of the Σ matrix as a square root of the block diagonal matrix $V^*(A^*A)V$.

(b) Block-SVD decomposition. Consider next a full rank matrix $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$. We want to find unitary U, V, and "block-diagonal" Σ, as smooth as A, such that $U^*(t)A(t)V(t) = \Sigma(t)$. Here $\Sigma(t)$ is partitioned as

$$\Sigma = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \\ 0 & 0 \end{bmatrix},$$

where S_1, S_2 are Hermitian positive definite matrices. We assume that the (positive) singular values of A can be arranged into two disjoint groups, like the eigenvalues in the block Schur decomposition above, and that initial conditions $U(0), \Sigma(0), V(0)$ are given. We proceed as usual, by differentiating the relation $U^*AV = \Sigma$. Denoting the skew-Hermitian matrices $H = U^*\dot{U}$, and $K = V^*\dot{V}$ we end up with the following equations

$$\begin{bmatrix} \dot{S}_1 & 0 \\ 0 & \dot{S}_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (U^*\dot{A}V)_{11} & (U^*\dot{A}V)_{12} \\ (U^*\dot{A}V)_{21} & (U^*\dot{A}V)_{22} \\ (U^*\dot{A}V)_{31} & (U^*\dot{A}V)_{32} \end{bmatrix} + \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ -K_{12}^* & K_{22} \end{bmatrix}$$
$$- \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ -H_{12}^* & H_{22} & H_{23} \\ -H_{13}^* & -H_{23}^* & H_{33} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \\ 0 & 0 \end{bmatrix}.$$

From the algebraic part in these equations we obtain

$$-H_{13}^*S_1 = (U^*\dot{A}V)_{31},$$
$$-H_{23}^*S_2 = (U^*\dot{A}V)_{32},$$

and

$$\begin{bmatrix} H_{12} \\ K_{12} \end{bmatrix} S_2 - \begin{bmatrix} 0 & S_1 \\ S_1 & 0 \end{bmatrix} \begin{bmatrix} H_{12} \\ K_{12} \end{bmatrix} = \begin{bmatrix} (U^*\dot{A}V)_{12} \\ ((U^*\dot{A}V)_{21})^* \end{bmatrix},$$

and thus we can uniquely determine H_{13} and H_{23} if $0 \notin \Lambda(S_1)$, $0 \notin \Lambda(S_2)$, and H_{12}, K_{12} if $\Lambda(S_1) \cap \Lambda(S_2) = \emptyset$. Note that in this full rank case we can consider the "standard" block SVD with S_1 and S_2 positive definite, since singular values do not become 0. H_{33} is an arbitrary skew-Hermitian matrix, and we may set it to 0. For blocks H_{ii}, K_{ii}, $i = 1, 2$, we can reason as follows. The differential equations for S_1 and S_2 are

$$\dot{S}_i = (U^*\dot{A}V)_{ii} - H_{ii}S_i + S_iK_{ii}, \; i = 1, 2,$$

and we want S_1 and S_2 to be Hermitian. Thus, we must have

$$S_i(H_{ii} - K_{ii}) + (H_{ii} - K_{ii})S_i = (U^*\dot{A}V)_{ii} - (U^*\dot{A}V)_{ii}^*, \; i = 1, 2,$$

and hence $H_{ii} - K_{ii}$, $i = 1, 2$ is uniquely determined and skew-Hermitian if $0 \notin \Lambda(S_i)$ (for example, can take $K_{11} = K_{22} = 0$). In summary, we obtain the desired smoothness for the block SVD, as long as A is full rank and the blocks S_1 and S_2 have disjoint spectra. Hence:

PROPOSITION 2.8. *Any full column rank $\mathcal{C}^k$–matrix with disjoint groups of singular values has a $\mathcal{C}^k$–block singular value decomposition.*

If a square A satisfies a linear system: $\dot{A} = BA$, the above relations simplify to

$$(2.17) \qquad \begin{aligned} (U^*BU)_{12}S_2 &= H_{12}S_2 - S_1K_{12}\,, \\ S_1(U^*BU)^*_{21} &= K_{12}S_2 - S_1H_{12}\,, \end{aligned}$$

and for $i = 1, 2$,

$$(2.18) \qquad \begin{aligned} \dot{S}_i &= (U^*BU)_{ii}S_i - H_{ii}S_i + S_iK_{ii} \\ S_i(H_{ii} - K_{ii}) + (H_{ii} - K_{ii})S_i &= (U^*BU)_{ii}S_i - S_i(U^*BU)^*_{ii}\,. \end{aligned}$$

The V factor needs not to be computed if only the blocks S_1 and S_2 are desired.

(c) Polar Factorization. With similar notation as above, we now consider differential equations for the polar factorization of a full rank $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$. That is, we want to write $A(t) = Q(t)P(t)$, where Q is orthonormal, P is Hermitian positive definite, and Q, P are as smooth as A. Differentiating $A = QP$, using orthonormality of Q, and letting as usual $H = Q^*\dot{Q}$, we obtain

$$(2.19) \qquad \begin{aligned} \dot{Q} &= QH\,, \\ \dot{P} &= Q^*\dot{A} - HP\,. \end{aligned}$$

Since we need $P = P^*$, we then must have

$$(2.20) \qquad PH + HP = Q^*\dot{A} - \dot{A}^*Q\,.$$

Since P is positive definite, (2.20) has a unique solution, H. Thus, we obtain the desired result since H satisfying (2.20) is skew-Hermitian.

Remark 2.8. Using (2.19) and (2.20) to obtain a smooth polar factorization of A, in contrast to passing through the smooth SVD, is not hampered by the need of noncoalescing singular values.

3. Extensions.

In this section, we extend the smooth factorizations to the case in which some singular behavior is encountered, such as rank deficiency, or eigenvalues' coalescing.

3.1. Rank deficient QR.

We begin with the case of the QR factorization of a matrix $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$. As we saw in §2, if A has full rank, then it admits a QR factorization where the Q and R factors are also $\mathcal{C}^k$. Next, we show that, under appropriate assumptions, A admits a QR factorization also in case it is rank deficient; however, some loss of differentiability may take place. Then we show that analytic (denote $\mathcal{C}^\omega$) matrices have analytic QR factorizations.

