# Lyapunov and other spectra: a survey*

*Luca Dieci*† *and Erik S. Van Vleck*‡

## 1 Introduction

Motivated by the basic problem of studying stability of nonautonomous linear systems, different definitions of spectra have been proposed over the years. We review some of these definitions, paying particular attention to how the spectra may be computed. Since the spectra are given by intervals of the real line, one would like to compute the endpoints of these intervals. As it turns out, these endpoints may coincide with the Lyapunov exponents (LEs) of the system. For this reason, we will review computational methods which have been used to approximate the LEs.

## 2 Lyapunov Exponents & Spectra

Ever since the seminal thesis of Lyapunov more than one century ago (reprinted in English in [20]), one of the most prolific and challenging areas of research in dynamical systems has been to unravel the stability structure of the linear nonautonomous system

$$\dot{x} = A(t)x\,, \quad 0 \le t\,. \tag{1}$$

Henceforth, we assume that $A:\ t \to I\!R^{n\times n}$ is a continuous and bounded function. In many situations of practical interest (1) is the linear variational equation about a trajectory of a nonlinear system.

Our purpose in this survey is twofold; (i) to review some of the different definitions of spectra that have been used to study (1), and (ii) to explain how numerical methods may be devised in order to approximate the spectra. Unlike its analytical counterpart, numerical work on the subject is of much more recent

*This work was supported in part under NSF Grants DMS-9973226 and DMS-9973393.

†School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332 (dieci@math.gatech.edu).

‡Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401 (evanvlec@mines.edu).

vintage. To date, the only techniques that have been studied to any degree of sophistication are based on analogs of tried and true linear algebra techniques, and heavily rely on orthogonal (time dependent) transformations to approximate the LEs of (1). For these techniques, some sufficient conditions can be given that clarify when the numerical methods may give the appropriate spectral information. Perhaps new methods lie ahead that will prove valuable in investigating the stability of (1).

## 2.1 Preamble

In the autonomous case ($A$ constant), the solution operator is $e^{At}$ and the stability problem for (1) is conceptually trivial. A Jordan form of $A$, $V^{-1}AV = J$, gives the entire shading of growth and decay factors, and the generalized eigenvectors (i.e., the columns of $V$) characterize the initial conditions $x_0$ that lead to the possible asymptotic behaviors. But in the nonautonomous case, the eigenstructure of $A$ provides no reliable information as the following example of Coppel [8] shows.

**Example 2.1.** *Take*

$$A(t) = U^{-1}(t)A_0U(t)\ ,\ A_0 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, U(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

*The eigenvalues of $A(t)$ are $-1$ and $-1$ for all $t$, but a matrix solution is*

$$\begin{pmatrix} e^t(\cos(t) + \sin(t)/2) & e^{-3t}(\cos(t) - \sin(t)/2) \\ e^t(\sin(t) - \cos(t)/2) & e^{-3t}(\sin(t) + \cos(t)/2) \end{pmatrix},$$

*so that any nonzero vector $x_0 \neq \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$ leads to a solution with exponential growth.*

In Example 2.1, $A(\cdot)$ is periodic of period $2\pi$, and it is well known that if $A(\cdot)$ is periodic in $t$ then Floquet theory can be used to reduces the problem to a constant coefficient problem. Recall that a *Lyapunov transformation* $T$ is a linear change of variables $y = T^{-1}x$, such that $\dot{T}, T$, and $T^{-1}$, are all bounded. All systems (1) that can be **reduced** by a Lyapunov transformation to a constant coefficient problem $\dot{y} = By$, $B$ constant (real or complex valued), allow for a trivial characterization of the asymptotic behavior of solutions of (1), and the spectrum is given by a set of $n$ numbers, namely the real parts of the eigenvalues of $B$. General nonautonomous problems are another matter.

**Example 2.2.** *Consider the scalar problem*

$$\dot{x} = (\sin(\ln(t)) + \cos(\ln(t)))x\ ,\quad 0 < t_0 \leq t\,. \tag{2}$$

*Observe that the coefficient in (2) assumes all values in $[-\sqrt{2}, \sqrt{2}]$, while the exact solution is $x(t) = \kappa(t_0)\exp(t\sin(\ln(t)))$, where $\kappa(t_0) = x(t_0)\exp(-t_0\sin(\ln(t_0)))$. Therefore, we may want to consider $[-1, 1]$ to be the spectrum, since all the growth factors in this interval, and only these, are attained infinitely often.*

Various attempts have been made to characterize growth behavior of the solutions of (1), some of them trying to work only with the function $A$ (which is desirable, since we can think of $A$ as known), in the past century. We refer to the delightful monograph [2], which we have used extensively for the background section on Lyapunov Exponents (LEs) in this survey.

**Remark 2.3.** *A favorite tool in the numerical analysis community (e.g., see [33]) is the logarithmic norm of $A$, which is not actually a norm, but it depends on a matrix norm:*

$$t \to \mu(A(t)) := \lim_{h \to 0^+} \frac{\|I + hA(t)\| - 1}{h} .$$

*For example, if the 2-norm is used, then $\mu_2(A(t))$ is the largest eigenvalue of the symmetric part of $A(t)$. Although the logarithmic norm of $A$ has proven to be a useful tool in many numerical studies (especially on contractive problems), it does not seem useful in the context under examination here, because: (i) $\mu(A(t))$ depends on the norm, while the asymptotic behavior of solutions of (1) does not, (ii) it only gives one growth factor, not the entire shading, and (iii) in general, it is not invariant under a Lyapunov change of variables $y \leftarrow T^{-1}x$, whereas the asymptotic behavior of solutions is.*

## 2.2 The Theory of Lyapunov Exponents

Consider the scalar differential equation

$$\dot{x} = a(t)x \ , \quad t \geq 0 \, , \tag{3}$$

where the function $a : \ t \geq 0 \to \mathbb{C}$ is bounded and continuous. Let $x(t, x_0)$ with $x(0, x_0) = x_0 \neq 0$ be the solution. To characterize the asymptotic behavior of $x$, we define the Lyapunov (or characteristic) exponents:

$$\lambda^s(x) = \limsup_{t \to \infty} \frac{1}{t} \ln |x(t)| \ , \qquad \lambda^i(x) = \liminf_{t \to \infty} \frac{1}{t} \ln |x(t)| \ . \tag{4}$$

These LEs are called *upper* and *lower* exponents, respectively. Notice that for all $t$, $|x(t)| = \exp(\ln(|x(t)|))$, so that $\lambda^s(x)$ and $\lambda^i(x)$ measure the endpoints of the interval of asymptotic growth factors of $|x|$. Since $a$ is bounded, these endpoints are finite. If $x$ is a vector valued function, then $\lambda^s(x) = \lambda^s(||x||)$, and similarly for $\lambda^i(x)$, where the vector norm is the Euclidean norm[1]. Further, the values of $\lambda^s(x)$, and $\lambda^i(x)$ are independent of the initial condition $x_0 \neq 0$. Finally, observe that the Lyapunov exponents are unaffected by the behavior of $x$ on a finite interval, and we could replace $t \geq 0$ with $t \geq T$ for any other (finite) value $T$.

**Example 2.4.** *To illustrate, we have $\lambda^{s,i}((t+1)^m) = 0$ for all $m$, and $\lambda^{s,i}(e^{\alpha t}) = \Re(\alpha)$ for all $\alpha \in \mathbb{C}$. On the other hand, in Example 2.2: $\lambda^s(\exp(t \sin(\ln(t))) = 1$,*

[1]Note that the values $\lambda^s(x)$ and $\lambda^i(x)$ are independent of the vector norm

*and* $\lambda^i(\exp(t\sin(\ln(t)))) = -1$. *Finally, from properties of the* lim sup *and* lim inf*, we have*

$$\lambda^i(x) \le \lambda^s(x)\ , \qquad \lambda^s(x) = -\lambda^i(1/x)\ , \qquad \lambda^i(x) = -\lambda^s(1/x)\,. \tag{5}$$

Based on (4), we define the **Lyapunov spectrum** of (3) to be the interval $[\lambda^i(x), \lambda^s(x)]$. If this spectrum reduces to a single point, we say that the solution $x$ (or the coefficient $a$) is **regular**. Example 2.2 is not regular.

