

# SLIDING MOTION ON THE INTERSECTION OF TWO MANIFOLDS: SPIRALLY ATTRACTIVE CASE.

L. DIECI

**ABSTRACT.** In this note, we consider sliding motion on the intersection  $\Sigma$  of two smooth manifolds in the case when the dynamics near the manifold  $\Sigma$  is *spiral-like*, and  $\Sigma$  is *spirally attractive*. We clarify the meaning of spiral-like dynamics around  $\Sigma$ , characterize what we mean by spiral attractivity of  $\Sigma$ , and finally discuss what to expect when  $\Sigma$  ceases to be attractive, with nearby orbits getting farther away from  $\Sigma$  through spiraling motion. Our characterization of spiral-attractivity of  $\Sigma$  is given by a single number, which plays a role similar to that of a *Floquet multiplier* for a smooth planar system.

## 1. INTRODUCTION

Piecewise smooth systems (differential systems with discontinuous right hand side) play an important role in many mechanical and engineering applications (e.g., see [1]), and present deep and complex mathematical questions. In particular, the well established Filippov convexification method (see [6]) gives a powerful mean to establish what to do when solution trajectories reach a co-dimension 1 manifold of discontinuity, but it is still not fully understood what happens when trajectories have to move on the intersection  $\Sigma$  of two smooth manifolds. To be of practical interest, such intersection  $\Sigma$  should enjoy some *attractivity* properties, that is nearby solution trajectories should reach  $\Sigma$  (in forward time), and solution trajectories starting on  $\Sigma$  ought to remain there, giving rise to so-called sliding motion. In [4], we characterized attractivity of  $\Sigma$  in the case of solution trajectories approaching it through sliding (see below). Our goal in this work is to complete the characterization of attractive  $\Sigma$  by treating the case of spiral attractivity of  $\Sigma$ . In short, our goal in this work is to give conditions characterizing a situation such as in Figure 1, where  $\Sigma$  is the vertical axis; the top (red) portion is motion out of  $\Sigma$ , the bottom (blue) is motion toward  $\Sigma$ , and there is sliding motion on (part of)  $\Sigma$  (green curve), which itself is spirally-attractive.

In this Introduction, we review the basic problem and set up the notation. In Section 2 we propose a characterization of what we mean by spiral-like behavior around  $\Sigma$ , where  $\Sigma$  is the intersection of two smooth co-dimension 1 manifolds. Then, in Section 3 we characterize spiral attractivity for  $\Sigma$ . In Section 4 we discuss what may happen when, still subject to spiral-like behavior of nearby dynamics,  $\Sigma$  loses attractivity.

**1.1. The problem and Filippov solutions.** We consider piecewise smooth differential systems of the following type:

$$(1.1) \quad \dot{x} = f(x), \quad f(x) = f_i(x), \quad x \in R_i, \quad i = 1, \dots, 4,$$

with initial condition  $x(0) = x_0 \in R_i$ , for some  $i$ . Here, the  $R_i \subseteq \mathbb{R}^n$  are open, disjoint and connected sets, and (locally)  $\mathbb{R}^n = \overline{\bigcup_i R_i}$ . Each  $f_i$  is assumed to be smooth in an open neighborhood of the closure of each  $R_i$ ,  $i = 1, \dots, 4$ .

Clearly, from (1.1), the vector field is not defined on the boundaries of the  $R_i$ 's.

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1991 *Mathematics Subject Classification.* 34A36.

*Key words and phrases.* Piecewise smooth systems, Filippov convexification, sliding motion, intersection of two manifolds, spiral attractivity, multiplier.

The author gratefully acknowledges the support provided by a Tao Aoqing Visiting Professorship at Jilin University, Changchun (CHINA).

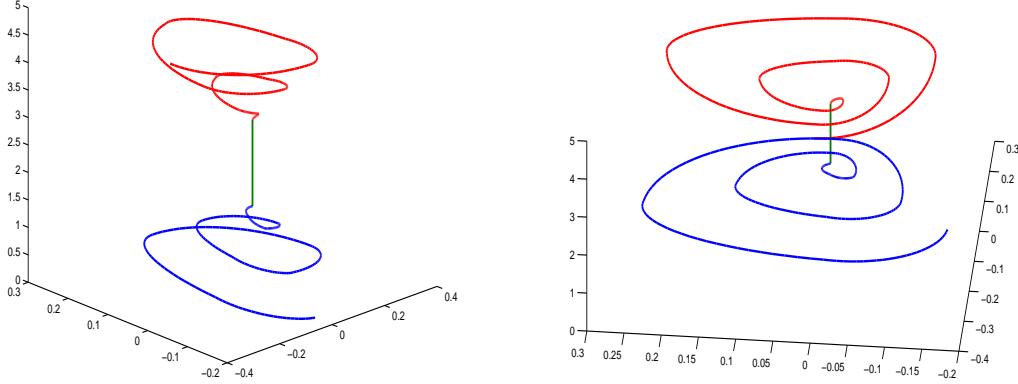


FIGURE 1. Spiraling around  $\Sigma$  (vertical axis), and sliding on (part of) it.

**1.2. Codimension 1 case: attractivity, existence and uniqueness.** The classical Filippov theory (see [6]) is concerned with the case of two regions  $R_1$  and  $R_2$ , separated by a manifold  $\Sigma$  defined as the 0-set of a smooth ( $\mathcal{C}^2$ ) scalar valued function  $h$ :

$$(1.2) \quad \begin{aligned} \dot{x} &= f_1(x), \quad x \in R_1, \quad \text{and} \quad \dot{x} = f_2(x), \quad x \in R_2, \\ \Sigma &:= \{x \in \mathbb{R}^n : h(x) = 0\}, \quad h: \mathbb{R}^n \rightarrow \mathbb{R}. \end{aligned}$$

