LYAPUNOV AND SACKER-SELL SPECTRAL INTERVALS *

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Abstract. In this work, we show that for linear upper triangular systems of differential equations, we can use the diagonal entries to obtain the *Sacker and Sell*, or *Exponential Dichotomy*, and also –under some restrictions– the *Lyapunov* spectral intervals. Since any bounded and continuous coeficient matrix function can be smoothly transformed to an upper triangular matrix function, our results imply that these spectral intervals may be found from scalar homogeneous problems. In line with our previous work [8], we emphasize the role of integral separation. Relationships between different spectra are shown, and examples are used to illustrate the results and define types of coefficient matrix functions that lead to continuous Sacker-Sell spectrum and/or continuous Lyapunov spectrum.

Key words. Exponential dichotomy, Sacker-Sell spectrum, Lyapunov exponents, integral separation.

AMS subject classifications. 35A, 65L

1. Introduction.

Since the thesis of Lyapunov more than one hundred years ago, see [16], different characterizations of spectra for linear nonautonomous systems of differential equations have been proposed. These spectra also characterize stability properties of a solution trajectory of a nonlinear system, via linearization about the solution trajectory itself.

In this paper, we consider several different definitions of spectrum for linear systems on the half line with bounded and continuous coefficients, and show relationships between the different spectra. Our main effort is directed towards two spectra: The *Exponential Dichotomy*, or *Sacker-Sell* spectrum [29], and the *Lyapunov* spectrum, which is defined in terms of the (upper) Lyapunov exponents of the given system and of its adjoint. Henceforth, we will label these spectra Σ_{ED} and Σ_{L}, respectively.

In recent years, we have been increasingly interested in viable ways to numerically approximate these spectra; see [7, 8]. And, to avoid working directly with the fundamental matrix solution of the system, we have been looking into techniques which dynamically find the QR factorization of the matrix solution, and hence end up working with triangular systems. The basic numerical approach was pioneered by Benettin et alia in [2] to approximate Lyapunov exponents, and it has traditionally been justified on the grounds of the Multiplicative Ergodic Theorem (see [22]), and as such restricted to so–called regular systems. Instead, we have favored the role played by the presence (or lack thereof) of integral separation of the fundamental matrix solution, which is much more closely related to the stability of the Lyapunov exponents (regularity does not ensure the stability/continuity of Lyapunov exponents). In particular, in [8], we showed that if the transformed triangular system has an integrally separated diagonal, then this diagonal can be used to approximate both Σ_{L} and Σ_{ED}.

Although integral separation of the system (i.e., of a fundamental matrix solution of the system) is both necessary and sufficient in order to have stability of distinct Lyapunov exponents, it is a condition which cannot be satisfied for many problems, e.g., when there are identical Lyapunov exponents. For this reason, in this work we investigate the situation in which there is not necessarily integral separation in the fundamental matrix solution. In the present work we will show that, regardless of integral separation, Σ_{ED} can always be obtained from the diagonal of bounded, continuous, upper triangular coefficient matrix functions, and that Σ_{L} can be obtained from the diagonal of the upper triangular coefficient matrix functions if the Lyapunov exponents are stable. Some of our results are similar in spirit to those of Palmer in [24, 25] where it is shown that an

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upper triangular system is integrally (exponentially) separated if and only if the corresponding diagonal system is integrally (exponentially) separated. Exponential separation was introduced in [5] and corresponds essentially to integral separation between two sets of columns each possibly containing more than a single column. An important aspect of our results is that we construct rather explicit forms of the diagonalizing transformations. These prove useful in the error analysis for Lyapunov exponents developed in [9, 10].

An outline of the paper is as follows. In section 2 we give the basic definitions and properties of $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$, and present other definitions of spectra, some of which are equivalent to $\Sigma_{\rm L}$ and $\Sigma_{\rm ED}$ in some cases, and can also be useful in their approximation. Section 3 summarizes the classical results due to Bylov, Izobov, and Millionshchikov, on stability of Lyapunov exponents for the case in which there is full integral separation and for the case in which there is not. Section 4 contains preparatory material for our main results of section 5, which justify the use of the diagonal to compute $\Sigma_{\rm ED}$ and $\Sigma_{\rm L}$. In section 6 we establish relationships between the different spectra. Section 7 contains examples to illustrate our theoretical results. We develop examples illustrating the type of behavior in diagonal coefficients that lead to continuous Sacker-Sell and Lyapunov spectra including an example where these spectra coincide in a certain limit. Conclusions are in section 8.