Our proof is based on a careful analysis of the Gram–Schmidt orthogonalization process. The idea behind the proof is that in order for the function A to have a QR factorization (with some degree of smoothness), then, at points where loss of rank occurs, the matrix obtained by differentiating the columns of A (possibly, derivatives of different orders for different columns) must have full rank.

For the $\mathcal{C}^k$ case, the following example is helpful in understanding what we should expect.

Example 3.1. Let $A = \begin{bmatrix} t^k|t| \\ t^d \end{bmatrix}$, $d, k \in \mathbb{N}$. Then A is k times continuously differentiable. Depending whether $d \le k$ or $d = k+1$ we have the QR–decompositions

$$A(t) = \begin{bmatrix} t^{k-d}\,|t|\,/\sqrt{1+t^{2(k-d+1)}} \\ 1/\sqrt{1+t^{2(k-d+1)}} \end{bmatrix}\; t^d\sqrt{1+t^{2(k-d+1)}}\;, \quad A(t) = \begin{bmatrix} \mathrm{sgn}(t)/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\; t^d\sqrt{2}\;,$$

respectively. In the first case Q is in $\mathcal{C}^{k-d}$, while in the latter it is not (and cannot be taken) continuous. Note that in the first case $A(t)^T A(t) = t^{2d}\,(t^{2(k-d+1)}+1)$, in particular $\lim_{t\to 0}\frac{1}{t^{2d}}A(t)^T A(t) = 1 > 0$.

THEOREM 3.1. *Let* $A \in \mathcal{C}^k(\mathbb{R},\mathbb{C}^{m\times n})$, $m \geq n$, *and assume* $d \leq k$ *is such that*[3]

$$\limsup_{\tau\to 0}\frac{1}{\tau^{2d}}\det(A^*A)(t+\tau) > 0 \tag{3.1}$$

for every t. *Given any QR-factorization* $A(t_0)=Q(t_0)R(t_0)$ *at a point* t_0 *where* A *has full rank, then there exists a* $\mathcal{C}^{k-d}$ *QR-factorization of* A *satisfying this initial condition. If* $R(t_0)$ *has real diagonal, this QR-factorization becomes unique, if we require the diagonal of* R *to be real.*

Proof. Write $A(t)=[a_1(t)\dots a_n(t)]$. If A has full rank at t, then $Q=[q_1\dots q_n]$ necessarily satisfies

$$\begin{aligned}
q_1(t) &= \frac{\eta_1(t)}{|a_1(t)|}\,a_1(t)\,, && P_1 = I - q_1q_1^*\\
q_2(t) &= \frac{\eta_2(t)}{|P_1(t)a_2(t)|}\,P_1(t)a_2(t)\,, && P_2 = P_1 - q_2q_2^*\\
&\vdots\\
q_j(t) &= \frac{\eta_j(t)}{|P_{j-1}(t)a_j(t)|}\,P_{j-1}(t)a_j(t)\,, && P_j = P_{j-1} - q_jq_j^*\ .
\end{aligned}$$

Here η_j's have to satisfy $|\eta_j(t)| = 1$ for all j, otherwise they can be chosen freely. Taking them smooth on intervals of full rank gives us Q and $R = Q^*A$ both $\mathcal{C}^k$ on these intervals. Further, real η_j is equivalent to real $R_{j,j}$, there.

Our assumption implies that the points where A does not have full rank are isolated.

The details of the proof for the case in which there is some possible loss of smoothness are based on the following

Claim: If $\hat t$ and $j\in\{0,1,\dots,k-1\}$ are such that $\limsup\limits_{\tau\to 0}\frac{\det(A^*A)(\hat t+\tau)}{\tau^{2j}} = 0$, then $\lim\limits_{\tau\to 0}\frac{\det(A^*A)(\hat t+\tau)}{\tau^{2(j+1)}}$ exists and is finite.

Proof of the claim. Case $j=0$: we can assume $\hat t = 0$, so that $\det(A(0)^*A(0)) = 0$. Applying a permutation to the columns of A we may also assume that $A(0) = [\,a_1^0 \quad A^0_{2\dots n}\,]$, where $a_1^0 = A^0_{2\dots n}\beta$ for some $\beta\in\mathbb{C}^{n-1}$. Then

$$A(t) = [\,A^0_{2\dots n}\beta + t\,a_1^\Delta(t) \quad A^0_{2\dots n} + t\,A^\Delta_{2\dots n}(t)\,]\ ,$$

where $[\,a_1^\Delta(t) \quad A^\Delta_{2\dots n}(t)\,] = \frac{1}{t}\,(A(t)-A(0))$. Then

$$\begin{aligned}
\det(A(t)^*A(t)) &= \det\!\left(\begin{bmatrix}1 & -\beta^*\\ 0 & I\end{bmatrix}\begin{bmatrix}\beta^*{A^0_{2\dots n}}^* + t\,a_1^\Delta(t)^*\\ {A^0_{2\dots n}}^* + t\,A^\Delta_{2\dots n}(t)^*\end{bmatrix}\right.\\
&\qquad\left.[\,A^0_{2\dots n}\beta + t\,a_1^\Delta(t) \quad A^0_{2\dots n} + t\,A^\Delta_{2\dots n}(t)\,]\begin{bmatrix}1 & 0\\ -\beta & I\end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} t\,(a_1^\Delta(t)^* - \beta^*A^\Delta_{2\dots n}(t)^*)\\ {A^0_{2\dots n}}^* + t\,A^\Delta_{2\dots n}(t)^*\end{bmatrix}[\,t\,(a_1^\Delta(t) - A^\Delta_{2\dots n}(t)\beta) \quad A^0_{2\dots n} + t\,A^\Delta_{2\dots n}(t)\,]\right)\\
&= t^2\det(\tilde A(t)^*\tilde A(t))\ ,
\end{aligned}$$

[3] This means that the limsup is either positive or $+\infty$

where $\tilde{A}(t) = [\, a_1^{\Delta}(t) - A_{2\ldots n}^{\Delta}(t)\beta \quad A_{2\ldots n}(t)\,]$. Hence the claim for $j = 0$ follows. Since[4], $\tilde{A} \in \mathcal{C}^{k-1}$ the proof for general j is completed by obvious induction.

