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## Lyapunov Exponents: Computation

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### History and Scope

In 1892, in his doctoral thesis *The general problem of the stability of motion* (reprinted in its original form in [33]), Lyapunov introduced several groundbreaking concepts to investigate stability in differential equations. These are collectively known as Lyapunov Stability Theory. Lyapunov was concerned with the asymptotic stability of solutions with respect to perturbations of initial data. Among other techniques (e.g., what are now known as first and second Lyapunov methods), he introduced a new tool to analyze the stability of solutions of linear time-varying systems of differential equations, the so-called characteristic numbers, now commonly and appropriately called *Lyapunov exponents*.

Simply put, these characteristic numbers play the role that the (real parts of the) eigenvalues play for time-invariant linear systems. Lyapunov considered the $n$-dimensional linear system

$$\dot{x} = A(t)x\,, \tag{1}$$

where $A$ is continuous and bounded: $\sup_t \|A(t)\| < \infty$. He showed that "if all characteristic numbers (see below for their definition) of (1) are negative, then the zero solution of (1) is asymptotically (in fact, exponentially) stable." He further proved an important characterization of stability relative to the perturbed linear system

$$\dot{x} = A(t)x + f(t,x)\,, \tag{2}$$

where $f(t,0) = 0$, so that $x = 0$ is a solution of (2), and further $f(t,x)$ is assumed to be "small" near $x = 0$ (this situation is what one expects from a linearized analysis about a bounded solution trajectory). Relative to (2), Lyapunov proved that "if the linear system (1) is *regular*, and all its characteristic numbers are negative, then the zero solution of (2) is asymptotically stable." About 30 years later, it was shown by Perron in [38] that the assumption of regularity cannot generally be removed.

### Definition

We refer to the monograph [1] for a comprehensive definition of Lyapunov exponents, regularity, and so forth. Here, we simply recall some of the key concepts.

Consider (1) and let us stress that the matrix function $A(t)$ may be either given or obtained as the linearization about the solution of a nonlinear differential equation; e.g., $\dot{y} = f(y)$ and $A(t) = Df(y(t))$ (note that in this case, in general, $A$ will depend on the initial condition used for the nonlinear problem). Now, let $X$ be a fundamental matrix solution of (1), and consider the quantities

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln ||X(t)e_i||\,,\ i = 1,\dots,n, \tag{3}$$

B. Engquist (eds.), *Encyclopedia of Applied and Computational Mathematics*,
DOI 10.1007/978-3-540-70529-1, © Springer-Verlag Berlin Heidelberg 2013

where $e_i$ denotes the $i$th standard unit vector, $i = 1, \ldots, n$. When $\sum_{i=1}^{n} \lambda_i$ is minimized with respect to all possible fundamental matrix solutions, then the $\lambda_i$ are called the characteristic numbers, or Lyapunov exponents, of the system. It is customary to consider them ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Similar definitions can be given for $t \to -\infty$ and/or with lim inf replacing the lim sup, but the description above is the prevailing one. An important consequence of *regularity* of a given system is that in (3) one has limits instead of lim sups.

### More Recent Theory

Given that the condition of regularity is not easy to verify for a given system, it was unclear what practical use one was going to make of the Lyapunov exponents in order to study stability of a trajectory. Moreover, even assuming that the system is regular, it is effectively impossible to get a handle on the Lyapunov exponents except through their numerical approximation. It then becomes imperative to have some comfort that what one is trying to approximate is robust; in other words, it is the Lyapunov exponents themselves that will need to be stable with respect to perturbations of the function $A$ in (1). Unfortunately, regularity is not sufficient for this purpose.

Major theoretical advances to resolve the two concerns above took place in the late 1960s, thanks to the work of Oseledec and Millionshchikov (e.g., see [36] and [34]). Oseledec was concerned with stability of trajectories on a (bounded) attractor, on which one has an invariant measure. In this case, Oseledec's *Multiplicative Ergodic Theorem* validates regularity for a broad class of linearized systems; the precise statement of this theorem is rather technical, but its practical impact is that (with respect to the invariant measure) almost all trajectories of the nonlinear system will give rise to a regular linearized problem. Millionshchikov introduced the concept of *integral separation*, which is the condition needed for stability of the Lyapunov exponents with respect to perturbations in the coefficient matrix, and further gave important results on the prevalence of this property within the class of linear systems.