Here,  $\nabla h$  is bounded away from 0 for all  $x \in \Sigma$ , and near  $\Sigma$ . Without loss of generality, we can label  $R_1$  such that  $h(x) < 0$  for  $x \in R_1$ , and  $R_2$  such that  $h(x) > 0$  for  $x \in R_2$ . Let us define

$$(1.3) \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \nabla h(x)^T f_1(x) \\ \nabla h(x)^T f_2(x) \end{bmatrix}, \quad x \in \Sigma,$$

for the projected vector fields. We say (see [6]) that  $\Sigma$  is *attractive* (in finite time) if, for some positive constant  $c$ , we have

$$w_1(x) \geq c > 0 \quad \text{and} \quad w_2(x) \leq -c < 0,$$

for  $x \in \Sigma$ . In this case, trajectories starting near  $\Sigma$  must reach it and remain there: this gives the so-called *sliding motion*. Filippov convexification method amounts to selecting as sliding vector field on  $\Sigma$  a convex combination of  $f_1$  and  $f_2$ ,  $f_F := (1-\alpha)f_1 + \alpha f_2$ , with  $\alpha$  chosen so that  $f_F \in T_\Sigma$  ( $f_F$  is tangent to  $\Sigma$  at each  $x \in \Sigma$ ):

$$(1.4) \quad x' = (1-\alpha)f_1 + \alpha f_2, \quad \alpha = \frac{\nabla h(x)^T f_1(x)}{\nabla h(x)^T f_1(x) - \nabla h(x)^T f_2(x)}.$$

Clearly, because of attractivity,  $\alpha \in (0, 1)$ . Whenever  $\alpha = 0$  (respectively  $\alpha = 1$ ), the vector field  $f_1$  (respectively  $f_2$ ), is itself tangent to  $\Sigma$ , and one should expect the trajectory to leave  $\Sigma$  to enter in  $R_1$  (respectively  $R_2$ ). These are tangential exits, predicted by the first order Filippov theory.

**Remark 1.1.** Observe that (1.4) gives a well defined sliding motion also in the case of *repulsive*  $\Sigma$ , that is when

$$w_1(x) \leq -c < 0 \quad \text{and} \quad w_2(x) \geq c > 0, \quad x \in \Sigma.$$

The difference in this case of repulsive sliding is that, for forward time, the sliding motion is unstable and there is no uniqueness, since one can also leave  $\Sigma$  at any instant of time with either  $f_1$  or  $f_2$ . These types of exits are non-tangential.  $\square$

**1.3. Intersection of two codimension 1 manifolds.** As we said, we are concerned with (1.1), where now the  $R_i$ 's are (locally) separated by two intersecting smooth manifolds of co-dimension 1,  $\Sigma_1 = \{x : h_1(x) = 0\}$  and  $\Sigma_2 = \{x : h_2(x) = 0\}$ . We have  $\Sigma = \Sigma_1 \cap \Sigma_2$ , and here  $h_1$  and  $h_2$  are  $\mathcal{C}^2$  functions, and  $\nabla h_1(x)$  and  $\nabla h_2(x)$  are linearly independent, for  $x$  on (and in a neighborhood of)  $\Sigma$ .

We have four different regions  $R_1$ ,  $R_2$ ,  $R_3$  and  $R_4$  with the four different vector fields  $f_i$ ,  $i = 1, \dots, 4$ , in these regions:

$$(1.5) \quad \dot{x} = f_i(x) , \quad x \in R_i , \quad i = 1, \dots, 4 .$$

Without loss of generality, we can label the regions as follows:

$$(1.6) \quad \begin{aligned} R_1 : f_1 &\text{ when } h_1 < 0 , h_2 < 0 , & R_2 : f_2 &\text{ when } h_1 < 0 , h_2 > 0 , \\ R_3 : f_3 &\text{ when } h_1 > 0 , h_2 < 0 , & R_4 : f_4 &\text{ when } h_1 > 0 , h_2 > 0 . \end{aligned}$$

We further let (cfr. with (1.3))

$$(1.7) \quad \begin{aligned} w_1^1 &= \nabla h_1^T f_1 , \quad w_2^1 = \nabla h_1^T f_2 , \quad w_3^1 = \nabla h_1^T f_3 , \quad w_4^1 = \nabla h_1^T f_4 , \\ w_1^2 &= \nabla h_2^T f_1 , \quad w_2^2 = \nabla h_2^T f_2 , \quad w_3^2 = \nabla h_2^T f_3 , \quad w_4^2 = \nabla h_2^T f_4 . \end{aligned}$$

In [4], the authors considered the case of  $\Sigma$  being *attractive through sliding*. By that, it is meant that solution trajectories starting near  $\Sigma$  will reach (in finite time) the intersection  $\Sigma$ , either directly, or (more likely) by first sliding on one of  $\Sigma_1$  or  $\Sigma_2$ , directed towards  $\Sigma$ . Formally, if we let  $\Sigma_1^\pm = \{x : h_1(x) = 0, h_2(x) \gtrless 0\}$ , and similarly for  $\Sigma_2^\pm$ , attractivity through sliding of  $\Sigma$  means that (near  $\Sigma$ ) there must be sliding motion on (at least) one of  $\Sigma_{1,2}^\pm$ , directed towards  $\Sigma$ , and no motion away from  $\Sigma$ ; note that the sliding motion on  $\Sigma_{1,2}^\pm$  would be taking place with a Filippov sliding vector field, call it  $f_{\Sigma_{1,2}^\pm}$ , defined as in Section 1.2, see (1.4).

Now, in the present case of  $\Sigma = \Sigma_1 \cap \Sigma_2$ , the general Filippov construction defines a sliding trajectory on  $\Sigma$  as an absolutely continuous function  $x$  such that, almost everywhere, its derivative is in the convex hull of the neighboring vector fields and lies on the tangent plane at  $\Sigma$ :  $\dot{x} \in T_{\Sigma(x)}$ . In other words, one must have

$$(1.8) \quad \begin{aligned} \dot{x} \in \mathcal{F}(x) &:= \{\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4, \quad \lambda_i \geq 0, \quad i = 1, \dots, 4, \quad \sum_{i=1}^4 \lambda_i = 1\} , \\ \nabla h_1^T \dot{x} &= \nabla h_2^T \dot{x} = 0 , \quad x \in \Sigma . \end{aligned}$$

As it is plainly obvious, the mere requirement of  $\dot{x}$  being on  $T_\Sigma$  is not in general sufficient to uniquely characterize a convex combination of the four vector fields  $f_1, \dots, f_4$ . Still, there are meaningful ways to select a unique, smoothly varying, sliding vector field  $f_F$ , e.g. by considering the bilinear interpolant of Example 1.2 below (other possibilities are surely possible, e.g., the choice recently examined in [3]).