Next consider (1), and a fundamental matrix solution $X$. As above, we have upper/lower LEs for any solution $Xx_0$. In particular, the following are well defined:

$$\lambda^s(Xe_j) \qquad \text{and} \qquad \lambda^i(Xe_j)\ , \quad j = 1, \ldots, n\ , \tag{6}$$

where the $e_j$'s are the standard unit vectors. Since it is always possible to permute the columns of $X$ to achieve it, we henceforth assume that

$$\lambda^s(Xe_1) \ge \ldots \ge \lambda^s(Xe_n)\,.$$

Consider these $\lambda^s(Xe_j)$, $j = 1, \ldots, n$. When $\sum_{j=1}^{n} \lambda^s(Xe_j)$ is minimized with respect to all possible fundamental matrix solutions, then Lyapunov called the $\lambda^s(Xe_j)$ *characteristic exponents*, and the corresponding matrix solution a *normal basis*. We henceforth refer to these as the **upper Lyapunov exponents**, and write simply $\lambda^s_j$. Lyapunov showed that a normal basis $X$ always exists and how to construct it from any other matrix solution $Z$.

**Theorem 2.5.** ([20]) *Let a matrix solution $Z$ be given such that $\lambda^s(Ze_1) \ge \ldots \ge \lambda^s(Ze_n)$. Then, there exists a (constant) unit upper triangular matrix $C$ such that $X(\cdot) = Z(\cdot)C$ is normal.*

Because of Theorem 2.5, we can assume we have a normal matrix solution such that the upper LEs are ordered:

$$\lambda^s_1 \ge \lambda^s_2 \ge \ldots \ge \lambda^s_n\,.$$

In a similar way, we may define the lower LEs. Alternatively, consider the adjoint system

$$\dot{z}(t) = -A^T(t)z(t)\,, \tag{7}$$

and let $\{-\mu^s_j\}_{j=1}^n$ be the (ordered) upper LEs for (7): $-\mu^s_1 \le -\mu^s_2 \le \ldots \le -\mu^s_n$. It is not hard to show that if $X$ is a normal basis for (1), then $X^{-T}$ is a normal basis for (7). As a consequence of this and of (5) we have

$$\lambda^i_j = -\mu^s_j\ , \quad j = 1, \ldots, n\,.$$

**Definition 2.6.** *The* **Lyapunov spectrum** $\Sigma_{\mathrm{L}}$ *for the linear system (1) is*

$$\Sigma_{\mathrm{L}} := \bigcup_{j=1}^{n} [\lambda^i_j, \lambda^s_j]\,. \tag{8}$$

If $\lambda_j^i = \lambda_j^s$, for all $j = 1, \ldots, n$, then the system is called **regular**[2], and we simply write $\lambda_j$ for the LEs, $j = 1, \ldots, n$. Of course, all systems reducible[3] to constant coefficients are regular.

**Example 2.7.** *A nonregular system (cf. Example 2.2) due to Lyapunov:*

$$\begin{array}{ll} \dot{x} & = (\sin(\ln(t+1)) + \cos(\ln(t+1))y,\ x(0) = x_0, \\ \dot{y} & = (\sin(\ln(t+1)) + \cos(\ln(t+1)))x,\ y(0) = y_0, \end{array} \tag{9}$$

*with solution*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(s(t)) & \sinh(s(t)) \\ \sinh(s(t)) & \cosh(s(t)) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad s(t) = (t+1)\sin(\ln(t+1))\,.$$

*Thus, the Lyapunov spectrum is the interval* $[-1, 1]$.

It is easier to work with regular systems since there is no extra difficulty caused by having to monitor lim inf and lim sup. Not surprisingly, most numerical works on approximation of LEs assume that the system under study is regular. The reason resides in the following result, due to Lyapunov,

**Theorem 2.8.** *The system:*

$$\dot{R} = B(t)R\,, \qquad \textit{with } B \textit{ bounded, continuous, and upper triangular}\,, \tag{10}$$

*is regular if and only if the limits*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t B_{jj}(s)ds\ ,\ j = 1, \ldots, n, \tag{11}$$

*exist, in which case they coincide with the* $\lambda_j$, $j = 1, \ldots, n$.

and the following fact:

**Fact 2.9.** *There always exists a Lyapunov transformation to an upper triangular form. Moreover, this transformation can be chosen to be orthogonal.*

Fact 2.9 has been known for a long time, see [15, 29]. A constructive verification goes as follows. For all $t$, we want to write $X(t) = Q(t)R(t)$ where $Q$ is orthogonal and $R$ is upper triangular. At any given $t$, say $t = 0$, we orthogonalize the columns of $X(0)$. Next, we use the Implicit Function Theorem. Differentiating $X = QR$, we obtain

$$AQR = Q\dot{R} + \dot{Q}R \quad \text{or} \quad \dot{Q} = AQ - QB\,, \quad B := Q^TAQ - Q^T\dot{Q}\,.$$

[2]the original definition of regularity due to Lyapunov is different than, but equivalent to, the one we have given, which is due to Perron

[3]regularity is preserved by Lyapunov transformations

Since we need $Q$ orthogonal and $B$ upper triangular, by letting $S(Q) := Q^T\dot{Q} = Q^TAQ - B$, then we define the strict lower triangular part of the skew symmetric function $S$ as the corresponding piece of $Q^TAQ$ and the rest of $S$ by skew-symmetry:

$$S(Q)_{ij} = \begin{cases} (Q^TAQ)_{ij}, & i > j, \\ 0, & i = j, \\ -(Q^TAQ)_{ji}, & i < j, \end{cases} .$$

Since $R$ is invertible ($X$ is full rank), the process is well posed and proves the claim.

Because of this, we can always assume that

**Fact 2.10.** *A fundamental matrix solution $X$ can be taken to be normal and upper triangular.*

To verify Fact 2.10, we reason as follows. Let $X$ be a fundamental matrix solution, which we assume has ordered upper (and lower) LEs: $\lambda_1^s \geq \lambda_2^s \geq \ldots \geq \lambda_n^s$. By Theorem 2.5, we can consider $X$ to be normal. The orthogonal change of variables $Q^TX \to R$ therefore gives a normal, upper triangular, fundamental matrix solution, since $\|X(t)e_j\| = \|Q^T(t)X(t)e_j\|$, $j = 1, \ldots, n$.

**Remark 2.11.** *In general, Fact 2.9 can be extended to only $p$ columns (with $p < n$) of a matrix solution $X$. In this case, $Q : t \to \mathbb{R}^{n\times p}$ is orthonormal ($Q^TQ = I_p$ for all $t$) and satisfies*

$$\dot{Q} = AQ - Q[Q^TAQ - S(Q)], \tag{12}$$

*where $S(Q)$ takes values in $\mathbb{R}^{p\times p}$. If $p = n$, (12) reduces to $\dot{Q} = QS(Q)$.*

The above considerations lead to the following algorithm.

**Algorithm 2.12 (***Computing LEs for a regular system***).**

- Given $\dot{x} = A(t)x$, integrate the $Q$–equation and perform the change of variables $R \leftarrow Q^TX$ so that

  $$B(t) := Q^T(t)A(t)Q(t) - Q^T\dot{Q}$$

  is upper triangular. Then, approximate the LEs at any time $T$, from a quadrature rule on the integral in (11).