### Further Uses of Lyapunov Exponents

Lyapunov exponents found an incredible range of applicability in several contexts, and both theory and computational methods have been further extended to discrete dynamical systems, maps, time series, etc. In particular:

(i) The largest Lyapunov exponent of (2), $\lambda_1$, characterizes the rate of separation of trajectories (with infinitesimally close initial conditions). For this reason, a positive value of $\lambda_1$ (coupled with compactness of the phase space) is routinely taken as an indication that the system is *chaotic* (see [37]).

(ii) Lyapunov exponents are used to estimate *dimension* of attractors through the Kaplan-Yorke formula (Lyapunov dimension):

$$\mathrm{Dim}_L = k + (\lambda_1 + \lambda_2 + \cdots + \lambda_k)/|\lambda_{k+1}|$$

where $k$ is the largest index $i$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_i > 0$. See [31] for the original derivation of the formula and [9] for its application to the 2-d Navier-Stokes equation.

(iii) The sum of all the positive Lyapunov exponents is used to estimate the entropy of a dynamical system (see [3]).

(iv) Lyapunov exponents have also been used to characterize persistence and degree of smoothness of invariant manifolds (see [26] and see [12] for a numerical study).

(v) Lyapunov exponents have even been used in studies of piecewise-smooth differential equations, where a formal linearized problem as in (1) does not even exist (see [27, 35]).

(vi) Finally, there has been growing interest also in approximating bases for the *growth directions* associated to the Lyapunov exponents. In particular, there is interest in obtaining representations for the stable (and unstable) subspaces of (1) and in their use to ascertain stability of traveling waves. For example, see [23, 39].

## Factorization Techniques

Many of the applications listed above are related to nonlinear problems, which in itself is witness to the power of linearized analysis based on the Lyapunov exponents. Still, the computational task of approximating some or all of the Lyapunov exponents for dynamical systems defined by the flow of a differential equation is ultimately related to the linear problem (1), and we will thus focus on this linear problem.

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Techniques for numerical approximation of Lyapunov exponents are based upon smooth matrix factorizations of fundamental matrix solutions $X$, to bring it into a form from which it is easier to extract the Lyapunov exponents. In practice, two techniques have been studied: based on the QR factorization of $X$ and based on the SVD (singular value decomposition) of $X$. Although these techniques have been adapted to the case of incomplete decompositions (useful when only a few Lyapunov exponents are needed) or to problems with Hamiltonian structure, we only describe them in the general case when the entire set of Lyapunov exponents is sought, the problem at hand has no particular structure, and the system is regular. For extensions, see the references.

## QR Methods

The idea of QR methods is to seek the factorization of a fundamental matrix solution as $X(t) = Q(t)R(t)$, for all $t$, where $Q$ is an orthogonal matrix valued function and $R$ is an upper triangular matrix valued function with positive diagonal entries. The validity of this factorization has been known since Perron [38] and Diliberto [25], and numerical techniques based upon the QR factorization date back at least to [4].

QR techniques come in two flavors, continuous and discrete, and methods for quantifying the error in approximation of Lyapunov exponents have been developed in both cases (see [15–17, 21, 40]).

### Continuous QR

Upon differentiating the relation $X = QR$ and using (1), we have

$$AQR = Q\dot{R} + \dot{Q}R \quad \text{or} \quad \dot{Q} = AQ - QB\,, \qquad (4)$$

where $\dot{R} = BR$; hence, $B$ must be upper triangular. Now, let us formally set $S = Q^T\dot{Q}$ and note that since $Q$ is orthogonal then $S$ must be skew symmetric. Now, from $B = Q^TAQ - Q^T\dot{Q}$ it is easy to determine at once the strictly lower triangular part of $S$ (and from this, all of it) and the entries of $B$. To sum up, we have two differential equations, for $Q$ and for $R$. Given $X(0) = Q_0R_0$, we have