**Example 1.2** (Bilinear sliding motion). Here, one seeks  $f_F$  of the form (e.g., see [4] and [7])

$$(1.9) \quad f_F = (1 - \alpha)[(1 - \beta)f_1 + \beta f_2] + \alpha[(1 - \beta)f_3 + \beta f_4] ,$$

where  $0 \leq \alpha, \beta \leq 1$  must be found so that –at each  $x \in \Sigma$ –  $f_F \in T_\Sigma$ . In [4], it was shown that this choice can always be done when  $\Sigma$  is attractive through sliding, and  $\alpha$  and  $\beta$  will depend smoothly on  $x \in \Sigma$ .  $\square$

However, our present concern is not how to properly define a unique Filippov sliding vector field. Our present concern is how to characterize the case of  $\Sigma$  being attracting when there is spiral motion of nearby trajectories, which was left as an open problem in [4]. As it turns out, the situation is substantially different from the case of  $\Sigma$  being attractive through sliding. Our main proposal in this work is that there is a number, a *multiplier*, which characterizes the present spiraling situation.

## 2. SPIRAL-LIKE DYNAMICS NEAR $\Sigma$

First, we characterize what we mean for the dynamics near  $\Sigma$  to be spiral-like. Informally, we mean that “*a solution trajectory starting near  $\Sigma$  in any of the regions  $R_i$ , for some  $i = 1, \dots, 4$ , will return to this same region  $R_i$  after having crossed transversally once (and once only) each of the submanifolds  $\Sigma_{1,2}^\pm$* ”.

To quantify the above statement, it is convenient to consider new variables in a neighborhood of  $\Sigma$ :

$$(2.1) \quad y_1 = h_1(x), \quad \text{and} \quad y_2 = h_2(x).$$

Now, note that, for  $i = 1, 2$ ,  $\frac{dy_i}{dt} = \nabla h_i^T \frac{dx}{dt}$ . So, letting  $w_i = \begin{bmatrix} w_i^1 \\ w_i^2 \end{bmatrix}$  as in (1.7), we have

$$(2.2) \quad \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = w_i, \quad x \in R_i, \quad i = 1, \dots, 4.$$

Thus, by spiral-like behavior in a neighborhood of  $\Sigma$  we mean that “*the y-trajectories starting in any of the four canonical quadrants will return to this quadrant after having crossed transversally once (and only once) the coordinate axes,  $y_j = 0$  (i.e.,  $h_j = 0$ ),  $j = 1, 2$ .*” See Figure 2.

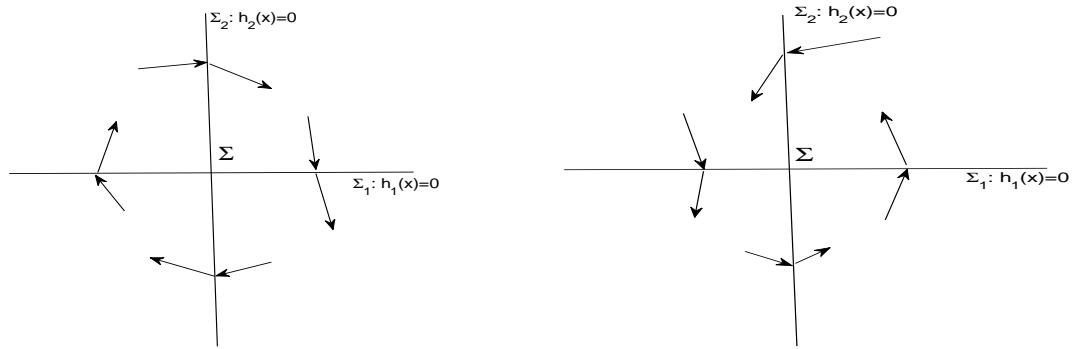


FIGURE 2. Transversal intersections while spiraling. Left, clockwise; right, counterclockwise.

So, we have to quantify two features: the transversal crossing, and the return to each region. Transversal crossings can be characterized (at first order) by one of the following two possible tables (Table 1 or 2) for the signs of the  $w_i^j$ 's upon hitting  $\Sigma_{1,2}^\pm$ .

TABLE 1. Signs of matrix  $W$  in clockwise spiraling around  $\Sigma$ .

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$w_i^1$	$w_1^1(\Sigma_1^-) > 0$	$w_2^1(\Sigma_1^+) < 0$	$w_3^1(\Sigma_1^-) > 0$	$w_4^1(\Sigma_1^+) < 0$
$w_i^2$	$w_1^2(\Sigma_2^-) < 0$	$w_2^2(\Sigma_2^-) < 0$	$w_3^2(\Sigma_2^+) > 0$	$w_4^2(\Sigma_2^+) > 0$

TABLE 2. Signs of matrix  $W$  in counterclockwise spiraling around  $\Sigma$ .

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$w_i^1$	$w_1^1(\Sigma_1^-) < 0$	$w_2^1(\Sigma_1^+) > 0$	$w_3^1(\Sigma_1^-) < 0$	$w_4^1(\Sigma_1^+) > 0$
$w_i^2$	$w_1^2(\Sigma_2^+) > 0$	$w_2^2(\Sigma_2^+) > 0$	$w_3^2(\Sigma_2^-) < 0$	$w_4^2(\Sigma_2^-) < 0$

**Remark 2.1.** To clarify, in each region  $R_i$ , the system is (1.5), and this is the system we have to solve. The conditions of Tables 1 or 2 guarantee that the  $x$ -trajectories have transversal intersections with  $\Sigma_{1,2}^\pm$ . The convenience of introducing the  $y$ -variables is that it allows us to use the 4 canonical quadrants in the  $y$ -plane in lieu of the regions of  $\mathbb{R}^n$  where  $h_{1,2} \geq 0$ . That said, it must be understood that, when we write (2.2), the vector field  $w_i$  is really evaluated along the solution of  $\dot{x} = f_i$ ,  $i = 1, \dots, 4$ . With this in mind, we can talk about evolution of the  $y$ -variables directly.  $\square$

The second requirement, that  $y$ -trajectories starting in any of the four canonical quadrants return to this quadrant after having crossed transversally once the coordinates axes, can be expressed as follows.