Let $\hat{t}$ be a point where A does not have full rank, and let $\hat{d}$ be the smallest integer for which

$$\limsup_{\tau\to 0} \frac{1}{\tau^{2\hat{d}}} \det(A^*A)(\hat{t}+\tau) > 0 \ .$$

Denote $A_j(t) = [\, a_1(t) \quad \ldots \quad a_j(t)\,]$. Using Fischer's inequality:

$$\det([\, B_1 \quad B_2\,]^* [\, B_1 \quad B_2\,]) \le \det(B_1^* B_1)\det(B_2^* B_2)$$

and the claim above, we find for every $j = 1, \ldots, n$ a $d_j \le \hat{d}$ such that

$$\lim_{\tau\to 0} \frac{1}{\tau^{2d_j}} \det(A_j^* A_j)(\hat{t}+\tau)$$

is finite and positive. Also $d_j \le d_{j+1}$. Set $e_1 = d_1$, $e_j = d_j - d_{j-1}$, $j > 1$.

Expand the first column of A:

$$a_1(\hat{t}+\tau) = \hat{a}_1 + \tau\, \hat{a}_1^{(1)} + \frac{\tau^2}{2!}\, \hat{a}_1^{(2)} + \cdots + \frac{\tau^{k-1}}{(k-1)!}\, \hat{a}_1^{(k-1)} + \frac{\tau^k}{k!}\, \tilde{a}_1^{(k)} \ ,$$

where $\hat{a}_1^{(i)} = \frac{d^i}{dt^i} a_1(t)\big|_{t=\hat{t}}$ and the entries of $\tilde{a}_1^{(k)}$ are the k^{th} derivatives of elements of a_1 at some points between $\hat{t}$ and $\hat{t}+\tau$.

Now, since $\lim_{\tau\to 0} \frac{1}{\tau^{2d_1}} \left| a_1(\hat{t}+\tau)\right|^2$ is positive, we have $\hat{a}_1 = \hat{a}_1^{(1)} = \cdots = \hat{a}_1^{(d_1-1)} = 0$ and $\hat{a}_1^{(d_1)} \neq 0$, and

$$\begin{aligned} q_1(t) &= \eta_1(\hat{t}+\tau)\, \frac{\frac{\tau^{d_1}}{d_1!}\, \hat{a}_1^{(d_1)} + \cdots + \frac{\tau^k}{k!}\, \tilde{a}_1^{(k)}}{\left| \frac{\tau^{d_1}}{d_1!}\, \hat{a}_1^{(d_1)} + \cdots + \frac{\tau^k}{k!}\, \tilde{a}_1^{(k)} \right|} \\ &= \eta_1(\hat{t}+\tau)\, \mathrm{sgn}(\tau)^{d_1}\, \frac{\frac{1}{d_1!}\, \hat{a}_1^{(d_1)} + \cdots + \frac{\tau^{k-d_1}}{k!}\, \tilde{a}_1^{(k)}}{\left| \frac{1}{d_1!}\, \hat{a}_1^{(d_1)} + \ldots + \frac{\tau^{k-d_1}}{k!}\, \tilde{a}_1^{(k)} \right|} \ . \end{aligned}$$

Choosing $\eta_1(\hat{t}+\tau)\,\mathrm{sgn}(\tau)^{d_1}$ smooth, i.e., changing the sign of η_1 at $\hat{t}$ whenever d_1 is odd, we have q_1 in $\mathcal{C}^{k-d_1}$ in a neighborhood of $\hat{t}$.

For $j \ge 2$ note first that $I - P_{j-1}(t)$ is the orthogonal projection onto the range of $A_{j-1}(t)$ so that

$$\begin{aligned} &\det([\, A_{j-1} \quad a_j\,]^* [\, A_{j-1} \quad a_j\,]) \\ &\quad = \det([\, (I-P_{j-1})A_{j-1} \quad P_{j-1}a_j\,]^* [\, (I-P_{j-1})A_{j-1} \quad P_{j-1}a_j\,]) \\ &\quad = \det\left(\begin{bmatrix} A_{j-1}^* A_{j-1} & 0 \\ 0 & (P_{j-1}a_j)^* P_{j-1}a_j \end{bmatrix} \right) = \det(A_{j-1}^* A_{j-1})\, \|P_{j-1}a_j\|^2 \end{aligned}$$

and $\lim_{\tau\to 0} \frac{1}{\tau^{e_j}} \left\| (P_{j-1}a_j)(\hat{t}+\tau) \right\|$ is finite and positive. Since $P_{j-1}a_j \in \mathcal{C}^{k-d_{j-1}}$ it follows that

$$(P_{j-1}a_j)(\hat{t}+\tau) = \tau^{e_j}\, \hat{b}_j + O(\tau^{e_j+1})$$

[4] Generally: if $f \in \mathcal{C}^k$ and $f(t) = f(0) + t\, f'(0) + \ldots + \frac{t^{d-1}}{(d-1)!}\, f^{(d-1)}(0) + \frac{t^d}{d!}\, g(t)$, $d \le k$, then $g(t) = d \int_0^1 (1-s)^{d-1}\, f^{(d)}(ts)\, ds$, which shows that $g \in \mathcal{C}^{k-d}$.

with $\hat{b}_j \neq 0$. Hence

$$q_j(\hat{t}+\tau) = \eta_j(\hat{t}+\tau)\,\mathrm{sgn}(\tau)^{e_j}\,\frac{\hat{b}_j + O(\tau)}{|\hat{b}_j + O(\tau)|}\ .$$

Again changing the sign of η_j at $\hat{t}$ whenever e_j is odd, we have q_j and consequently also $P_j = P_{j-1} - q_j q_j^*$ in $\mathcal{C}^{k-d_j}$, in a neighborhood of $\hat{t}$.

This way we get the whole Q and hence also R in $\mathcal{C}^{k-d}$.