**Remark 2.13.** *It must be stressed that $X$ does not need to be known (nor found) in order to obtain the transformed triangular function $B$. It is sufficient to find $Q$ by integrating (12).*

Algorithm 2.12 is a sensible strategy, but, in spite of its appeal, regularity is not the most natural property to assume if we are interested in computing the LEs. Rather, we require that the spectrum be **stable** with respect to perturbations. We now discuss this aspect by restricting focus on the upper Lyapunov exponents. By

working with the adjoint system, all results have an immediate analog for the lower exponents, and hence for $\Sigma_{\mathrm{L}}$.

Stability of the LEs is tantamount to continuity with respect to perturbations in the function of coefficients. Formally, we have the following definition.

**Definition 2.14.** *The upper LEs, $\lambda_1^s \geq \lambda_2^s \geq \ldots \geq \lambda_n^s$, are stable if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\sup_{t \in \mathbb{R}^+} \|E(t)\| < \delta$ implies*

$$|\lambda_j^s - \nu_j^s| < \epsilon \ , \ j = 1, \ldots, n,$$

*where the $\nu_j^s$'s are the (ordered) upper LEs of the differential equation $\dot{x} = [A(t) + E(t)]x$.*

It is relatively straightforward to show that stability of the upper (and lower) LEs is invariant under Lyapunov transformation, such us an orthogonal change of variables. Because of Fact 2.9, it is thus reasonable to approximate the LEs after triangularization of the system. But if the system is not regular, can one approximate the LEs from the diagonal of the transformed system? Can we do it if the LEs are stable? Should we expect the LEs to be stable?

First of all, it is well known that the LEs are not stable in general; see the many examples in [2] for what can go wrong. Indeed, for many years, mathematicians tried to characterize the linear systems with stable upper LEs as thoroughly as possible. Major contributions bear the names of Perron, Lipschitz, Bylov, Vinograd, Millionshchikov, Grobman, Malkin, and several others. Eventually, from the mid '60s to the mid '70s (and beyond) the problem was resolved, chiefly thanks to the contributions of Bylov, Izobov, and Millionshchikov (see references at the end). The key assumption is **integral separation**.

**Definition 2.15.** *A fundamental solution $X$ is said to be integrally separated if for $j = 1, \ldots, n-1$, there exist $a > 0$ and $d > 0$ such that*

$$\frac{\|X(t)e_j\|}{\|X(s)e_j\|} \cdot \frac{\|X(s)e_{j+1}\|}{\|X(t)e_{j+1}\|} \geq de^{a(t-s)} \ ,$$

*for all $t, s$ with $t \geq s$. The columns of an integrally separated fundamental solution are said to be integrally separated.*

Below, we collect a number of consequences of integral separation: 1. to 3. can be found in the cited works of Millionshchikov or in [2], 4. is in [13], and 5. is in [27] (but see [21]).

1. Integral separation is invariant under Lyapunov transformations.
2. Integrally separated systems have distinct upper LEs: $\lambda_1^s > \ldots > \lambda_n^s$. Similarly, for the lower LEs.
3. Distinct upper LEs $\lambda_1^s > \ldots > \lambda_n^s$ are stable if and only if there exists a fundamental matrix solution with integrally separated columns.

4. If a system has a matrix solution with integrally separated columns, then so does the adjoint. As a consequence, the lower LEs are also stable.

5. Consider the Banach space $\mathcal{B}$ of continuous bounded matrix valued functions $A$, with norm $\|A\| = \sup_{t\geq 0} \|A(t)\|$. Then, the systems with integral separation form an open and dense subset of $\mathcal{B}$. That is, integral separation is a *generic* property in $\mathcal{B}$.

As a consequence of 1., 3., 4., and 5. above, and within the class of systems in 5., we have the following result.

**Theorem 2.16.** *Generically, the Lyapunov spectrum (8) is stable.*

In light of Theorem 2.16, it is natural to assume that we have to deal with integrally separated fundamental matrix solutions. Because of Fact 2.10, we can thus restrict consideration to a fundamental matrix solution that is normal, upper triangular, and (generically) integrally separated. At this point, if the system is regular, we can use Algorithm 2.12 to approximate the LEs. So, once again, we must address the following questions.

(i) Is regularity sufficient for stability? Is regularity generic?

(ii) If the system is not regular, but the LEs are stable, can we approximate the LEs from the diagonal of the transformed system?

In general, the answers to these questions are: (i): No, and (ii): "Yes". The following example, modeled after one in [2], shows that regularity does not ensure stability of the LEs.

**Example 2.17.** *Take the system with diagonal matrix valued function of coefficients*

$$A(t) = \begin{pmatrix} 1 - \frac{\pi}{2}\sin(\pi\sqrt{t}) & 0 \\ 0 & 0 \end{pmatrix}.$$

*A normal matrix solution is*

$$X(t) = \begin{pmatrix} t - \frac{1}{\pi}\sin(\pi\sqrt{t}) + \sqrt{t}\cos(\pi\sqrt{t}) & 0 \\ 0 & 1 \end{pmatrix}.$$

*The system is regular with distinct LEs $\lambda_1 = 1$, $\lambda_2 = 0$. However, the LEs are not stable, since $X$ is not integrally separated.*

**Remark 2.18.** *Unlike integral separation, regularity is not a generic property for linear systems. Still, regular systems are prevalent in a certain measure theoretic sense, see the work [23] and [26]. For example, the work [26] implies that if (1) comes from linearization of a trajectory of a nonlinear system, and the orbit through the initial condition $x_0$ generates an ergodic measure, then the LEs exist as limits (and the system is regular) and they are the same for almost every $x_0$ with respect to the ergodic measure. This is a very remarkable fact, but we cannot see that it implies*

*that the LEs are stable: for this to occur, integral separation remains necessary and sufficient.*

The answer to the above question (ii) is the content of the following result, which is in [13].

**Theorem 2.19.** *Let the triangular system $\dot{R} = BR$, with $B$ bounded and continuous, have an integrally separated, and triangular, fundamental matrix solution $R$, and let $\lambda_j^s$, $j = 1, \ldots, n$, be the ordered upper LEs of the system. Then for any given $\epsilon > 0$ and any $j = 1, \ldots, n$, there exists a permutation $\pi$ such that $|\lambda^s_{\pi(j)} - \limsup_{t\to\infty} \frac{1}{t}\int_0^t B_{jj}(s)ds| < \epsilon$.*

In other words, for an upper triangular, integrally separated system, and hence a system with stable LEs, the upper (and lower) LEs can be in principle recovered to arbitrary accuracy from the diagonal entries. The caveat is that the diagonal system $\dot{D} = \mathrm{diag}(B)D$ does not necessarily have stable LEs. For this to be the case, we need the diagonal system itself to be integrally separated, a condition that can be rephrased in terms of the functions $B_{jj}$, $j = 1, \ldots, n$.

**Definition 2.20.** *The functions $B_{jj}$, $i = 1, \ldots, n$, are integrally separated if for $j = 1, \ldots, n-1$, there exist $a > 0$ and $d \in \mathbb{R}$ such that*

$$\int_s^t (B_{j,j}(\tau) - B_{j+1,j+1}(\tau))d\tau \geq a(t-s) - d, t \geq s \ .$$

This definition, and Theorem 2.19, suggest the following alternative to the Lyapunov spectrum (see [13]).

**Definition 2.21.** *For an upper triangular system $\dot{R} = BR$, with $B$ bounded and continuous, the* **computed Lyapunov spectrum** $\Sigma_{\mathrm{CL}}$ *is*

$$\Sigma_{\mathrm{CL}} := \bigcup_{j=1}^{n} [\lambda^i_{jj}, \lambda^s_{jj}] \ , \quad \text{where for } \ j = 1, \ldots, n :$$

$$\lambda^i_{jj} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t B_{jj}(s)ds \ , \ \ \lambda^s_{jj} = \limsup_{t\to\infty} \frac{1}{t}\int_0^t B_{jj}(s)ds \, .$$

Clearly, $\Sigma_{\mathrm{CL}} \subseteq \Sigma_{\mathrm{L}}$. The following result gives a condition for these spectra to coincide.