Let  $\phi_i^t(z)$ ,  $z \in R_i$ , be the evolution in each region  $R_i$ ,  $i = 1, 2, 3, 4$ . Consider an initial condition  $x_0 \in \Sigma_2^-$ , and let  $y^{(0)} = \begin{bmatrix} h_1(x_0) \\ h_2(x_0) \end{bmatrix} = \begin{bmatrix} y_1^{(0)} \\ 0 \end{bmatrix}$  with  $y_1^{(0)} < 0$ . Then, spiral-like behavior can be written as the requirement that there exist first, and unique, times (which of course depend on  $x_0$ )  $t_1, t_2, t_3, t_4$ , such that we have one of the following:

(a): Clockwise

$$(2.3) \quad \begin{aligned} x(t_1) &:= \phi_1^{t_1}(x_0) : y^{(1)} = \begin{bmatrix} h_1(x(t_1)) \\ h_2(x(t_1)) \end{bmatrix} = \begin{bmatrix} 0 \\ y_2^{(1)} \end{bmatrix}, y_2^{(1)} < 0 ; \\ x(t_2) &:= \phi_3^{t_2}(x(t_1)) : y^{(2)} = \begin{bmatrix} h_1(x(t_2)) \\ h_2(x(t_2)) \end{bmatrix} = \begin{bmatrix} y_1^{(2)} \\ 0 \end{bmatrix}, y_1^{(2)} > 0 ; \\ x(t_3) &:= \phi_4^{t_3}(x(t_2)) : y^{(3)} = \begin{bmatrix} h_1(x(t_3)) \\ h_2(x(t_3)) \end{bmatrix} = \begin{bmatrix} 0 \\ y_2^{(3)} \end{bmatrix}, y_2^{(3)} > 0 ; \\ x(t_4) &:= \phi_2^{t_4}(x(t_3)) : y^{(4)} = \begin{bmatrix} h_1(x(t_4)) \\ h_2(x(t_4)) \end{bmatrix} = \begin{bmatrix} y_1^{(4)} \\ 0 \end{bmatrix}, y_1^{(4)} < 0 ; \end{aligned}$$

(b): Counter-clockwise

$$\begin{aligned} x(t_1) &:= \phi_2^{t_1}(x_0) : y^{(1)} = \begin{bmatrix} h_1(x(t_1)) \\ h_2(x(t_1)) \end{bmatrix} = \begin{bmatrix} 0 \\ y_2^{(1)} \end{bmatrix}, y_2^{(1)} > 0 ; \\ x(t_2) &:= \phi_4^{t_2}(x(t_1)) : y^{(2)} = \begin{bmatrix} h_1(x(t_2)) \\ h_2(x(t_2)) \end{bmatrix} = \begin{bmatrix} y_1^{(2)} \\ 0 \end{bmatrix}, y_1^{(2)} > 0 ; \\ x(t_3) &:= \phi_3^{t_3}(x(t_2)) : y^{(3)} = \begin{bmatrix} h_1(x(t_3)) \\ h_2(x(t_3)) \end{bmatrix} = \begin{bmatrix} 0 \\ y_2^{(3)} \end{bmatrix}, y_2^{(3)} < 0 ; \\ x(t_4) &:= \phi_1^{t_4}(x(t_3)) : y^{(4)} = \begin{bmatrix} h_1(x(t_4)) \\ h_2(x(t_4)) \end{bmatrix} = \begin{bmatrix} y_1^{(4)} \\ 0 \end{bmatrix}, y_1^{(4)} < 0 . \end{aligned}$$

We summarize our proposal to characterize (at first order) spiral-like dynamics around  $\Sigma$ .

**Definition 2.2.** *We say that there is clockwise (respectively, counterclockwise) spiral-like dynamics around  $\Sigma$ , if:*

- (a) Table 1 and (2.3)-(a) hold (clockwise case), or
- (b) Table 2 and (2.3)-(b) hold (counterclockwise case).

□

**Remark 2.3.** Observe that the conditions of Table 1 and (2.3)-(a) imply the following for the mean values of the derivative of  $y$  in each region:

$$(2.4) \quad \begin{aligned} R_1 : \quad \int_0^1 w_1^1(\phi_1^{st_1}(x_0)) ds &> 0 , & R_3 : \quad \int_0^1 w_3^2(\phi_3^{st_2}(x(t_1))) ds &> 0 , \\ R_4 : \quad \int_0^1 w_4^1(\phi_4^{st_3}(x(t_2))) ds &< 0 , & R_2 : \quad \int_0^1 w_2^2(\phi_2^{st_4}(x(t_3))) ds &< 0 . \end{aligned}$$

Naturally, similar relations hold if we assume the conditions of Table 2 and (2.3)-(b), namely:

$$(2.5) \quad \begin{aligned} R_2 : \quad \int_0^1 w_2^1(\phi_2^{st_1}(x_0)) ds &> 0 , & R_4 : \quad \int_0^1 w_4^2(\phi_4^{st_2}(x(t_1))) ds &< 0 , \\ R_3 : \quad \int_0^1 w_3^1(\phi_3^{st_3}(x(t_2))) ds &< 0 , & R_1 : \quad \int_0^1 w_1^2(\phi_1^{st_4}(x(t_3))) ds &> 0 . \end{aligned}$$

□

**Remark 2.4.** In [6, Section 20], Filippov considers the use of spiraling toward the origin in  $\mathbb{R}^2$ , and gives sufficient conditions for asymptotic stability of this fixed point (in [6, Section 23] an extension of this analysis to  $\mathbb{R}^3$  is also discussed). In the context of intersection of two manifolds in  $\mathbb{R}^n$ , there appears to have been little work on spiraling behavior around, and toward, this intersection. A proposal was made in [5]. There, the authors gave the sufficient conditions expressed by the signs constraints of Tables 3 and 4 below, requiring these to hold (uniformly) in a neighborhood of  $\Sigma$ , in the two clockwise and counterclockwise cases:

TABLE 3. Signs of matrix  $W$  in clockwise spiraling around  $\Sigma$  from [5].