The uniqueness follows from the fact that the sign changes in η_j's are also necessary for smoothness. ∎

Remark 3.1. If $m = n$, then in the previous theorem it suffices that the matrix A_{n-1} satisfies the assumptions. This is because the n^{th} column of Q is then determined (up to η_n with $|\eta_n| = 1$) by the previous columns.

Remark 3.2. The matrix $A = \begin{bmatrix} t^k|t| & 1 \\ t^d & 0 \end{bmatrix}$, $d \leq k+1$, see Example 3.1, shows that no extra smoothness can be generally expected for the R factor.

In the case of a real analytic matrix we don't need any extra assumptions (a different proof of this result is in [6]):

THEOREM 3.2. *Any* $A \in \mathcal{C}^\omega(\mathbb{R}, \mathbb{R}^{m\times n})$, $m \geq n$, *has a* $\mathcal{C}^\omega$ QR *decomposition. Similarly for* $A \in \mathcal{C}^\omega(\mathbb{R}, \mathbb{C}^{m\times n})$.

Proof. The proof is by induction with respect to n .
Assume $n = 1$. If $A \equiv 0$ set $Q = e_1$, $R = 0$. Else all the zeroes of A^*A are of finite order and the proof of Theorem 3.1 gives us an analytic QR decomposition.
Assume the assertion is true for all $m \times n$ matrices with $m > n$. Then for analytic $A = [\,A_n \quad a_{n+1}\,]$ take an analytic QR decomposition $A_n = Q_n R_n$ and set $r = Q_n^* a_{n+1}$, $b = a_{n+1} - Q_n r$. If $b \equiv 0$ take an analytic unit q_{n+1} such that $Q_n^* q_{n+1} \equiv 0$ and set $Q = [\,Q_n \quad q_{n+1}\,]$,$R = \begin{bmatrix} R_n & r \\ 0 & 0 \end{bmatrix}$. Else take q_{n+1} to be the analytic unit vector in the direction of b as in the proof of Theorem 3.1. ∎

3.2. Schur decomposition of a Hermitian matrix with multiple eigenvalues. Consider again the Schur decomposition of $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}_H^{n\times n})$. As we saw in §2.2, if the eigenvalues of A are simple then A admits a smooth Schur decomposition. Under reasonable assumptions, we show next that A admits a Schur decomposition with some possible loss of smoothness also in case some of its eigenvalues coalesce.

From the book of Kato ([12]) we know that Hermitian real analytic matrices have analytic eigendecompositions. Also, it is true (a theorem of Rellich, see [13]) that $\mathcal{C}^1$ Hermitian matrices have $\mathcal{C}^1$ eigenvalues. Generally, however, even $\mathcal{C}^\infty$ Hermitian matrices don't have $\mathcal{C}^2$ eigenvalues or continuous Schur decompositions as the following example shows.

Example 3.2 (Wasow). Let α, $\beta > 0$ and

$$A(t) = \begin{bmatrix} e^{-(\alpha+\beta)/|t|} & e^{-\beta/|t|}\,\sin(1/t) \\ e^{-\beta/|t|}\,\sin(1/t) & -e^{-(\alpha+\beta)/|t|} \end{bmatrix}\ .$$

Then $A \in \mathcal{C}^\infty$. The eigenvalues are

$$\lambda_\pm(t) = \pm e^{-\beta/|t|}\,\left(e^{-2\alpha/|t|} \ + \ \sin^2(1/t)\right)^{\frac{1}{2}}\ .$$

These are in $\mathcal{C}^1$ but not in $\mathcal{C}^2$, if $\alpha \geq \beta$. Notice that $\lambda_+(t) - \lambda_-(t) = o(|t|^d)$, $\forall d$.

Here we show that the unitary diagonalization can be made smooth, provided that the order of coalescing of the eigenvalues is not more than k.

Denote by W_e the class of matrices depending on a real parameter, for which the eigenvalues do not have contacts of order higher than e. More precisely; if an $n \times n$ matrix $A \in W_e$ and $\lambda_1(t), \ldots, \lambda_n(t)$ are the continuous eigenvalues of $A(t)$, then for any t and $i \neq j$ we have

$$\liminf_{\tau \to 0} \frac{|\lambda_i(t+\tau) - \lambda_j(t+\tau)|}{|\tau^e|} \in (0, \infty] \ . \tag{3.2}$$

THEOREM 3.3. *Any Hermitian matrix $A \in \mathcal{C}^k \cap W_e$, $e \leq k$, has a $\mathcal{C}^{k-e}$ Schur decomposition.*

Proof. The proof is by induction on e. Proposition 2.4 gives the theorem for $e = 0$ (i.e., for distinct eigenvalues).

Assume now that the assertion is true for W_{e-1}–matrices and let $A \in \mathcal{C}^k \cap W_e$. We show first that A has a $\mathcal{C}^{k-e}$ decomposition in a neighborhood of any $\hat{t}$. We may assume that $\hat{t} = 0$. Let λ_0 be a p–fold eigenvalue of $A(0)$ and let Q_0 be a $\mathcal{C}^k$ orthonormal $n \times p$ matrix obtained as one column block of a block Schur decomposition (see Prop. 2.6) corresponding to the eigenvalues close to λ_0. So, Q_0 is defined in a neighborhood of 0 such that

$$Q_0(t)^* A(t) Q_0(t) = \lambda_0 I + t\, S_1(t) \ ,$$

where $S_1 \in \mathcal{C}^{k-1}$. The eigenvalues of $A(t)$ near λ_0 are of the form $\lambda_0 + t\,\mu(t)$, where $\mu(t)$ is an eigenvalue of $S_1(t)$. If $\tilde{\mu}(t)$ is another eigenvalue of $S_1(t)$, we have

$$\frac{|\mu(t) - \tilde{\mu}(t)|}{|t^{e-1}|} = \frac{|\lambda(t) - \tilde{\lambda}(t)|}{|t^e|}$$

where $\lambda(t)$, $\tilde{\lambda}(t)$ are eigenvalues of $A(t)$. Hence $S_1 \in \mathcal{C}^{k-1} \cap W_{e-1}$ so that by induction hypothesis it has a $\mathcal{C}^{k-1-(e-1)} = \mathcal{C}^{k-e}$ Schur decomposition. This is true for all blocks corresponding to different eigenvalues of $A(0)$ and hence for A.