**Theorem 2.22.** *If the diagonal of the upper triangular system $\dot{R} = BR$ is integrally separated, then $\Sigma_{\mathrm{L}} = \Sigma_{\mathrm{CL}}$, and it is stable.*

***Proof.*** A proof is in [13]. We only remark that the assumption allows the construction of a Lyapunov transformation that reduces $\dot{R} = BR$ to the diagonal system $\dot{S} = \mathrm{diag}(B)S$. □

The relevance of Theorem 2.22 is that the condition of integral separation of the diagonal of $B$ is –in principle– verifiable without explicit knowledge of the matrix solution (see Remark 2.13), while the condition of integral separation of a fundamental matrix solution apparently requires the matrix solution itself. To verify integral separation of the diagonal of $B$, we may use a construction based on so-called Steklov functions. Recall that given a function $f$, and positive $H > 0$, the Steklov function $f^H$ is defined by averaging: $f^H(t) = \frac{1}{H}\int_t^{t+H} f(\tau)d\tau$. A proof of the next theorem is in [2].

**Theorem 2.23.** *Two functions $f_1$ and $f_2$ are integrally separated if and only if there exists $H > 0$ such that their Steklov difference is positive. That is, with*

$$f_1^H(t) - f_2^H(t) \equiv \frac{1}{H}\int_t^{t+H} (f_1(\tau) - f_2(\tau))d\tau \ , \tag{13}$$

*then, for $H$ sufficiently large, $f_1^H(t) - f_2^H(t) \geq a > 0$, for all $t \geq 0$.*

## 2.3 Other spectra

An alternative concept of spectrum to the ones previously presented is due to Sacker and Sell, see [32]. It is based on the concept of exponential dichotomy.

**Definition 2.24.** *The fundamental matrix solution $X$ admits an exponential dichotomy if there exist a projection $P$ and constants $\alpha, \beta > 0$, and $K, L \geq 1$, such that*

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \|X(t)(I-P)X^{-1}(s)\| &\leq Le^{\beta(t-s)}, \quad t \leq s. \end{aligned} \tag{14}$$

**Definition 2.25.** *The* **exponential dichotomy**, *or Sacker & Sell,* **spectrum** $\Sigma_{\rm ED}$ *is given by those values $\lambda \in \mathbb{R}$ such that the shifted system $\dot{x}_\lambda = [A(t) - \lambda I]x_\lambda$ does not have exponential dichotomy. The complement of $\Sigma_{\rm ED}$ is called the* **resolvent** *set.*

Sacker and Sell proved that $\Sigma_{\rm ED}$ is given by the union of at most $n$ closed intervals:

$$\Sigma_{\rm ED} := [a_1, b_1] \cup \cdots \cup [a_k, b_k], \ \ (k \leq n) \ ,$$

and they further proved that $\Sigma_{\rm ED}$ is stable (see [32, pp. 342-346]): "$\forall \epsilon > 0$, if $\rho$ is in the resolvent of $\dot{x} = (A - \lambda I)x$, then there exists $\delta$ such that if $\|A - C\|_\infty < \delta$, then $\rho$ is in the resolvent of $\dot{x} = (C - \lambda I)x$."

Exponential dichotomy plays a fundamental role in many studies in dynamical systems and has been a widely used assumption also in many numerical works of recent vintage. For example, shadowing results all need –in one form or another– exponential dichotomy; see [28] and [30]. Obviously, if $0 \notin \Sigma_{\rm ED}$ then the system has exponential dichotomy, and thus being able to approximate $\Sigma_{\rm ED}$ is of considerable interest. It is natural to inquire whether or not the end points of the $\Sigma_{\rm ED}$–intervals

coincide with the upper/lower LEs. Although some results in [32] suggest that the end-points of the intervals in $\Sigma_{\rm ED}$ are often given by the upper/lower LEs, in general we only know that

$$\Sigma_{\rm L} \subseteq \Sigma_{\rm ED} \ .$$

This makes it tricky to compute $\Sigma_{\rm ED}$, since the endpoints of the intervals in $\Sigma_{\rm ED}$ must come about from something else than mere growth behavior of the solutions. The difficulty is caused by the request for uniformity (for all $t \geq s$) in Definitions 2.25 and 2.24. The following example illustrates some of these aspects.

**Example 2.26.** *Consider the scalar problem of Example 2.2: $\dot{x} = c(t)x,\ t \geq t_0 > 0$, with $c(t) = (\sin(\ln(t)) + \cos(\ln(t)))$ and solution $x(t) = \kappa(t_0)\exp(t\sin(\ln(t)))$, where $\kappa(t_0) = x(t_0)\exp(-t_0\sin(\ln(t_0)))$. Clearly, the function $c$ takes all values in $[-\sqrt{2},\sqrt{2}]$, while $\Sigma_{\rm CL} = \Sigma_{\rm L} = [-1,+1]$, and $\Sigma_{\rm L}$ is stable. We now verify that $\Sigma_{\rm ED} = [-\sqrt{2},\sqrt{2}]$. Thus, we show an explicit connection between $\Sigma_{\rm ED}$ and the coefficient $c$ of the differential equation.*

*First take $\lambda > 1$, so that having an exponential dichotomy means that for all $t$ and $s$ with $t \geq s \geq t_0$, there are $K \geq 1$ and $\alpha > 0$ such that*

$$e^{-\lambda(t-s)}e^{\int_s^t c(r)dr} = x_\lambda(t)x_\lambda^{-1}(s) \leq Ke^{-\alpha(t-s)}.$$

*But this is equivalent to*

$$\frac{e^{\lambda t}}{e^{\lambda s}}\ \frac{e^{\int_{t_0}^s cd\tau}}{e^{\int_{t_0}^t cd\tau}} \geq \frac{1}{K}e^{\alpha(t-s)}\,,$$

*which is equivalent to requiring that the following system be integrally separated:*

$$\dot{X} = \begin{pmatrix} \lambda & 0 \\ 0 & c(t) \end{pmatrix} X\,. \tag{15}$$

*The latter condition (recall Definition 2.20) is equivalent to the existence of $a > 0$ and $d \in \mathbb{R}$ such that, for all $t \geq s \geq t_0$,*

$$\lambda(t-s) - (t\sin(\ln(t)) - s\sin(\ln(s))) \geq a(t-s) - d\,.$$

*Now, consider the following sequences for $t$ and $s$,*

$$t_k = \exp(2k\pi + \tau + h)\,,\ \ s_k = \exp(2k\pi + \tau)\ .$$

*The requirement for integral separation can be rewritten as*

$$0 < a(e^h - 1) \leq \lambda(e^h - 1) - (e^h\sin(\tau + h) - \sin(\tau)) + de^{-2k\pi}\,.$$

*Taking $h > 0$ small, and setting $\tau = \pi/4 - h/2$, so that*

$$\begin{aligned} e^h - 1 &= h(1 + h/2 + h^2/6 + \dots)\,, \\ e^h\sin(\tau + h) - \sin(\tau) &= \sqrt{2}h(1 + h/2 + h^2/16 + \dots)\,, \end{aligned}$$

*we should find* $a > 0$ *and* $d \in \mathbb{R}$ *such that*

$$\begin{aligned} 0 < \quad & ah(1 + h/2 + h^2/6 + \dots) \leq \lambda h(1 + h/2 + h^2/6 + \dots) \\ & -\sqrt{2}h(1 + h/2 + h^2/16 + \dots) + de^{-2k\pi} . \end{aligned}$$