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$w_i^1$	+	-	+	-
$w_i^2$	-	-	+	+

TABLE 4. Signs of matrix  $W$  in counterclockwise spiraling around  $\Sigma$  from [5].

Component	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$w_i^1$	-	+	-	+
$w_i^2$	+	+	-	-

Clearly, the conditions of Table 3 (respectively Table 4) imply those expressed by Table 1 (respectively Table 2), and void the need for (2.3). However, this characterization from [5] is too restrictive, since it requires a monotone behavior of the variables  $y_1$  and  $y_2$  in each region  $R_i$ ,  $i = 1, 2, 3, 4$ , which is not needed for spiral-like dynamics around  $\Sigma$ .  $\square$

### 3. SPIRAL ATTRACTIVITY

The conditions of Definition 2.2 give a spiral-like behavior of orbits near  $\Sigma$ , but are not sufficient to characterize whether or not  $\Sigma$  is attractive. Our next goal is to arrive at such characterization in terms of the projected vector fields  $w_i^j$ 's. Moreover, if we are on  $\Sigma$ , can we decide when  $\Sigma$  loses attractivity while there is spiraling behavior around it? In this section and the next, we address these questions.

Let  $x \in \Sigma$ , and observe that –for  $x \in \Sigma$ – Tables 1 and 3 give the same information, and so do Tables 2 and 4.

What follows is the plan of our argument to characterize *spiral attractivity* of  $\Sigma$ .

- (i) Take a point  $x_0$  at  $h$ -distance  $\rho$  from  $\Sigma$ . Here, the  $h$ -distance of a point  $x$  from  $\Sigma$  is defined as the 1-norm of  $h(x)$ :  $\|h(x)\|_1 = |h_1(x)| + |h_2(x)|$ . Without loss of generality, then, we can take  $x_0$  such that  $h_1(x_0) = -\rho$  and  $h_2(x_0) = 0$ , and therefore we will assume that  $x_0 \in \Sigma_2^-$ .
- (ii) We are only interested in the evolution of the  $y$ -variables (starting at  $x_0$ ), that is in the  $h$ -distance from  $\Sigma$ . We reiterate that one has all transversal crossings of the manifolds  $h_j = 0$  (i.e.,  $y_j = 0$ ),  $j = 1, 2$ .
- (iii) After going around once, and returning to  $\Sigma_2^-$ , we will monitor the  $h$ -distance of the return point and compare it to the initial point. Attractivity of  $\Sigma$  would be implied by a decrease of this distance.

We first consider a simplified situation, where the end result will be arrived at more easily, and we obtain a necessary and sufficient condition, then we will consider the general case.

**3.1. Attractivity of  $\Sigma$  under simplified spiral-like conditions off  $\Sigma$ .** First, consider the case when the spiraling conditions of Table 3 hold in a neighborhood of  $\Sigma$ , and the  $w_i^j$ 's in that Table are constant ( $i = 1, \dots, 4$ ,  $j = 1, 2$ ); for example, this situation occurs if (at least locally) the original vector fields  $f_i$ ,  $i = 1, \dots, 4$ , are constant, and the manifolds  $\Sigma_{1,2}$  are planes.

We consider the evolution starting with a point at distance  $\rho$  from a point  $x \in \Sigma$ . We take 4 steps of a Euler discretization of the system (which is exact, on account of the assumption that the  $w_i^j$ 's are constant), which we will do so that on the first step we go from  $\Sigma_2^-$  to  $\Sigma_1^-$ , then from there to  $\Sigma_2^+$ , then to  $\Sigma_1^+$ , until the last step will make us return to  $\Sigma_2^-$ . By comparing initial and final point, we will see if distances have decreased (or not). This process gives the following.

(1) [From  $\Sigma_2^-$  to  $\Sigma_1^-$ ]. Let  $y^0$  be the  $y$ -value of  $x_0$ , that is  $y^0 = \begin{bmatrix} -\rho \\ 0 \end{bmatrix}$ . We take a Euler step until the first component of  $y$  becomes 0:

$$y^1 = y^0 + \tau_1 w_1, \quad \begin{bmatrix} y_1^1 \\ y_2^1 \end{bmatrix} = \begin{bmatrix} y_1^0 + \tau_1 w_1^1 \\ \tau_1 w_1^2 \end{bmatrix},$$

from which we get (since  $y_1^1 = 0$ ,  $y_1^0 < 0$ , and  $w_1^1 > 0$ ):

$$(3.1) \quad \tau_1 = -\frac{y_1^0}{w_1^1}.$$

(2) [From  $\Sigma_1^-$  to  $\Sigma_2^+$ ]. Now we obtain

$$y^2 = y^1 + \tau_2 w_3, \quad \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} = \begin{bmatrix} \tau_2 w_3^1 \\ \tau_1 w_1^2 + \tau_2 w_3^2 \end{bmatrix},$$

and therefore (since  $y_2^2 = 0$ ,  $w_1^2 < 0$ , and  $w_3^2 > 0$ ):

$$(3.2) \quad \tau_2 = -\tau_1 \frac{w_1^2}{w_3^2}.$$

(3) [From  $\Sigma_2^+$  to  $\Sigma_1^+$ ]. We have

$$y^3 = y^2 + \tau_3 w_4, \quad \begin{bmatrix} y_1^3 \\ y_2^3 \end{bmatrix} = \begin{bmatrix} \tau_2 w_3^1 + \tau_3 w_4^1 \\ \tau_3 w_4^2 \end{bmatrix},$$