$$\dot{Q} = QS(Q,A)\,, \quad Q(0) = Q_0\,, \qquad (5)$$

$$\dot{R} = B(t)R\,, \quad R(0) = R_0\,,$$
$$B := Q^TAQ - S(Q,A) \qquad (6)$$

The diagonal entries of $R$ are used to retrieve the exponents:

$$\lambda_i = \lim_{t\to\infty}\frac{1}{t}\int_0^t (Q^T(s)A(s)Q(s))_{ii}\,ds\,,\ i = 1,\ldots,n. \qquad (7)$$

A unit upper triangular representation for the growth directions may be further determined by $\lim_{t\to\infty}\operatorname{diag}(R^{-1}(t))R(t)$ (see [13, 22, 23]).

### Discrete QR

Here one seeks the QR factorization of the fundamental matrix $X$ at discrete points $0 = t_0 < t_1 < \cdots < t_k < \cdots$, where $t_k = t_{k-1} + h_k$, $h_k \geq \hat{h} > 0$. Let $X_0 = Q_0R_0$, and suppose we seek the QR factorization of $X(t_{k+1})$. For $j = 0,\ldots,k$, progressively define $Z_{j+1}(t) = X(t,t_j)Q_j$, where $X(t,t_j)$ solves (1) for $t \geq t_j$, $X(t_j,t_j) = I$, and $Z_{j+1}$ is the solution of

$$\begin{cases} \dot{Z}_{j+1} = A(t)Z_{j+1}\,, & t_j \leq t \leq t_{j+1} \\ Z_{j+1}(t_j) = Q_j\,. \end{cases} \qquad (8)$$

Update the QR factorization as

$$Z_{j+1}(t_{j+1}) = Q_{j+1}R_{j+1}\,, \qquad (9)$$

and finally observe that

$$X(t_{k+1}) = Q_{k+1}\,[R_{k+1}R_k \cdots R_1R_0] \qquad (10)$$

is the QR factorization of $X(t_{k+1})$. The Lyapunov exponents are obtained from the relation

$$\lim_{k\to\infty}\frac{1}{t_k}\sum_{j=0}^{k}\log(R_j)_{ii}\,, i = 1,\ldots,n\,. \qquad (11)$$

## SVD Methods

Here one seeks to compute the SVD of $X$: $X(t) = U(t)\Sigma(t)V^T(t)$, for all $t$, where $U$ and $V$ are orthogonal and $\Sigma = \operatorname{diag}(\sigma_i,\ i = 1\ldots,n)$, with $\sigma_1(t) \geq \sigma_2(t) \geq \cdots \geq \sigma_n(t)$. If the singular values are distinct, the following differential equations $U$, $V$, and $\Sigma$ hold. Letting $G = U^TAU$, they are

$$\dot{U} = UH, \quad \dot{V}^T = -KV^T, \quad \dot{\Sigma} = D\Sigma, \qquad (12)$$

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where $D = \mathrm{diag}(G)$, $H^T = -H$, and $K^T = -K$, and for $i \neq j$,

$$H_{ij} = \frac{G_{ij}\sigma_j^2 + G_{ji}\sigma_i^2}{\sigma_j^2 - \sigma_i^2}, \quad K_{ij} = \frac{(G_{ij} + G_{ji})\sigma_i\sigma_j}{\sigma_j^2 - \sigma_i^2}. \tag{13}$$

From the SVD of $X$, the Lyapunov exponents may be obtained as

$$\lim_{t\to\infty} \frac{1}{t} \ln \sigma_i(t) \,. \tag{14}$$

Finally, an orthogonal representation for the growth directions may be determined by $\lim_{t\to\infty} V(t)$ (see [10, 13, 22, 23]).

## Numerical Implementation

Although algorithms based upon the above techniques appear deceivingly simple to implement, much care must be exercised in making sure that they perform as one would expect them to. (For example, in the continuous QR and SVD techniques, it is mandatory to maintain the factors $Q$, $U$, and $V$ orthogonal.) Fortran software codes for approximating Lyapunov exponents of linear and nonlinear problems have been developed and tested extensively and provide a combined state of the knowledge insofar as numerical methods suited for this specific task. See [14, 20, 24].