*Taking* $1 < \lambda < \sqrt{2}$*,* $h$ *sufficiently small, and* $k$ *sufficiently large, so that*

$$\lambda h(1 + h/2 + h^2/6 + \dots) - \sqrt{2}h(1 + h/2 + h^2/16 + \dots) + de^{-2k\pi} < 0 ,$$

*shows that such an* $a > 0$ *cannot exist. On the other hand, from*

$$e^{-\lambda(t-s)}e^{\int_s^t c(r)dr} \leq e^{-\lambda(t-s)}e^{\sqrt{2}(t-s)} ,$$

*it follows that if* $\lambda > \sqrt{2}$ *then the system has an exponential dichotomy. The arguments for* $\lambda < -1$ *are similar, leading to the conclusion that* $\Sigma_{\mathrm{ED}} = [-\sqrt{2}, \sqrt{2}]$*. (For the last inference, we have used that* $\Sigma_{\mathrm{ED}}$ *is a closed interval).*

We now extend the idea in Example 2.26 to the case of a general diagonal linear system, thereby providing a link between $\Sigma_{\mathrm{ED}}$ and integral separation. We then point out the implications for a general integrally separated system.

So, consider

$$\dot{x} = D(t)x, \quad \text{where} \quad D = \mathrm{diag}(D_{jj},\ j = 1, \dots, n) . \tag{16}$$

For all $j = 1, \dots, n$, and for $\lambda \in \mathbb{R}$, consider the planar systems

$$\dot{y}_j = \begin{pmatrix} \lambda & 0 \\ 0 & D_{jj}(t) \end{pmatrix} y_j \tag{17}$$

and

$$\dot{y}_j = \begin{pmatrix} D_{jj}(t) & 0 \\ 0 & \lambda \end{pmatrix} y_j \tag{18}$$

We now introduce the **integral separation spectrum**.

**Definition 2.27.** *The integral separation spectrum is* $\Sigma_{\mathrm{IS}} = \bigcup_{j=1}^n \Lambda_j$ *where* $\Lambda_j = \Lambda_j^+ \cap \Lambda_j^-$ *with* $\Lambda_j^+ = \{\lambda \in \mathbb{R} : (17)$ *is not integrally separated*$\}$ *and* $\Lambda_j^- = \{\lambda \in \mathbb{R} : (18)$ *is not integrally separated*$\}$*, for all* $j = 1, \dots, n$*.*

**Remark 2.28.** *A simple rewriting gives, in the above definition,*

$$\Lambda_j = \{\lambda \in \mathbb{R} : (17) \text{ and } (18) \text{ are not integrally separated}\},\ j = 1, \dots, n.$$

We then have

**Theorem 2.29.** *For (16),* $\Sigma_{\mathrm{IS}} = \Sigma_{\mathrm{ED}}$*.*

***Proof.*** Given $\lambda \in I\!R$, we have $\lambda \notin \Sigma_{\mathrm{ED}}$ if for all $j = 1, \ldots, n$, either

$$e^{-\lambda(t-s)} e^{\int_s^t D_{jj}(\tau)d\tau} \leq K e^{-\alpha(t-s)}, \ t \geq s, \tag{19}$$

or

$$e^{\lambda(t-s)} e^{-\int_s^t D_{jj}(\tau)d\tau} \leq L e^{-\beta(t-s)}, \ t \geq s. \tag{20}$$

If (19) holds then (17) is integrally separated. If (20) holds then (18) is integrally separated. In either case, we have $\lambda \notin \Sigma_{\mathrm{ED}} \Rightarrow \lambda \notin \Sigma_{\mathrm{IS}}$. Conversely, if $\lambda \notin \Sigma_{\mathrm{IS}}$, then for all $j = 1, \ldots, n$, either (17) or (18) is integrally separated and hence either (19) or (20) hold. □

**Remark 2.30.** *The relevance of Theorem 2.29 is that, for (16), we may attempt to approximate $\Sigma_{\mathrm{ED}}$ by approximating $\Sigma_{\mathrm{IS}}$. The latter task, in principle, could be carried out by relying on Theorem 2.23.*

The following result of Bylov (see [2, Corollary 5.3.2]) provides the link between $\Sigma_{\mathrm{ED}}$ (via $\Sigma_{\mathrm{IS}}$) and general (not necessarily diagonal) integrally separated systems.

**Theorem 2.31.** *Let $X$ be an integrally separated matrix solution for the system $\dot{x} = A(t)x$. Then the transformation $T$ given for all $t$ by*

$$T(t) = \{X(t)e_1/\|X(t)e_1\|, \ldots, X(t)e_n/\|X(t)e_n\|\},$$

*is Lyapunov and $Y = T^{-1}X$ is diagonal. The $y$-system becomes $\dot{y} = D(t)y$, where $D_{jj}(t) = \frac{d}{dt} \ln \|X(t)e_j\|$, $j = 1, \ldots, n$, and the functions $D_{jj}$ are integrally separated.*

Unfortunately, in general, Theorem 2.31 is not conducive to a constructive diagonalization procedure for the system, and hence to a way to approximate $\Sigma_{\mathrm{ED}}$ via $\Sigma_{\mathrm{IS}}$, because constructing the Lyapunov transformation $T$ requires knowledge of the matrix solution $X$. On the other hand, we know that $X$ can transformed to upper triangular with an orthogonal function $Q$, and that the transformed coefficient function $B := Q^T A Q - Q^T \dot{Q}$ can be found without explicit knowledge of $X$. Thus, it is natural to ask if we can obtain $\Sigma_{\mathrm{IS}}$ from knowledge of $B$. We now show that this is possible under the same assumptions we needed in order to approximate the LEs from the diagonal of an integrally separated triangular system (see Theorem 2.22).

**Theorem 2.32.** *Consider the upper triangular system $\dot{R} = BR$, and let $D = \mathrm{diag}(B)$. Let $\Sigma_{\mathrm{IS}}$ be defined from $D$. If $D$ is integrally separated, then $\Sigma_{\mathrm{ED}} = \Sigma_{\mathrm{IS}}$.*

***Proof.*** Similarly to the proof of Theorem 2.22, we reduce the problem to the diagonal form $\dot{S} = DS$, with $D = \mathrm{diag}(B)$. The transformation is Lyapunov and does not change $\Sigma_{\mathrm{ED}}$. For this diagonal system, $\Sigma_{\mathrm{IS}}$ coincides with $\Sigma_{\mathrm{ED}}$ because of Theorem 2.29. □

# 3 Computational Aspects

In this section, we briskly review some of the techniques that have been used to approximate Lyapunov exponents. Then, we discuss approximation of spectral intervals.

To date, numerical techniques for approximating LEs have rested on the assumption that the given system is regular and that the LEs exist as limits. The most widely adopted techniques mimic Algorithm 2.12, and therefore attempt to find the function $Q$ that performs the change of variable to upper triangular form. We refer to [3, 4, 10, 11, 16, 17] for different variations of this basic approach. In the end, existing techniques may be classified as either continuous or discrete. The continuous methods approximate $Q$ by integrating its differential equation, and hence avoid direct computation of the fundamental matrix solution. The discrete techniques, instead, approximate the QR factorization of the fundamental solution at grid points as product of transition matrices, followed by reorthogonalization. Though in exact arithmetic these two classes of methods are equivalent, in finite precision they perform differently. In particular, the discrete techniques have potential difficulties in approximating (large and) negative Lyapunov exponents (see [10, Theorem 4.4, pp. 413]). Below, we restrict to the continuous QR technique.