and therefore (since  $y_1^3 = 0$ ,  $w_1^3 < 0$ , and  $w_4^3 > 0$ ):

$$(3.3) \quad \tau_3 = -\tau_2 \frac{w_3^1}{w_4^3}.$$

(4) [From  $\Sigma_1^+$  to  $\Sigma_2^-$ ]. We have

$$y^4 = y^3 + \tau_4 w_2, \quad \begin{bmatrix} y_1^4 \\ y_2^4 \end{bmatrix} = \begin{bmatrix} \tau_4 w_2^1 \\ \tau_3 w_4^2 + \tau_4 w_2^2 \end{bmatrix},$$

and therefore (since  $y_2^4 = 0$ ,  $w_2^2 < 0$ , and  $w_4^2 > 0$ ):

$$(3.4) \quad \tau_4 = -\tau_3 \frac{w_4^2}{w_2^2}.$$

At this point, we compare  $y_1^4$  with  $y_1^0$  (which was equal to  $-\rho$ ). Using the expressions found above for  $\tau_1, \dots, \tau_4$ , we easily obtain that

$$y_1^4 = y_1^0 \frac{w_1^2 w_3^1 w_4^2 w_2^1}{w_1^1 w_3^2 w_4^1 w_2^2}.$$

Therefore, we have obtained the key quantity to monitor (the multiplier):

$$(3.5) \quad \mu = \frac{w_1^2 w_3^1 w_4^2 w_2^1}{w_1^1 w_3^2 w_4^1 w_2^2}.$$

Notice that, because of Table 3, we always have  $\mu > 0$ .

Next, we consider the case of Table 4, and we look at 4 steps of the Euler discretization so that on each step we will go from  $\Sigma_2^-$  to  $\Sigma_1^+$ , from there to  $\Sigma_2^+$ , and then to  $\Sigma_1^-$ , until we return to  $\Sigma_2^-$ . Computations are similar to the previous case, and we now obtain:

$$y_1^4 = y_1^0 \frac{w_2^2 w_4^1 w_3^2 w_1^1}{w_2^1 w_4^2 w_3^1 w_1^2}.$$

Therefore, we obtain the multiplier in the present case:

$$(3.6) \quad \nu = \frac{w_2^2 w_4^1 w_3^2 w_1^1}{w_2^1 w_4^2 w_3^1 w_1^2}.$$

**Remark 3.1.** Because of Table 4, again we always have  $\nu > 0$ . Observe that  $\nu$  in (3.6) is precisely  $1/\mu$  with  $\mu$  in (3.5); this is in agreement with what we expect to have when we change the direction of motion.  $\square$

We formally summarize the above considerations in the following.

**Theorem 3.2.** *Let the conditions of Table 3 be valid in a neighborhood of  $\Sigma$ , and let the  $w_i^j$ 's be constant. Then,  $\Sigma$  is clockwise spiraling attractive if and only if  $\mu < 1$ , with  $\mu$  given by (3.5). Similarly, if the conditions of Table 4 hold, and the  $w_i^j$ 's are constant, then  $\Sigma$  is counterclockwise spiraling attractive if and only if  $\nu < 1$ , with  $\nu$  in (3.6).*

*Proof.* In the clockwise case, the proof is a consequence of the expression  $y_1^4 = \mu y_1^0$ . Similarly in the counterclockwise case.  $\square$

**3.2. Attractivity of  $\Sigma$  under general spiral-like conditions off  $\Sigma$ .** Here, we consider the case of spiral-like behavior off  $\Sigma$  under the general conditions of Definition 2.2.

The idea is similar to what we did in Section 3.1; however, we cannot use the Euler discretization since we will generally fail to have constant  $w_i^j$ 's. For this reason, we proceed as follows, first in the case of Table 1 and (2.3)-(a).

Let  $x_0 \in \Sigma_2^-$ , that is  $y(0) \equiv y^{(0)} = \begin{bmatrix} y_1^{(0)} \\ 0 \end{bmatrix} = \begin{bmatrix} -\rho \\ 0 \end{bmatrix}$ , and let us look at the solution at times  $t_1, t_2, t_3, t_4$ , defined as in (2.3)-(a). We stress that these times depend smoothly on  $x_0$ .

Recall that while the trajectory is in region  $R_i$ , then  $\dot{y} = w_i(\phi_i^t(\cdot))$ ,  $i = 1, 2, 3, 4$ . So, on account of (2.4), we have the following.

(1) [From  $\Sigma_2^-$  to  $\Sigma_1^-$ ]. We have  $x(t) = \phi_1^t(x_0)$ ,  $0 \leq t \leq t_1$ , and  $\dot{y} = w_1(x)$ , and so

$$y(t_1) = y(0) + t_1 \int_0^1 w_1(\phi_1^{st_1}(x_0)) ds,$$

and since  $y(t_1) = \begin{bmatrix} 0 \\ y_2(t_1) \end{bmatrix}$ ,  $y_1(0) = -\rho < 0$ , and  $\int_0^1 w_1(\phi_1^{st_1}(x_0)) ds > 0$ , we write

$$(3.7) \quad t_1 = -\frac{y_1(0)}{\int_0^1 w_1(\phi_1^{st_1}(x_0)) ds}.$$

(2) [From  $\Sigma_1^-$  to  $\Sigma_2^+$ ]. Now we have  $x(t) = \phi_3^t(x(t_1))$ ,  $0 \leq t \leq t_2$ , and  $\dot{y} = w_3(x)$ , and we obtain

$$y(t_2) = y(t_1) + t_2 \int_0^1 w_3(\phi_3^{st_2}(x(t_1))) ds.$$

Since  $y(t_2) = \begin{bmatrix} y_1(t_2) \\ 0 \end{bmatrix}$ ,  $y_2(t_1) < 0$ , and  $\int_0^1 w_3^2(\phi_3^{st_2}(x(t_1))) ds > 0$ , we write

$$(3.8) \quad t_2 = -\frac{y_2(t_1)}{\int_0^1 w_3^2(\phi_3^{st_2}(x(t_1))) ds}.$$