## References

1. Adrianova, L.Ya.: Introduction to Linear Systems of Differential Equations (Trans. from the Russian by Peter Zhevandrov). Translations of Mathematical Monographs, vol. 146, pp. x+204. American Mathematical Society, Providence (1995)
2. Aston, P.J., Dellnitz, M.: The computation of Lyapunov exponents via spatial integration with application to blowout bifurcations. Comput. Methods Appl. Mech. Eng. **170**, 223–237 (1999)
3. Barreira, L., Pesin, Y.: Lyapunov Exponents and Smooth Ergodic Theory. University Lecture Series, vol. 23. American Mathematical Society, Providence (2001)
4. Benettin, G., Galgani, L., Giorgilli, A., Strelcyn, J.-M.: Lyapunov exponents for smooth dynamical systems and for Hamiltonian systems: a method for computing all of them. Part 1: Theory, and . . . Part 2: Numerical applications. Meccanica **15**, 9–20, 21–30 (1980)
5. Bridges, T., Reich, S.: Computing Lyapunov exponents on a Stiefel manifold. Physica D **156**, 219–238 (2001)
6. Bylov, B.F., Vinograd, R.E., Grobman, D.M., Nemyckii, V.V.: The Theory of Lyapunov Exponents and Its Applications to Problems of Stability. Nauka, Moscow (1966)
7. Calvo, M.P., Iserles, A., Zanna, A.: Numerical solution of isospectral flows. Math. Comput. **66**(220), 1461–1486 (1997)
8. Christiansen, F., Rugh, H.H.: Computing Lyapunov spectra with continuous Gram-Schmidt orthonormalization. Nonlinearity **10**, 1063–1072 (1997)
9. Constantin, P., Foias, C.: Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations. Commun. Pure Appl. Math. **38**, 1–27 (1985)
10. Dieci, L., Elia, C.: The singular value decomposition to approximate spectra of dynamical systems. Theoretical aspects. J. Differ. Equ. **230**(2), 502–531 (2006)
11. Dieci, L., Lopez, L.: Smooth SVD on symplectic group and Lyapunov exponents approximation. CALCOLO **43**(1), 1–15 (2006)
12. Dieci, L., Lorenz, J.: Lyapunov type numbers and torus breakdown: numerical aspects and a case study. Numer. Algorithms **14**, 79–102 (1997)
13. Dieci, L., Van Vleck, E.S.: Lyapunov spectral intervals: theory and computation. SIAM J. Numer. Anal. **40**(2), 516–542 (2002)
14. Dieci, L., Van Vleck, E.S.: LESLIS and LESLIL: codes for approximating Lyapunov exponents of linear systems. Technical report, Georgia Institute of Technology. http://www.math.gatech.edu/~dieci (2004)
15. Dieci, L., Van Vleck, E.S.: On the error in computing Lyapunov exponents by QR methods. Numer. Math. **101**(4), 619–642 (2005)
16. Dieci, L., Van Vleck, E.S.: Perturbation theory for approximation of Lyapunov exponents by QR methods. J. Dyn. Differ. Equ. **18**(3), 815–840 (2006)
17. Dieci, L., Van Vleck, E.S.: On the error in QR integration. SIAM J. Numer. Anal. **46**(3), 1166–1189 (2008)
18. Dieci, L., Russell, R.D., Van Vleck, E.S.: Unitary integrators and applications to continuous orthonormalization techniques. SIAM J. Numer. Anal. **31**(1), 261–281 (1994)
19. Dieci, L., Russell, R.D., Van Vleck, E.S.: On the computation of Lyapunov exponents for continuous dynamical systems. SIAM J. Numer. Anal. **34**, 402–423 (1997)
20. Dieci, L., Jolly, M., Van Vleck, E.S.: LESNLS and LESNLL: codes for approximating Lyapunov exponents of nonlinear systems. Technical report, Georgia Institute of Technology. http://www.math.gatech.edu/~dieci (2005)
21. Dieci, L., Jolly, M., Rosa, R., Van Vleck, E.: Error on approximation of Lyapunov exponents on inertial manifolds: the Kuramoto-Sivashinsky equation. J. Discret. Contin. Dyn. Syst. Ser. B **9**(3–4), 555–580 (2008)
22. Dieci, L., Elia, C., Van Vleck, E.S.: Exponential dichotomy on the real line: SVD and QR methods. J. Differ. Equ. **248**(2), 287–308 (2010)
23. Dieci, L., Elia, C., Van Vleck, E.S.: Detecting exponential dichotomy on the real line: SVD and QR algorithms. BIT **51**(3), 555–579 (2011)
24. Dieci, L., Jolly, M.S., Van Vleck, E.S.: Numerical techniques for approximating Lyapunov exponents and their implementation. ASME J. Comput. Nonlinear Dyn. **6**, 011003–1–7 (2011)
25. Diliberto, S.P.: On systems of ordinary differential equations. In: Lefschetz, S. (ed.) Contributions to the Theory of AU2