**Remark 3.1.** *As an alternative to QR based techniques, people have also investigated methods based on the SVD of the matrix solution $X$, for which [26] provide the theoretical foundation. Also these approaches come in two flavors: continuous and discrete. We have not had extensive computational experience with these techniques, and refer the interested reader to the works [1, 13, 16, 18, 34] for various aspects of these SVD-based methods.*

## 3.1 The Continuous QR Method: Lyapunov Exponents of Regular Systems

How to implement Algorithm 2.12? The fundamental points to resolve are: approximating $Q$, and truncating time to a finite interval.

Now, $Q$ is the solution of (12), and we may think that it suffices to integrate it as any other differential equation. However, this is not true, since a naive integration scheme for (12) will perform unsatisfactorily, if nothing else because it will fail to maintain the computed solution orthogonal at the grid points. For this reason, many recent works have been concerned with appropriate ways to approximate $Q$. The works [4, 9, 11, 12, 14, 19, 25] give a good overview into different ideas that have been explored. Some of the techniques work with no modifications for both cases $p = n$ and $p < n$ in (12), others require $p = n$. It is not the purpose of this review to discuss the relative merits of these different techniques, but it suffices to stress once more that integration of (12) must be done carefully.

Once we have $Q$, we can use a quadrature rule to approximate $\frac{1}{T}\int_0^T B_{jj}(s)ds$, $j = 1, \ldots, n$, where $B$ is defined in (10). A simple way to do this, which we have successfully adopted in [13], goes as follows. For $j = 1, \ldots, n$, let $\gamma_j(t) =$

$\int_0^t B_{jj}(s)ds$, so that $\lambda_j = \lim_{t\to\infty} \frac{1}{t}\gamma_j(t)$. Observe that

$$\dot{\gamma}_j = B_{jj}, \quad \gamma_j(0) = 0, \quad j = 1, \ldots, n, \tag{21}$$

and (21) can be integrated at once along with $Q$, yielding the desired quadrature.

**Remark 3.2.** *The LEs are often considered for nonlinear systems. This requires working with the linear variational equation about a (approximate) solution to the nonlinear system. In this setting, the LEs are useful in determining instability in the system, classifying invariant sets, and approximating the dimension of strange attractors or other nontrivial invariant sets (see [31]). In applications to nonlinear problems, we can use an algorithm similar to Algorithm 2.12.*

**Algorithm 3.3 (***LEs for Nonlinear Regular System***).** To approximate $p$ LEs relatively to the trajectory $x(t, x_0)$, integrate

$$\begin{array}{rll} \dot{x} = & f(x)\ , & \\ \dot{Q} = & AQ - Q[Q^T AQ - S(Q)]\ , & A(t) := f_x(x(t, x_0)) \\ \dot{\gamma}_j = & B_{jj}, \quad B_{jj}(t) = (Q^T(t)A(t)Q(t))_{jj}\ , & j = 1, \ldots, p\ . \end{array} \tag{22}$$

The exponents are $\lambda_j = \lim_{t\to\infty} \frac{1}{t}\gamma_j(t)$.

Truncating time to a finite interval is a tricky issue. In general, it is hard to decide how to truncate time without some extra assumptions on the problem, such as recurrency, quasi-periodicity, or the like. The difficulty is inherent in the task itself: the LEs are defined as limits, say $\lambda = \lim_{t\to\infty} \frac{1}{t}c(t)$ (here, $c(t) = \int_0^t b(s)ds$). Define the function $\lambda(t) = \frac{1}{t}c(t)$, for all $t$; in practice, we can only compute on a finite interval, say $[0, T]$, and thus at best we compute $\lambda(T)$. How good is $\lambda(T)$ as an approximation of $\lambda$? In theory, arbitrarily poor, even if $T$ is very large.

**Example 3.4.** *Take* $b(t) = \begin{cases} 1, & 0 \le t \le \tau \\ t - \tau + 1, & \tau \le t \le \tau + 1 \\ 2, & \tau + 1 \le t \end{cases}$ *so that* $\lambda = 2$. *However, if* $T \le \tau$ *then* $\lambda(T) = 1$.

Clearly, we need to require that the asymptotic behavior of $\lambda(t)$ is determined by the finite time interval on which we compute. In practice, we do not see a foolproof way to verify this fact computationally. A simple strategy people have adopted with apparent success is to compute on progressively longer time intervals, while monitoring the variation in the answers obtained; convergence is declared if the variation goes to 0.

## 3.2 Approximation of Spectral Intervals

Unlike approximation of the LEs, the approximation of spectral intervals has not received much attention, except in the case of regular systems or of *point spectrum*:

the intervals in the definition $\Sigma_{\rm ED}$ reduce to a single point[4]. Perhaps this is because many problems appear to be regular or even to have point spectrum, while at the same time approximation of lim sups and lim infs is an even more difficult task than approximation of a limit. However, there are good reasons for looking into approximation of spectral intervals, so we now discuss this issue.

Clearly, the different characterizations of spectrum have advantages and disadvantages: for example, the stability of $\Sigma_{\rm ED}$ comes at the price of a seemingly greater complexity in approximating it than, say, $\Sigma_{\rm CL}$. It is a "no win" situation: the stability derives from the Roughness Theorem for exponential dichotomies, but the complexity derives from the requirement of uniformity in the exponential dichotomy definition. Still, $\Sigma_{\rm ED}$ gives access to information that is not generally attainable from $\Sigma_{\rm CL}$ or $\Sigma_{\rm L}$. In our opinion, the most important piece of information we can retrieve from $\Sigma_{\rm ED}$ is that: "*If $0 \notin \Sigma_{\rm ED}$, then there exists a bounded solution to the inhomogeneous equation $\dot{x} = A(t)x + b(t)$ defined for $t \geq 0$, for any $b$ bounded and continuous*" (see [8]), because it has important implications for shadowing. Naturally, if $0 \in \Sigma_{\rm CL}$ or $\Sigma_{\rm L}$, then the system does not have an exponential dichotomy, since $\Sigma_{\rm CL} \subseteq \Sigma_{\rm L} \subseteq \Sigma_{\rm ED}$. But, in general, we cannot infer an exponential dichotomy from knowing that $0 \notin \Sigma_{\rm L}$.

**Example 3.5.** *Take a shifted version of Example 2.26: $\dot{x} = [c(t) - 1.2]x$, $t \geq t_0 > 0$, with $c(t) = (\sin(\ln(t)) + \cos(\ln(t)))$, and solution $x(t) = \kappa(t_0)\exp[t(\sin(\ln(t)) - 1.2)]$, where $\kappa(t_0) = x(t_0)\exp[-t_0(\sin(\ln(t_0)) - 1.2)]$. Thus, $\Sigma_{\rm CL} = \Sigma_{\rm L} = [-2.2, -0.2]$, and $0 \notin \Sigma_{\rm L}$, but $0 \in \Sigma_{\rm ED} = [-\sqrt{2} - 1.2, \sqrt{2} - 1.2]$.*

We now list some aspects that must be considered when approximating spectral intervals.

(a) In principle, $\Sigma_{\rm CL}$ is the simplest spectrum to compute. The price of simplicity is the potential lack of stability. However, it is possible to determine if $\Sigma_{\rm CL}$ is stable by checking integral separation of the diagonal of $B$ (the transformed, triangular, function of coefficients); see Theorem 2.22.

(b) The Lyapunov spectrum $\Sigma_{\rm L}$ is at least guaranteed to be stable in a generic situation (see Theorem 2.16), but it is much harder to determine if a matrix solution is integrally separated than it is to check if $\mathrm{diag}(B)$ is integrally separated. For example, take the case of a triangular system $\dot{R} = BR$. To check integral separation of the matrix solution $R$, in case in which $\mathrm{diag}(B)$ is not integrally separated, requires monitoring the off diagonal elements of $R$, which is at best cumbersome and computationally intensive.