The assumption implies that the points where $A(t)$ has multiple eigenvalues are isolated. Cover $\mathbb{R}$ with a countable set of intervals $I_j = (\alpha_j, \beta_j)$ such that on each of these A has a $\mathcal{C}^{k-e}$ Schur decomposition $A(t) = U_j(t)\Lambda_j(t)U_j(t)^*$, $I_{j-1} \cap I_{j+1} = \emptyset$ for all $j \in \mathbb{Z}$, and A has simple eigenvalues on each (α_j, β_{j-1}).

Set $U = U_0$, $\Lambda = \Lambda_0$, on $[\beta_{-1}, \alpha_1]$, and $\Pi_0 = I$. For $j = 1, 2, \ldots$ let $\tau_j = \frac{1}{2}(\alpha_j + \beta_{j-1})$. Since $A(\tau_j)$ has simple eigenvalues, there exists a permutation matrix Π_j such that

$$\Pi_j^T \Lambda_j(\tau_j) \Pi_j = \Pi_{j-1}^T \Lambda_{j-1}(\tau_j) \Pi_{j-1} \ .$$

Take $D_j \in \mathcal{C}^{k-e}((\alpha_j, \alpha_{j+1}], \mathbb{C}^{n \times n})$ such that $D_j(t)$ is diagonal and unitary for all t, and

$$D_j(t) = \begin{cases} \Pi_j^T U_j(t)^* U_{j-1}(t) \Pi_{j-1} & \text{on } (\alpha_j\,,\, \frac{2}{3}\alpha_j + \frac{1}{3}\beta_{j-1}) \\ I & \text{on } (\frac{1}{3}\alpha_j + \frac{2}{3}\beta_{j-1}\,,\, \alpha_{j+1}] \ . \end{cases}$$

Then $U_j \Pi_j D_j$ is smooth and equals to $U_{j-1}\Pi_{j-1}$ on the beginning part and $U_j \Pi_j$ on the final part of $(\alpha_j, \alpha_{j+1}]$. Set $U = U_j \Pi_j D_j$ and $\Lambda = \Pi_j \Lambda_j \Pi_j$ on this interval. This gives U and Λ smooth on $[\beta_{-1}, \alpha_{j+1}]$.

Similarly continue for $j = -1, -2, \ldots$. □

Our next task is to prove that in the situation of the previous theorem the eigenvalues are in fact $\mathcal{C}^k$ functions. For this we need a closer look at the functions involved in the Taylor remainder terms.

For a function $f \in \mathcal{C}^0([-T,T])$ set $m_e(f)(t) = t^e f(t)$.

For $k \geq e$ denote by $\mathcal{C}^k_e$ the set of $f \in \mathcal{C}^0([-T,T])$ for which $m_e(f) \in \mathcal{C}^k([-T,T])$. With the norm

$$\|f\|_{k,e} = \max_{\substack{0\leq j\leq e\\ 0\leq l\leq k-e+j}} \left\| m_j(f)^{(l)} \right\|_\infty ,$$

$\mathcal{C}^k_e$ becomes a Banach space.

Proposition 3.4. *If $f, g \in \mathcal{C}^k_e$, then*

$$i) \quad fg \in \mathcal{C}^k_e , \quad ii) \quad f(t) \neq 0 \,\forall t \implies 1/f \in \mathcal{C}^k_e , \quad iii) \quad f(t) > 0 \,\forall t \implies \sqrt{f} \in \mathcal{C}^k_e .$$

Further, the operations $i) - iii)$ are continuous.

Proof. Clearly $\mathcal{C}^{k+1}_{e+1} \subset \mathcal{C}^k_e$ with $\|f\|_{k,e} \leq \|f\|_{k+1,e+1}$.

For $f \in \mathcal{C}^{k+1}_{e+1}$, $0 \leq d \leq e$, set $h_d(f)(t) = t^{-d} m_{d+1}(f)'(t)$. We show first that h_d is continuous $\mathcal{C}^{k+1}_{e+1} \to \mathcal{C}^k_e$. Take $0 \leq j \leq e$ and $0 \leq l \leq k-e+j$.
If $j < d$ then by the footnote on page 12

$$\begin{aligned}
\left| m_j(h_d(f))^{(l)}(t) \right| &= \left| \frac{d^l}{dt^l} \big(t^{j-d} m_{d+1}(f)'(t) \big) \right| \\
&= \left| \frac{d^l}{dt^l} \frac{1}{(d-j-1)!} \int_0^1 (1-s)^{d-j-1} m_{d+1}(f)^{(d-j+1)}(ts)\, ds \right| \\
&\leq C \left\| m_{d+1}(f)^{(d-j+1+l)} \right\|_\infty \leq C \|f\|_{k+1,e+1}
\end{aligned}$$

since $d+1 \leq e+1$ and $d-j+1+l \leq k+1-(e+1)+d+1$.
If $j \geq d$ then

$$\begin{aligned}
\left| m_j(h_d(f))^{(l)}(t) \right| &= \left| \frac{d^l}{dt^l} m_{j-d}\big(m_{d+1}(f)' \big)(t) \right| \\
&= \left| \frac{d^l}{dt^l} \big(m_{j+1}(f)'(t) - (j-d) m_j(f)(t) \big) \right| \\
&= \left| m_{j+1}(f)^{(l+1)}(t) - (j-d) m_j(f)^{(l)}(t) \right| \leq C \|f\|_{k+1,e+1}
\end{aligned}$$

since $j+1 \leq e+1$ and $l+1 \leq k+1-(e+1)+d+1$. Hence $\|h_j(f)\|_{k,e} \leq C \|f\|_{k+1,e+1}$, for some constant C depending only on T, k, and e.