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Nonlinear Oscillations. Annals of Mathematics Studies, vol. 20, pp. 1–38. Princeton University Press, Princeton (1950)

26. Fenichel, N.: Persistence and smoothness of invariant manifolds for flows. Indiana Univ. Math. J. **21**, 193–226 (1971)
27. Galvanetto, U.: Numerical computation of Lyapunov exponents in discontinuous maps implicitly defined. Comput. Phys. Commun. **131**, 1–9 (2000)
28. Geist, K., Parlitz, U., Lauterborn, W.: Comparison of different methods for computing Lyapunov exponents. Prog. Theor. Phys. **83**, 875–893 (1990)
29. Goldhirsch, I., Sulem, P.L., Orszag, S.A.: Stability and Lyapunov stability of dynamical systems: a differential approach and a numerical method. Physica D **27**, 311–337 (1987)
30. Greene, J.M., Kim, J.-S.: The calculation of Lyapunov spectra. Physica D **24**, 213–225 (1987)
31. Kaplan, J.L., Yorke, J.A.: Chaotic behavior of multidimensional difference equations. In: Peitgen, H.-O., Walter, H.-O. (eds.) Functional Differential Equations and Approximations of Fixed Points. Lecture Notes in Mathematics, vol. 730. Springer, Berlin (1979)
32. Leimkuhler, B.J., Van Vleck, E.S.: Orthosymplectic integration of linear Hamiltonian systems. Numer. Math. **77**(2), 269–282 (1997)
33. Lyapunov, A.: Problém Géneral de la Stabilité du Mouvement. Int. J. Control **53**, 531–773 (1992)
34. Millionshchikov, V.M.: Systems with integral division are everywhere dense in the set of all linear systems of differential equations. Differ. Uravn. **5**, 1167–1170 (1969)
35. Müller, P.: Calculation of Lyapunov exponents for dynamic systems with discontinuities. Chaos Solitons Fractals **5**, 1671–1681 (1995)
36. Oseledec, V.I.: A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. Trans. Mosc. Math. Soc. **19**, 197–231 (1968)
37. Ott, E.: Chaos in Dynamical Systems, 2nd edn. Cambridge University Press, Cambridge (2002)
38. Perron, O.: Die Ordnungszahlen Linearer Differentialgleichungssysteme. Math. Z. **31**, 748–766 (1930)
39. Sandstede, B.: Stability of travelling waves. In: Hasselblatt, B., Katok, A.B. (eds.) Handbook of Dynamical Systems, vol. 2, pp. 983–1055. North-Holland, Amsterdam (2002)
40. Van Vleck, E.S.: On the error in the product QR decomposition. SIAM J. Matrix Anal. Appl. **31**(4), 1775–1791 (2009/2010)
41. Wiesel, W.E.: Continuous-time algorithm for Lyapunov exponents: Part 1, and Part 2. Phys. Rev. E **47**, 3686–3697 (1993)

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