2. Sacker-Sell, Lyapunov, and other Spectra.

Consider the system

$$\dot{x} = A(t)x,\ \ t \geq 0, \tag{2.1}$$

where $A : I\!\!R^+ \to I\!\!R^{n\times n}$ is continuous and bounded, uniformly in t, and the corresponding inhomogeneous equation

$$\dot{x} = A(t)x + f(t),\ \ t \geq 0, \tag{2.2}$$

with $f : I\!\!R^+ \to I\!\!R^n$ bounded and continuous.

It is natural to ask for a definition of spectrum $\Sigma \subset I\!\!R$ associated to (2.1) such that one or more of the following properties hold:

Property 1: For $\Sigma \cap (-\infty, 0) \neq \emptyset$ and $0 \notin \Sigma$, there exists a non-zero bounded solution to (2.1).

Property 2: For $0 \notin \Sigma$, there exists a bounded solution to (2.2) for any bounded, continuous, **non-zero** f.

Property 3: For $\Sigma \cap (-\infty, 0) \neq \emptyset$ and $0 \notin \Sigma$, there exists a bounded solution to (2.2) for any bounded, continuous f.

These three properties are important in different contexts. Property 1 is related to stability or the saddle property of the zero solution. Properties 2 and 3 are both concerned with the inhomogeneous problem and are central to topics such as shadowing (see e.g. [26, 28]). As it turns out, the Sacker-Sell spectrum gives all three properties, while the Lyapunov spectrum only gives Property 1.

Clearly, in the autonomous case ($A(t) \equiv A \in I\!\!R^{n\times n}$ for all t), $\Sigma_{\rm EIG} := \{\lambda \in I\!\!R : \lambda$ is the real part of an eigenvalue of $A\}$ defines a spectrum for which the above properties hold. In the nonautonomous case, different definitions of spectra have been used. We recall some of them.

In [29], Sacker and Sell introduced a spectrum for (2.1) based upon exponential dichotomy: The Sacker-Sell, or Exponential Dichotomy, spectrum ($\Sigma_{\rm ED}$) is given by those values $\lambda \in I\!\!R$ such that the shifted system $\dot{x} = [A(t) - \lambda I]x$ does not have exponential dichotomy. Recall that the system (2.1) has *exponential dichotomy* if for a fundamental matrix solution X there exists a projection P and constants $\alpha, \beta > 0$, and $K, L \geq 1$, such that

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \|X(t)(I-P)X^{-1}(s)\| &\leq Le^{\beta(t-s)}, \quad t \leq s. \end{aligned} \tag{2.3}$$

It is shown in [29] that Σ_{ED} is given by the union of at most n closed, disjoint, intervals. Thus, it can be written, for some $1 \le k \le n$, as

$$\Sigma_{\mathrm{ED}} := [a_1, b_1] \cup \cdots \cup [a_k, b_k], \tag{2.4}$$

where the intervals are disjoint. The complement of Σ_{ED} is called the **resolvent**: It is given by all values $\lambda \in I\!R$ for which the shifted system has exponential dichotomy.

To define a spectrum in terms of Lyapunov exponents, let X be a fundamental matrix solution of (2.1) and consider the quantities

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln ||X(t)e_i|| \ , \ i = 1, \dots, n, \tag{2.5}$$

where e_i denotes the i-th standard unit vector. When $\sum_{i=1}^{n} \lambda_i$ is minimized with respect to all possible fundamental matrix solutions, then the λ_i's are called the upper Lyapunov exponents, or simply Lyapunov exponents or characteristic numbers, of the system and the corresponding fundamental matrix solution is called **normal**.

The Lyapunov exponents satisfy

$$\sum_{i=1}^{n} \lambda_i \ge \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}(A(s))ds \tag{2.6}$$

where $\mathrm{Tr}(A(\cdot))$ is the trace of $A(\cdot)$. Linear systems for which the Lyapunov exponents exist as limits were called regular by Lyapunov.