If $f, g \in \mathcal{C}^{k+1}_{e+1}$ and $0 \leq j \leq e$, then

$$\begin{aligned}
m_{j+1}(fg)'(t) &= \frac{d}{dt} \big(t^{-j-1} m_{j+1}(f)(t) m_{j+1}(g)(t) \big) \\
&= (h_j(f) m_j(g))(t) + (m_j(f) h_j(g))(t) \\
&\qquad -(j+1) t^{-j} (m_j(f) m_j(g))(t) \\
&= \big(m_j(h_j(f) g) + m_j(f h_j(g)) - (j+1) m_j(fg) \big)(t) .
\end{aligned} \tag{3.3}$$

Now, $i)$ is trivially true for any k if $e = 0$. Assume it is true for k and e. Then (3.3) with $j = e$ shows that $m_{e+1}(fg)' \in \mathcal{C}^k$, i.e., $fg \in \mathcal{C}^{k+1}_{e+1}$.

Similarly for $ii)$ and $iii)$. If they are true for k, e and if $f \in \mathcal{C}^{k+1}_{e+1}$ vanishes nowhere, then

$$m_{e+1}(1/f)'(t) = \frac{d}{dt} (t^{2e+2} / m_{e+1}(f)) = \left((2e+2) m_e(1/f) - m_e \left(h_e(f) \frac{1}{f} \frac{1}{f} \right) \right)(t) .$$

So, by $i)$ we get $m_{e+1}(1/f)' \in \mathcal{C}^k$ and $1/f \in \mathcal{C}^{k+1}_{e+1}$. Further, if f is positive, then differentiating $m_{e+1}(\sqrt{f}\,)(t)^2 = t^{e+1} m_{e+1}(f)(t)$ gives

$$m_{e+1}(\sqrt{f}\,)' = \frac{e+1}{2}\, m_e(\sqrt{f}\,) + \frac{1}{2} m_e(h_e(f)\,1/\sqrt{f}\,)$$

in $\mathcal{C}^k$ and $\sqrt{f} \in \mathcal{C}^{k+1}_{e+1}$.

Next, we prove continuity of the multiplication. We show inductively that

$$\|fg\|_{k,e} \leq C\, \|f\|_{k,e}\, \|g\|_{k,e} \ , \tag{3.4}$$

which is trivially true for any k if $e = 0$. Assume it is true for k and e and let $f, g \in \mathcal{C}^{k+1}_{e+1}$. From (3.3) we get for $j+1 \leq e+1$, $l+1 \leq k+1-(e+1)+j+1$

$$\begin{aligned}\left\| m_{j+1}(fg)^{(l+1)} \right\|_\infty &\leq \left\| m_j(h_j(f)g)^{(l)} \right\|_\infty + \left\| m_j(fh_j(g))^{(l)} \right\|_\infty + (j+1) \left\| m_j(fg)^{(l)} \right\|_\infty \\ &\leq C\big(\|h_j(f)\|_{k,e}\, \|g\|_{k,e} + \|f\|_{k,e}\, \|h_j(g)\|_{k,e} + (j+1)\, \|f\|_{k,e}\, \|g\|_{k,e} \big) \\ &\leq C\, \|f\|_{k+1,e+1}\, \|g\|_{k+1,e+1} \ .\end{aligned}$$

Thus, continuity of the multiplication follows from (3.4) by

$$\left\| \tilde{f}\tilde{g} - fg \right\|_{k,e} \leq \left\| (\tilde{f} - f)\tilde{g} \right\|_{k,e} + \|f(\tilde{g} - g)\|_{k,e} \ .$$

Proofs of continuity of $f \to 1/f$ and $f \to \sqrt{f}$ are similar to that of multiplication and hence omitted. ∎

With these tools we can show the following:

THEOREM 3.5. *Under the assumptions of* Theorem 3.3*, the eigenvalues can be taken to be* $\mathcal{C}^k$ *functions.*

Proof. From the proof of Theorem 3.3 we get that the eigenvalues of $A(t)$ are of the form $\lambda_0 + t\,\mu_1(t)$, where $\mu_1(t)$ is an eigenvalue of $S_1(t)$, and $S_1 \in \mathcal{C}^k_1$. If $\mu_1(0)$ is a $q-$fold eigenvalue of $S_1(0)$, let $P_1(t)$ be the eigenprojector corresponding to the q eigenvalues of $S_1(t)$ close to $\mu_1(0)$. Then (see [12])

$$P_1(t) = -\frac{1}{2\pi i} \int_\Gamma (S_1(t) - \zeta I)^{-1} d\zeta \ ,$$

where Γ is the boundary of a small disk containing only these eigenvalues. By $i)$ and $ii)$ of Proposition 3.4 $\zeta \to (S_1(\cdot) - \zeta I)^{-1}$ defines a continuous function $\Gamma \to \mathcal{C}^k_1([-T,T], \mathbb{C}^{q\times q})$ so that also $P_1 \in \mathcal{C}^k_1$. Take $Q_1(0)$ the columns of which form an orthonormal basis for the range of $P_1(0)$, and let $Q_1(t)R_1(t) = P_1(t)Q_1(0)$ be a QR–decomposition which again by Prop. 3.4 is in $\mathcal{C}^k_1$. The columns of $Q_1(t)$ form an orthonormal basis for the $S_1(t)-$invariant subspace corresponding to the eigenvalues close to $\mu_1(0)$. Hence, we get that

$$Q_1^*(t)S_1(t)Q_1(t) = \mu_1(0)\; I + t\; S_2(t) \ ,$$

where $S_2 \in \mathcal{C}^k_2$. So, the eigenvalues of $A(t)$ are of the form $\lambda_0 + t\,\mu_1(0) + t^2\,\mu_2(t)$, where μ_2 is an eigenvalue of S_2. Continuing this way, we get that the eigenvalues of $A(t)$ are of the form

$$\lambda(t) = \lambda_0 + t\,\mu_1(0) + t^2\,\mu_2(0) + \cdots + t^e \mu_e(t) \ ,$$

where μ_e is an eigenvalue of $S_e \in \mathcal{C}^k_e$. By assumption, it is a simple eigenvalue. Then, as above, the corresponding eigenvector Q_e is in $\mathcal{C}^k_e$ and the same holds for $\mu_e(t) = Q_e(t)^* S_e(t) Q_e(t)$. Hence $m_e(\mu_e)$ and consequently also λ is in $\mathcal{C}^k$. ∎