(c) To approximate $\Sigma_{\rm CL}$ and $\Sigma_{\rm L}$, we need to approximate lim sup and lim inf as $t \to \infty$. Just as for the case of a limit, we do not see how this can be done without some assumptions on the function whose limits are sought. A simple approach is discussed in Example 4.1.

[4] we note that point spectrum implies regularity

(d) As far as we know, there is no computational work on the approximation of $\Sigma_{\rm ED}$. We believe that this task gets greatly simplified if we can reduce it to computing $\Sigma_{\rm IS}$. Constructively speaking, then, we may proceed by triangularizing the problem to the form $\dot{R} = BR$, and further approximate $\Sigma_{\rm IS}$ from $\mathrm{diag}(B)$. We illustrate this approach in Example 4.1.

### 3.3 Testing Integral Separation

From the preceding discussion, it is clearly important to infer integral separation of two functions. This is needed to gain some confidence in the answers we obtain for the LEs, and also when we attempt to find $\Sigma_{\rm IS}$. We have successfully adopted a construction based on Theorem 2.23. That is, to determine if two functions $f_1$ and $f_2$ are integrally separated, we check that, for sufficiently large $H$, their Steklov functions are separated:

$$f_1^H(t) - f_2^H(t) \geq a > 0,\ t \geq 0\,. \tag{23}$$

We refer to Examples 4.1 and 4.2 for practical considerations.

## 4 Numerical Experiments

We present two examples. The first is a linear system for which we compute the different spectra[5]. The second example is the Lorenz system: in this case we report on some experiments from [13].

**Example 4.1.** This is a linear nonregular system for which we approximate $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$. We have $\dot{x} = A(t)x$, with

$$A(t) = \begin{pmatrix} \cos(\ln(t+1)) + \sin(\ln(t+1)) - 2 & 2 \\ 2 & \sin(\ln(t+1)) + \cos(\ln(t+1)) - 2 \end{pmatrix}.$$

The spectra can be found analytically, which is useful for testing our computational results. To get the exact solution, it is enough to observe that the constant rotation with $Q = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ brings $Q^T A(t) Q$ into the diagonal form

$$D(t) = \begin{pmatrix} \cos(\ln(t+1)) + \sin(\ln(t+1)) & 0 \\ 0 & \cos(\ln(t+1)) + \sin(\ln(t+1)) - 4 \end{pmatrix}.$$

Clearly, $D$ has integrally separated diagonal, and –just as for Example 2.26– we can find that $\Sigma_{\rm L} = \Sigma_{\rm CL} = [-1, 1] \cup [-5, -3]$ and $\Sigma_{\rm ED} = \Sigma_{\rm IS} = [-\sqrt{2}, \sqrt{2}] \cup [-4 - \sqrt{2}, -4 + \sqrt{2}]$. Now we pretend to have no idea of the solution to check limits of computability of the spectra. The numerical method we implement is the following.

We solve for $Q$ using the code `QRINT` (see [12]) on the interval $[0, T]$, and local error control on $Q$ with a tolerance of $10^{-4}$. With initial conditions on $Q$ set

[5] as far as we know, our algorithm based on approximation of $\Sigma_{\rm IS}$ is the first computational technique used for approximating $\Sigma_{\rm ED}$

to the identity matrix, we observe that $Q$ rather quickly goes towards the matrix $Q = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, which is thereafter correctly approximated at 5 digits. To approximate $\Sigma_{\mathrm{L}}$, we compute $\lambda_j(t)$ for $j = 1, 2$, and then find the maximum and minimum as a function of $t$, for $\tau_0 \le t \le T$. To approximate $\Sigma_{\mathrm{IS}}$, we use the values of the diagonal system that are obtained numerically, and approximate their Steklov averages for a given $H > 0$. Then, we compute the maximum and minimum of these Steklov averages as a function of $t$ to obtain an approximation of the upper and lower endpoints of $\Sigma_{\mathrm{IS}}$. All the integrals used to determine the spectra are approximated using the composite trapezoidal rule.

In Table 1, we record our results for $\Sigma_{\mathrm{L}}$. We let $T$ denote the endpoint of integration, $\tau_0$ the value used to compute the approximation to the liminf and limsup, and $[\lambda_j^i, \lambda_j^s]$ is the computed $j$-th Lyapunov spectral interval ($j = 1, 2$), at four significant digits. Clearly, $\Sigma_{\mathrm{L}}$ is approximated quite well.

Table 1. Example 4.1: $\Sigma_{\mathrm{L}}$.

| $T$ | $\tau_0$ | $[\lambda_1^i, \lambda_1^s]$ | $[\lambda_2^i, \lambda_2^s]$ |
|---|---|---|---|
| $10^5$ | $10^2$ | $[-1.019, 1.]$ | $[-5.019, -3.]$ |
| $10^5$ | $10^3$ | $[-1., 1.]$ | $[-5., -3.]$ |
| $10^6$ | $10^2$ | $[-1.019, 1.]$ | $[-5.019, -3.]$ |
| $10^6$ | $10^3$ | $[-1., 1.]$ | $[-5., -3.]$ |
| $10^6$ | $10^4$ | $[-1., 0.9487]$ | $[-5., -3.051]$ |

In Table 2, we record our results for $\Sigma_{\mathrm{ED}}$. With $H$ we indicate the length used to compute Steklov averages of the diagonal entries. For $j = 1, 2$, we denote by $[a_j, b_j]$ the computed approximation to the $j$-th Sacker-Sell interval, at four significant digits. Again, $\Sigma_{\mathrm{ED}}$ is approximated quite well.

Table 2. Example 4.1: $\Sigma_{\mathrm{ED}}$.

| $T$ | $H$ | $[a_1, b_1]$ | $[a_2, b_2]$ |
|---|---|---|---|
| $10^5$ | $10^2$ | $[-1.414, 1.414]$ | $[-5.414, -2.586]$ |
| $10^5$ | $10^3$ | $[-1.414, 1.373]$ | $[-5.414, -2.627]$ |
| $10^6$ | $10^2$ | $[-1.414, 1.414]$ | $[-5.414, -2.586]$ |
| $10^6$ | $10^3$ | $[-1.414, 1.414]$ | $[-5.414, -2.586]$ |
| $10^6$ | $10^4$ | $[-1.406, 1.414]$ | $[-5.406, -2.586]$ |

**Example 4.2.** Consider the Lorenz equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \sigma(y - x) \\ \rho x - xz - y \\ xy - \beta z \end{pmatrix},$$

with parameter values $\sigma = 16$, $\beta = 4.0$ and $\rho = 45.92$. With initial condition $(x(0), y(0), z(0)) = (0, 1, 0)$, we computed the LEs according to Algorithm 3.3. A summary of results is in Table 3 below. Integration was done with local error control on the trajectory $x$, on $Q$, and on $\gamma_j$, $j = 1, 2, 3$, using local error tolerance $10^{-6}$.