Definition 2.1. *A system is* **regular** *(Lyapunov) if the time average of the trace has a finite limit and equality holds in (2.6).*

To define a spectrum based upon Lyapunov exponents, along with (2.1), we will also need to consider the associated adjoint equation

$$\dot{y}(t) = -A^T(t)y(t) \tag{2.7}$$

with Lyapunov exponents $\{-\mu_i\}_{i=1}^n$; these are also called *lower Lyapunov exponents* of (2.1). Now, take the λ_i's and $-\mu_i$'s which we can assume to be ordered: $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ where, in fact, $\lambda_j \ge \mu_j$ (see e.g. [8]). We define the Lyapunov spectrum, Σ_{L}, as

$$\Sigma_{\mathrm{L}} := \bigcup_{j=1}^{n} [\lambda_j^i, \lambda_j^s] \tag{2.8}$$

where $\lambda_j^i = \mu_j$ and $\lambda_j^s = \lambda_j$ for $j = 1, \dots, n$.

There are examples of systems for which Σ_{L} is a collection of points, while Σ_{ED} is a continuum. E.g., Millionshchikov, [21], and Johnson, [12], construct regular systems where the Lyapunov exponents are the endpoints of the interval making up Σ_{ED}. In general $\Sigma_{\mathrm{L}} \subseteq \Sigma_{\mathrm{ED}}$ (see Section 6), and both spectra may be a continuum (intervals), not just points.

Example 2.2. [16, 7]. For $\dot{x} = a(t)x$ with $a(t) = \sin(\ln(t))$, or also $a(t) = \cos(\ln(t))$, Σ_{ED} is $[-1, 1]$ and Σ_{L} is $[-1/\sqrt{2}, 1/\sqrt{2}]$.

Example 2.3. This simple example highlights that Lyapunov spectrum on the negative real axis does not imply bounded solutions for inhomogeneous problems. In fact, we consider an inhomogeneous scalar problem having an unbounded solution: The system is regular, with $\Sigma_{\mathrm{L}} = -\alpha < 0$, but does not have exponential dichotomy.

Consider the inhomogeneous differential equation $\dot{x} = a(t)x + f(t)$ where $a(t) = \sin(\omega_j(t - a_j)) - \alpha$ for $a_j \leq t < b_j$ with $0 < \alpha < 1$ and $a_1 = 0$, $b_j = a_j + 2\pi/\omega_j$, $a_j = b_{j-1}$ for $j = 1, \cdots$. For simplicity we will set $\omega_j = 1/j$ so that $\Sigma_{\rm L} = -\alpha$ and $\Sigma_{\rm ED} = [-1-\alpha, 1-\alpha]$ as will be illustrated in Example 7.1. We will also set $f(t) = 1$ for all t, but the same discussion will hold for any bounded (in the sup norm) f that is bounded away from zero. We show that any solution of the inhomogeneous equation is unbounded.

Denote by $\Phi(t) = \exp(\int_0^t a(\tau))d\tau = \exp(j(1-\cos((t-a_j)/j))-\alpha t)$ the principal matrix solution to the homogeneous problem $\dot{x} = a(t)x$. It is sufficient to consider $x(0) = 0$ so that the solution to the inhomogeneous equation is $x(t) = \Phi(t)[\int_0^t \Phi^{-1}(s)ds]$. Thus, for t such that $a_k \leq t < b_k$ we have

$$
\begin{aligned}
\int_0^t \Phi^{-1}(s)ds &= \sum_{j=1}^{k-1} \int_{a_j}^{b_j} \Phi^{-1}(s)ds + \int_{a_k}^{t} \Phi^{-1}(s)ds \\
&= \sum_{j=1}^{k-1} \int_{a_j}^{b_j} e^{(-(j(1-\cos((s-a_j)/j))))}ds + \int_{a_k}^{t} e^{(-(k(1-\cos((s-a_k)/k))))}ds \\
&\geq 2\pi \sum_{j=1}^{k-1} je^{-j}e^{\alpha a_j}e^{jg(u^*)} + u^* ke^{-k}e^{\alpha a_k}e^{kg(u^*)}
\end{aligned}
\tag{2.9}
$$

where $g(u) = \cos(u) + \alpha u$, u^* is the minimum of g on the interval $0 < u < 2\pi$, and t is chosen to be $t = ku^* + a_k$. For this t we have $\Phi(t) = e^k e^{-\alpha^k} e^{-kg(u^*)}$ so that

$$
|x(t)| = \Phi(t)\int_0^t \Phi^{-1}(s)ds \geq 2\pi\Phi(t)\sum_{j=1}^{k-1} je^{-j}e^{\alpha a_j}e^{jg(u^*)} + u^*k \to \infty \tag{2.10}
$$

as $k \to \infty$.