3.3. SVD in the rank deficient case and with multiple singular values. Here we consider a smooth singular value decomposition of $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$, $m \geq n$, allowing now multiple singular values and/or loss of rank at some points. In parallel to sections 3.1 and 3.2 we assume that at any point the order of coalescing of the squares of the singular values is at most $e \leq k$ (as in (3.2)):

$$\liminf_{\tau\to 0} \frac{\left|\sigma_i^2(t+\tau) - \sigma_j^2(t+\tau)\right|}{|\tau^e|} \in (0,\infty]\,, \ \ \forall t\,,$$

and that for every t

$$\limsup_{\tau\to 0} \frac{1}{\tau^{2d}} \det(A^*A)(t+\tau) > 0\ . \tag{3.5}$$

We also assume that only one of the singular values can become zero, i.e., the rank of A is always at least $n-1$.

To get smooth decomposition one has to allow sign changes in the singular values as in the analytic case of [3]. The result might be more properly called a *signed* SVD.

THEOREM 3.6. *With the above assumptions there exists a $\mathcal{C}^{k-\max(d,e)}$–singular value decomposition of A. Moreover, the singular values can be taken to be $\mathcal{C}^k$ functions.*

Proof. According to Theorem 3.3, we get that the Hermitian positive semidefinite matrix A^*A has a $\mathcal{C}^{k-e}$ – Schur decomposition

$$A^*A = \tilde V \Sigma^2 \tilde V^*\ , \qquad \Sigma^2 = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}.$$

Let $\hat t$ be a point where A loses rank. We can assume that σ_n^2 is the smallest eigenvalue near $\hat t$. Then, by the smooth block Schur result (Proposition 2.6) we can write $\tilde V = [\,\tilde V_1 \quad \tilde v_n\,]$, where $\tilde v_n \in \mathcal{C}^k$. Write also $\Sigma^2 = \left[\begin{smallmatrix}\Sigma_1^2 & \\ & \sigma_n^2\end{smallmatrix}\right]$. Similarly, we can make the decomposition

$$AA^* = [\,U_1 \quad \tilde U_2\,] \begin{bmatrix} \Sigma_1^2 & \\ & B^*B \end{bmatrix} [\,U_1 \quad \tilde U_2\,]^*\ ,$$

where $U_1 \in \mathcal{C}^{k-e}$, $\tilde U_2 \in \mathcal{C}^k$, and $B = A^*\tilde U_2 \in \mathcal{C}^k$.

Since $w = A\tilde v_n$ is in $\mathcal{C}^k$ and $\left\|w(\hat t+\tau)\right\| \geq c\,|\tau|^{\hat d}$ for some $c>0$, $\hat d \leq d$, we get $w(\hat t+\tau) = \tau^{\hat d} w_0(\tau)$ where $w_0 \in \mathcal{C}^k_{\hat d}$ and $w_0(0) \neq 0$. So, by Proposition 3.4, $s(\tau) = \|w_0(\tau)\|$ is in $\mathcal{C}^k_{\hat d}$. Hence, $\sigma_n = m_{\hat d}(s) \in \mathcal{C}^k$. In Σ_1 we can take the square roots of $\sigma_1^2, \ldots, \sigma_{n-1}^2$ with any combination of signs, just keeping them constant until a possible sign change of one that becomes zero.

Since $\|B\|^2 = \sigma_n^2$, we have

$$B(\hat t+\tau) = \tau^{\hat d} \hat B(\tau)\ ,$$

where $\hat B \in \mathcal{C}^{k-\hat d}$, and $\hat B(0)$ has rank one. Then we have the $\mathcal{C}^{k-\hat d}$ block Schur decomposition:

$$W\hat B^* \hat B W^* = \begin{bmatrix} s^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Set $U = [U_1 \quad \tilde{U}_2 W] = [U_1 \quad u_n \quad U_0] \in \mathcal{C}^{k-\max(\hat{d},e)}$. Then we want to find a matrix V such that $A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^*$. We look for it in the form: $V = \tilde{V} D$, where $D = \mathrm{diag}(d_1, \ldots, d_n)$ satisfies $|d_j| = 1$, $\forall j$. We need:

$$\begin{bmatrix} U_1^* \\ u_n^* \end{bmatrix} A \; [\tilde{V}_1 \quad \tilde{v}_n] = \begin{bmatrix} \Sigma_1 & \\ & \sigma_n \end{bmatrix} \bar{D} ,$$

from which $\mathrm{diag}(\bar{d}_1, \ldots, \bar{d}_{n-1}) = \Sigma_1^{-1} U_1^* A \tilde{V}_1 \in \mathcal{C}^{k-e}$. Finally, from $\sigma_n \bar{d}_n = u_n^* w$ we get $d_n \in \mathcal{C}^{k-\max(\hat{d},e)}$ by

$$\bar{d}_n(\hat{t}+\tau) = \frac{u_n(\hat{t}+\tau)^* w_0(\tau)}{s(\tau)} .$$

Hence $\Sigma \in \mathcal{C}^k$ and $U, V \in \mathcal{C}^{k-\max(d,e)}$. □

Remark 3.3. If in the previous theorem $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{n\times n})$, then condition (3.5) is not needed and we get a $\mathcal{C}^{k-e}$ SVD. This is because U_1 and V_1 are in $\mathcal{C}^{k-e}$ and they determine (up to sign) uniquely the last columns $u_n, v_n \in \mathcal{C}^{k-e}$.

4. Genericity of smooth factorizations. In this section we discuss how smooth factors a generic one-parameter family of matrices has. For example, we show that, generically, $\mathcal{C}^k$ matrices have $\mathcal{C}^k$ QR decomposition and $\mathcal{C}^k$ singular value decomposition.

In what follows, for $\mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$ we take the Whitney topology (see [11], it is called the "fine" topology in [1]). Also, recall that a *generic* property is one that holds for a set that contains a countable intersection of open and dense sets.