(3) [From  $\Sigma_2^+$  to  $\Sigma_1^+$ ]. Now we have  $x(t) = \phi_4^t(x(t_2))$ ,  $0 \leq t \leq t_3$ , and  $\dot{y} = w_4(x)$ , and we obtain

$$y(t_3) = y(t_2) + t_3 \int_0^1 w_4(\phi_4^{st_3}(x(t_2))) ds .$$

Since  $y(t_3) = \begin{bmatrix} 0 \\ y_2(t_3) \end{bmatrix}$ ,  $y_1(t_2) > 0$ , and  $\int_0^1 w_4(\phi_4^{st_3}(x(t_2))) ds < 0$ , we have

$$(3.9) \quad t_3 = -\frac{y_1(t_2)}{\int_0^1 w_4^1(\phi_4^{st_3}(x(t_2))) ds} .$$

(4) [From  $\Sigma_1^+$  to  $\Sigma_2^-$ ]. Finally, here we have  $x(t) = \phi_2^t(x(t_3))$ ,  $0 \leq t \leq t_4$ , and  $\dot{y} = w_2(x)$ , and we obtain

$$y(t_4) = y(t_3) + t_4 \int_0^1 w_2(\phi_2^{st_4}(x(t_3))) ds .$$

Since  $y(t_4) = \begin{bmatrix} y_1(t_4) \\ 0 \end{bmatrix}$ ,  $y_2(t_3) > 0$ , and  $\int_0^1 w_2^2(\phi_2^{st_4}(x(t_3))) ds < 0$ , we obtain

$$(3.10) \quad t_4 = -\frac{y_2(t_3)}{\int_0^1 w_2^2(\phi_2^{st_4}(x(t_3))) ds} .$$

Now we compare  $y_1(t_4) = t_4 \int_0^1 w_2^1(\phi_2^{st_4}(x(t_3))) ds$  with  $y_1(0)$ . Using the above for  $t_1, \dots, t_4$ , and the fact that

$$y_2(t_1) = t_1 \int_0^1 w_1^2(\phi_1^{st_1}(x_0)) ds, \quad y_1(t_2) = t_2 \int_0^1 w_3^1(\phi_3^{st_2}(x(t_1))) ds, \quad y_2(t_3) = t_3 \int_0^1 w_4^2(\phi_4^{st_3}(x(t_2))) ds,$$

we obtain

$$(3.11) \quad y_1(t_4) = y_1(0) \frac{\int_0^1 w_1^2(\phi_1^{st_1}(x_0)) ds \int_0^1 w_3^1(\phi_3^{st_2}(x(t_1))) ds \int_0^1 w_4^2(\phi_4^{st_3}(x(t_2))) ds \int_0^1 w_2^1(\phi_2^{st_4}(x(t_3))) ds}{\int_0^1 w_1^1(\phi_1^{st_1}(x_0)) ds \int_0^1 w_3^2(\phi_3^{st_2}(x(t_1))) ds \int_0^1 w_4^1(\phi_4^{st_3}(x(t_2))) ds \int_0^1 w_2^2(\phi_2^{st_4}(x(t_3))) ds} .$$

Now, as the distance  $\rho \rightarrow 0$ , the point  $x_0$  approaches a value  $x \in \Sigma$ . Since all the integrands above converge to their values at this point  $x \in \Sigma$  (and the integrands are bounded), all integrals reduce to evaluations made at the point  $x \in \Sigma$ , and we obtain what we call *instantaneous rate of attractivity* of  $\Sigma$  as characterized by the multiplier  $\mu$  below (cfr. with (3.5)):

$$(3.12) \quad \mu(x) = \frac{w_1^2(x)w_3^1(x)w_4^2(x)w_2^1(x)}{w_1^1(x)w_3^2(x)w_4^1(x)w_2^2(x)}, \quad x \in \Sigma .$$

Similarly, when we proceed in the counterclockwise case, under the conditions of Table 2 and (2.3)-(b), we end up with the instantaneous rate of attractivity as characterized by the multiplier  $\nu$  (cfr. with (3.6)), which is in fact  $1/\mu(x)$ :

$$(3.13) \quad \nu(x) = \frac{w_2^2(x)w_4^1(x)w_3^2(x)w_1^1(x)}{w_2^1(x)w_4^2(x)w_3^1(x)w_1^2(x)}, \quad x \in \Sigma .$$

We summarize in the following result.

**Theorem 3.3.** *Assume that for  $x \in \Sigma$  the signs of Table 3 hold, and we have  $\mu < 1$ , with  $\mu$  given by (3.12). Then  $\Sigma$  is clockwise spirally attractive for solution trajectories starting at a point  $x_0$  in a sufficiently small neighborhood of  $\Sigma$ . Similarly, if for  $x \in \Sigma$  the signs of Table 4 hold, and we have  $\nu < 1$ , with  $\nu$  given by (3.13), then  $\Sigma$  is counterclockwise spiraling attractive for solution trajectories starting at a point  $x_0$  in a sufficiently small neighborhood of  $\Sigma$ .*

*Proof.* In the clockwise case, the result follows from (3.11), using continuity –with respect to  $x_0$ – of the  $w_i^j$ ’s and of the times  $t_1, \dots, t_4$ , and permanence of the signs in (2.4). Similarly in the counterclockwise case.  $\square$

**Remark 3.4.** Finally, we observe that it is a simple computation to verify that the bilinear vector field of Example 1.2 (and, for that matters, the moments sliding vector fields of [3]) is well defined and varies smoothly when  $\Sigma$  is spiraling attractive according to the above characterization. This consideration prompts the next definition.  $\square$

**Definition 3.5.** *We say that the motion (i.e., the sliding regime) on  $\Sigma$  is “clockwise spiraling attractive” if for  $x \in \Sigma$  the signs of Table 3 hold, and we have  $\mu < 1$ , with  $\mu$  given by (3.12). Similarly, we say that the sliding regime is “counterclockwise spiraling attractive” if for  $x \in \Sigma$  the signs of Table 4 hold, and we have  $\nu < 1$ , with  $\nu$  given by (3.13).*  $\square$

**Remarks 3.6.**

- (a) When  $\mu = 1$  (which is the same as  $\nu = 1$ ), at first order  $\Sigma$  loses attractivity, and thus generically (see below) solution trajectories would leave  $\Sigma$ .
- (b) In [5, Table 3], to have attractive clockwise spiral-like sliding on  $\Sigma$ , the authors required Table 3 to hold as well as the following sufficient conditions:

$$(3.14) \quad w_1^1 > -w_1^2, w_2^1 > w_2^2, w_3^2 > w_3^1, -w_4^1 > w_4^2.$$

Quite clearly, (3.14), and Table 3, will give  $\mu < 1$ . At the same time, the conditions (3.14) are much too stringent.