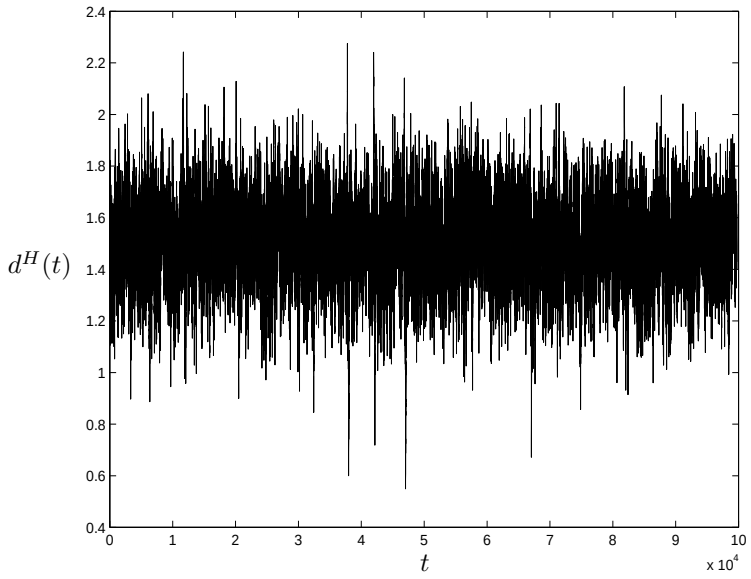


**Figure 1.** *Plot of* $(t, d^H(t))$ *where* $d^H(t) = b_{11}^H(t) - b_{22}^H(t)$ *for* $H = 20$.

| Table 3. Lorenz system. | | | | |
|---|---|---|---|---|
| $t_{\text{end}}$ | Steps | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
| 1.E2 | 8.6E3 | 1.415 | 3.E-2 | -22.466 |
| 1.E3 | 8.6E4 | 1.4892 | 4.64E-3 | -22.494 |
| 1.E4 | 8.6E5 | 1.499 | 4.64E-4 | -22.499 |
| 1.E5 | 8.6E6 | 1.5027 | 4.07.E-5 | -22.5027 |
| 1.E6 | 8.6E7 | 1.5024 | 7.6E-6 | -22.5024 |

The LEs seem to be converging towards $\lambda_1 \approx 1.5$, $\lambda_2 = 0$, and $\lambda_3 \approx -22.5$.

In order to infer stability of the exponents, we checked that the assumptions of Theorem 2.22 are satisfied, in particular that the functions $b_{11}, b_{22}, b_{33}$, are integrally separated. A simple inspection of the diagonal of the transformed function $B$ shows that $b_{22}$ and $b_{33}$ are integrally separated. For $b_{11}$ and $b_{22}$, we resorted to checking their Steklov difference; i.e., to check if (23) holds for $H$ sufficiently large. To form $b_{11}^H$ and $b_{22}^H$, we approximate the integrals by the composite trapezoidal rule. For $t \in [0, 10000]$, the value $H = 20$ gives sufficient separation; see Figure 1.

# Bibliography

[1] H. D. I. Abarbanel, R. Brown, and M. B. Kennel, "Local Lyapunov exponents computed from observed data," *J. Nonlin. Sci.* **2** (1992), pp. 343–366.

[2] L. Ya. Adrianova, *Introduction to linear systems of differential equations*, Translations of Mathematical Monographs Vol. 146, AMS, Providence, R.I. (1995).

[3] G. Benettin, L. Galgani, A. Giorgilli and J.-M. Strelcyn, "Lyapunov exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1: theory ", and " ... Part 2: numerical applications ", *Meccanica* **15** (1980), pp. 9–20, 21–30.

[4] T. Bridges and S. Reich, "Computing Lyapunov exponents on a Stiefel manifold", *Physica D* **156** (2001), pp. 219–238.

[5] B.F. Bylov, R.E. Vinograd, D.M. Grobman and V.V. Nemyckii, *The theory of Lyapunov exponents and its applications to problems of stability*, Nauka Pub., Moscow (1966).

[6] B.F. Bylov, "On the reduction of systems of linear equations to the diagonal form," *Math. Sb.* **67** (1965), pp. 338–334.

[7] B.F. Bylov and N.A. Izobov, "Necessary and sufficient conditions for stability of characteristic exponents of a linear system," *Differentsial'nye Uravneniya* **5** (1969), pp. 1794–1903.

[8] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics 629, Springer-Verlag, Berlin (1978).

[9] L. Dieci, R. D. Russell, and E. S. Van Vleck, "Unitary integrators and applications to continuous orthonormalization techniques," *SIAM J. Numer. Anal.* **31** (1994), pp. 261–281.

[10] L. Dieci, R. D. Russell, and E. S. Van Vleck, "On the computation of Lyapunov exponents for continuous dynamical systems," *SIAM J. Numer. Anal.* **34** (1997), pp. 402–423.

[11] L. Dieci and E.S. Van Vleck, "Computation of a few Lyapunov exponents for continuous and discrete dynamical systems," *Appld. Numer. Math.* **17** (1995), pp. 275–291.

[12] L. Dieci and E.S. Van Vleck, "Orthonormal integrators based on Householder and Givens transformations," *submitted* (2001).

[13] L. Dieci, and E. S. Van Vleck. "Lyapunov spectral intervals: theory and computation", *submitted* (2001).

[14] F. Diele, L. Lopez, and R. Peluso, "The Cayley transform in the numerical solution of unitary differential systems," *Adv. Comp. Math.* **8** (1998), pp. 317–334.

[15] S.P. Diliberto, "On systems of ordinary differential equations," in *Contributions to the Theory of Nonlinear Oscillations* (Ann. of Math. Studies 20), Princeton Univ. Press, Princeton (1950), pp. 1–38.

[16] K. Geist, U. Parlitz and W. Lauterborn, "Comparison of different methods for computing Lyapunov exponents," *Prog. Theor. Phys.* **83** (1990), pp. 875–893.

[17] I. Goldhirsch, P. L. Sulem, and S. A. Orszag, "Stability and Lyapunov stability of dynamical systems: a differential approach and a numerical method," *Physica D* **27** (1987), pp. 311–337.

[18] J. M. Greene and J-S. Kim, "The calculation of Lyapunov spectra," *Physica D* **24** (1987), pp. 213–225.

[19] D. Higham, "Time-stepping and preserving orthonormality," *BIT* **37** (1997), pp. 24–36.

[20] A. Lyapunov, "Problém géneral de la stabilité du mouvement," *Int. J. Control* **53** (1992), pp. 531–773.

[21] V. M. Millionshchikov, "Systems with integral division are everywhere dense in the set of all linear systems of differential equations," *Differents. Uravneniya* **5** (1969), pp. 1167–1170.

[22] V. M. Millionshchikov, "Structurally stable properties of linear systems of differential equations," *Differents. Uravneniya* **5** (1969), pp. 1775–1784.

[23] V. M. Millionshchikov, "Linear systems of ordinary differential equations," *Differents. Uravneniya* **7** (1971), pp. 387–390.

[24] V. M. Millionshchikov, "The stochastic stability of Lyapunov's characteristic exponent," *Differents. Uravneniya* **11** (1975), pp. 581–583.

[25] H. Munthe-Kaas, "Runge-Kutta methods on Lie groups," *BIT* **38** (1998), pp. 92–111.

[26] V. I. Oseledec, "A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems", *Trans. Moscow Mathem. Society*, 19:197, 1968.

[27] K. J. Palmer, "The structurally stable systems on the half-line are those with exponential dichotomy," *J. Diff. Eqn.* **33** (1979), pp. 16–25.

[28] K. J. Palmer, *Shadowing in dynamical systems: theory and applications*, Mathematics and its applications v. 501, Kluwer Academic Publishers, Dordrecht-Boston (2000).

[29] O. Perron, "Die Ordnungszahlen Linearer Differentialgleichungssysteme," *Math. Zeits.* **31** (1930), pp. 748–766.

[30] S. Yu. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Math. 1706, Springer–Verlag, Berlin–Heidelberg (1999).

[31] D. Ruelle, *Chaotic evolution and strange attractors*, Cambridge University Press, Cambridge (1989).

[32] R. J. Sacker and G. R. Sell, "A spectral theory for linear differential systems," *J. Diff. Eqn.* **7** (1978), pp. 320–358.

[33] G. Söderlind and R. M. Mattheij, "Stability and asymptotic estimates in nonautonomous linear differential equations," *SIAM J. Math. Anal.* **16** (1985), pp. 69–92.

[34] D.E. Stewart, "A new algorithm for the SVD of a long product of matrices and the stability of products, " *ETNA* **5** (1997), pp. 29–47.