4.1. QR **decomposition.** Here we show that the QR decomposition is, generically, as smooth as the family.

THEOREM 4.1. *A generic* $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{m\times n})$, $k \geq 1$, $m \geq n$, *has a* $\mathcal{C}^k$ QR *decomposition. Similarly for generic* $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{m\times n})$.

Proof. The set V of $m \times n$ real matrices having rank $r < n$ is a stratified manifold, where each stratum has codimension $(m-r)(n-r)$ (see [11]). So, for $m > n$, the codimension is at least 2, and by the weak transversality theorem a generic one parameter family does not meet this set (again, see [11]). So, it has full rank for all t. In the case $m = n$, for generic $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{n\times n})$ we have that $[a_1 \ldots a_{n-1}]$ is of full rank for all t, and by Remark 3.1 A has a $\mathcal{C}^k$ QR decomposition.
In the complex case, each stratum has (real) codimension $2(m-r)(n-r)$, and hence a generic one parameter family has full rank for all t. □

4.2. Schur decomposition. For a generic matrix $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n\times n})$, the eigenvalues are simple and we have a $\mathcal{C}^k$ Schur decomposition (and $\mathcal{C}^k$ eigendecomposition).
On the other hand, not even for a generic family of analytic *real* matrices we can expect smooth eigenvalues. For example, any smooth *real* perturbation of

$$A(t) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}$$

will have a defective eigenvalue and nondifferentiable eigenvalues at some t near 0.

More interesting is the case of Hermitian matrices. For the $\mathcal{C}^k$ case, the next theorem shows that generically there is no loss of smoothness.

THEOREM 4.2. *A generic* $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}_H^{n\times n})$, $k \geq 1$ *(or complex Hermitian) has a* $\mathcal{C}^k$ *Schur decomposition.*

Proof. We show that a generic one parameter family of real symmetric matrices has simple eigenvalues for every t. The set W_0 of symmetric matrices with a double eigenvalue is the image of the map $(U,D) \to UDU^T$, where U is orthogonal, D is diagonal with $d_{11} = d_{22}$. This map is real analytic and proper (compact sets have compact preimages). Hence by [10] W_0 has a Whitney stratification. The dimension of the set of $n \times n$ orthogonal matrices is $n(n-1)/2$. If $\tilde U = U \left[\begin{smallmatrix} c & s & \\ -s & c & \\ & & I \end{smallmatrix}\right]$ where $c^2 + s^2 = 1$, then $\tilde U D \tilde U^T = UDU^T$. Hence the maximum dimension of the strata of W_0 is

$$n(n-1)/2 - 1 + n - 1 = n(n+1)/2 - 2 \ .$$

Thus W_0 has codimension two and by the weak transversality theorem a generic one parameter family does not meet W_0.
Similarly, let now W_0 be the set of complex Hermitian matrices with a double eigenvalue. This is the image of the map $(U,D) \to UDU^*$, with U unitary, and D real diagonal with $d_{11} = d_{22}$. By viewing this as the equivalent real map $\psi :\ (U_r,\ U_i,\ D) \to (U_rDU_r^T + U_iDU_i^T\ ,\ U_iDU_r^T - U_rDU_i^T)$, where $U = U_r + iU_i^T$, similarly to the previous case one infers that W_0 admits a stratification. Now, the (real) dimension of the set of $n \times n$ unitary matrices is n^2. If we let $\tilde U = U \left[\begin{smallmatrix} V & 0 \\ 0 & \Phi \end{smallmatrix}\right] = \tilde U_r + i\tilde U_i$ where $V \in \mathbb{C}^{2\times 2}$ is unitary, and $\Phi = \mathrm{diag}(\eta_1, \ldots, \eta_{n-2})$, $|\eta_j| = 1$, then $\psi(\tilde U_r,\ \tilde U_i,\ D) = \psi(U_r,\ U_i,\ D)$. Hence the maximum dimension of the strata of W_0 is

$$n^2 + (n-1) - 4 - (n-2) \ = \ n^2 - 3\,,$$

and since the set of Hermitian matrices has real dimension n^2 we see that W_0 has codimension 3 and generically a one parameter family does not meet it. ∎

Example 4.1. Let $A(t) = \left[\begin{smallmatrix} t & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The perturbation $A_\varepsilon(t) = \left[\begin{smallmatrix} t & \varepsilon \\ \varepsilon & 0 \end{smallmatrix}\right]$ has simple eigenvalues $\frac{1}{2}(t \pm \sqrt{t^2 + 4\varepsilon^2})$ for every t. This example also shows that analytic symmetric matrices of *two* parameters do not necessarily have smooth eigenvalues.

4.3. Singular value decomposition. Here we show, in the $\mathcal{C}^k$ case, that generically the singular value decomposition is as smooth as the family.

THEOREM 4.3. *A generic* $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{m\times n})$, $k \geq 1$ *(or complex valued) has a* $\mathcal{C}^k$ *singular value decomposition.*

Proof. We can assume that $m \geq n$, otherwise apply the following to A^T. We show first that generically a one parameter family of real matrices has simple singular values for every t, i.e., we have $e = 0$ in Theorem 3.6.

Similarly to the previous proof, the set V_0 of real $m \times n$ matrices with a double singular value is the image of the proper analytic map $(U, \Sigma, V) \to U\Sigma V^T$, where $U \in \mathbb{R}^{m\times n}$ is orthonormal, Σ is diagonal with $\sigma_{11} = \sigma_{22}$, and V is orthogonal. As above

$$U \begin{bmatrix} c & s & \\ -s & c & \\ & & I \end{bmatrix} \Sigma \begin{bmatrix} c & -s & \\ s & c & \\ & & I \end{bmatrix} V^T = U\Sigma V^T$$

so that the dimensions of the strata of V_0 do not exceed

$$mn - n(n+1)/2 + n - 1 + n(n-1)/2 - 1 = mn - 2 \ .$$

Hence a generic one parameter family does not intersect V_0. The complex case is handled similarly to Theorem 4.2.

If $m > n$ then as in the proof of Theorem 4.1 we generically get $d = 0$ for Theorem 3.6. This is also true for complex $n \times n$ matrices. For real $n \times n$ matrices we get the result by Remark 3.3. ∎

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