- (c) For completeness, we emphasize that the choice of initial condition on  $\Sigma_2^-$  made in the above derivation is not a restriction, and that the same value of  $\mu$  is arrived at regardless of where we place our initial condition at  $h$ -distance  $\rho$  from  $\Sigma$ .  $\square$

#### 4. LOSS OF ATTRACTIVITY AND LEAVING $\Sigma$

In this section, we conclude with a discussion of what to expect when  $\Sigma$  loses attractivity, while there continues to be spiral-like motion around it; namely, say the spiral-like conditions of Table 3 (or Table 4) hold for  $x \in \Sigma$ , and there is clockwise (counterclockwise) spiraling motion around  $\Sigma$ .

So, suppose that, say under the conditions of Table 3, we have a smooth trajectory moving on  $\Sigma$ , which reaches a point  $\bar{x} \in \Sigma$ , where  $\mu(\bar{x}) = 1$ . We may actually assume that this trajectory itself is the solution of a differential equation  $\dot{x} = f_F(x)$ , where the vector field  $f_F$  is a smooth Filippov sliding vector field (i.e., a smooth selection in (1.8)). For example, this could be the bilinear vector field of (1.9) (see Remark 3.4), but other choices are also possible; in any case, we are assuming to have  $\bar{x} = x(\bar{t})$ , and  $\mu(\bar{x}) = 1$ , while  $\mu(x(t)) < 1$  for  $t < \bar{t}$ .

Now, as long as the sign-conditions of Table 3 continue to hold, none of the sliding vector fields  $f_{\Sigma_{1,2}^\pm}$  on  $\Sigma_{1,2}^\pm$  is also tangent to  $\Sigma$  (hence, it is not an admissible sliding vector field on  $\Sigma$ ), and therefore the first order exit conditions identified in [4] do not provide an exit mechanism. Indeed, for example, the vector field  $f_F$  in (1.9) and the associated trajectory continue to be well defined past the point  $\bar{x}$ .

So,  $\mu(x(t))$  is itself well defined for  $t > \bar{t}$ , and  $t - \bar{t}$  sufficiently small. Since  $\mu$  (viewed as function of  $x(t)$ ) is smooth, generically we will have that the function  $\mu - 1$  changes sign at  $\bar{t}$ . As a consequence,  $\Sigma$  will cease to be attractive past  $\bar{x}$ , and sliding motion on  $\Sigma$  –albeit existing– would become repulsive, hence ill-posed. So, the following question naturally arises: which motion, off  $\Sigma$ , is well posed?

In the present case, the answer is that **any** of the vector fields  $f_1, f_2, f_3, f_4$ , gives a possible choice. Indeed, there is no preferred direction along which we should leave  $\Sigma$ . One can take a perturbed value off  $\Sigma$  so that the resulting motion –since the conditions of Table 1 hold– will be of spiraling type and away from  $\Sigma$ ; hence, purely on qualitative grounds, it does not seem to be crucial in which region one takes the perturbed value.

The situation just described bears some similarity to the case of repulsive sliding on a co-dimension 1 manifold, see Remark 1.1. There, one could do one of three things at any point during repulsive sliding: (i) continue with the ill-posed repulsive sliding motion, (ii) exit with  $f_1$ , or (iii)

exit with  $f_2$ . Note that these exits are not tangential, and also note that during repulsive sliding the coefficient  $\alpha$  in (1.4) verifies  $0 < \alpha < 1$ . In our present situation, we have a similar scenario, but five different things can occur: we can (i) continue with an ill-posed repulsive sliding motion, or (ii)-(v) exit with any of  $f_1, f_2, f_3, f_4$ . Again, none of these exits is tangential. Again, during repulsive sliding motion one has well defined coefficients of the convex combination (e.g.,  $\alpha$  and  $\beta$  in (1.9) are smoothly varying and in  $(0, 1)$ ). But there are also important differences between these two cases. For one, in the case of sliding on a co-dimension 1 manifold, it is not possible to arrive at repulsive sliding through attractive sliding, except if both  $f_1$  and  $f_2$  become simultaneously tangential<sup>1</sup> to  $\Sigma$ ; in the present case of sliding on the intersection  $\Sigma$  of 2 manifolds while there is spiraling motion around  $\Sigma$ , instead, one can arrive at repulsive sliding through a genuine loss of attractivity of the sliding motion on  $\Sigma$ , without any of the subsliding vector fields having placed itself tangent to  $\Sigma$ . Moreover, in the case of repulsive sliding regime on a co-dimension 1 manifold, the dynamics resulting from leaving with one of  $f_1$  or  $f_2$  are surely qualitatively different, whereas in the present case of sliding on the intersection  $\Sigma$  while there is spiraling motion away from  $\Sigma$ , exiting with any of the  $f_i$ 's ( $i = 1, 2, 3, 4$ ) gives qualitatively similar dynamics.

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SCHOOL OF MATHEMATICS, GEORGIA TECH, ATLANTA, GA 30332 U.S.A.

*E-mail address:* dieci@math.gatech.edu

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<sup>1</sup>This is a degenerate situation in  $\mathbb{R}^2$ , but already in  $\mathbb{R}^3$  it is generic; see [8, 2].