Lyapunov exponents and their stability properties (and hence $\Sigma_{\rm L}$) are preserved under **Lyapunov transformations**.

Definition 2.4. *A smooth invertible change of variables $y \leftarrow T^{-1}x$ is called a Lyapunov transformation if T, T^{-1}, and $\dot{T}$, are bounded.*

Under a Lyapunov transformation, (2.1) is transformed to

$$
\dot{y} \;=\; B(t)y \ , \quad B(t) := T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t) \ , \ \forall t \geq 0 \ . \tag{2.11}
$$

It is obvious that $\Sigma_{\rm ED}$ is invariant under a Lyapunov transformation. Indeed, any sensible characterization of spectrum for (2.1) ought to be invariant under Lyapunov transformations, which can then be used to simplify the form of the system at hand. In our mind, the most important transformation is based on the QR factorization of a fundamental matrix solution X. In fact, it has been known for a long time (see [27] and [11]) that there exists an orthogonal, Lyapunov, change of variables for which the transformed system (B in (2.11)) is upper triangular. To see this, for all t we write a fundamental matrix solution $X(t)$ as $Q(t)R(t)$ where Q is orthogonal and R is upper triangular with positive diagonal entries. Upon differentiating the relation $X = QR$, we have

$$
AQR = Q\dot{R} + \dot{Q}R \quad \text{or} \quad \dot{Q} = AQ - QB \quad \text{with} \quad B = Q^TAQ - Q^T\dot{Q}. \tag{2.12}
$$

Since $\dot{R} = BR$, B is upper triangular. Since Q is orthogonal, if we let $S(Q) := Q^T\dot{Q} = Q^TAQ - B$, then the strictly lower triangular piece of the skew symmetric function S can be defined as the corresponding piece of Q^TAQ and the rest of S is given by skew-symmetry.

A third spectrum we consider is what we call the *computed Lyapunov spectrum*, since it is close to what traditionally has been approximated. It is defined for an upper triangular system: $\dot{R} =$

BR, with B and the fundamental matrix R both triangular. We define the computed Lyapunov spectrum, written $\Sigma_{\rm CL}$, as

$$\Sigma_{\rm CL} := \bigcup_{j=1}^{n} [\lambda_{jj}^i, \lambda_{jj}^s] \ , \quad \lambda_{jj}^i = \liminf_{t\to\infty} \frac{1}{t}\int_0^t B_{jj}(s)ds \ , \quad \lambda_{jj}^s = \limsup_{t\to\infty} \frac{1}{t}\int_0^t B_{jj}(s)ds \, . \tag{2.13}$$

Other types of spectrum may be defined for linear upper triangular systems, and we next review some which are defined from the diagonal system extracted from the triangular one. In particular, the next spectrum we give highlights the parallel between exponential dichotomy and integral separation. To introduce it, we need the following.

Definition 2.5. *Two bounded, continuous function $f(t)$ and $g(t)$ defined for $t \geq 0$ are said to be* **integrally separated** *if there exists constants $a > 0$ and $d \geq 0$ such that*

$$\int_s^t (f(\tau) - g(\tau))d\tau \geq a(t-s) - d, \ \ t \geq s \geq 0. \tag{2.14}$$

Now, consider a linear diagonal differential system,

$$\dot{x} = D(t)x, \quad \text{where} \ \ D = \text{diag}(B_{jj}, \ j = 1, \dots, n) \, . \tag{2.15}$$

We will say that the diagonal system (2.15) is **integrally separated** if, for $j = 1, \dots, n-1$, the functions $B_{jj}, B_{j+1,j+1}$, are integrally separated in the sense of (2.14).

Next, for all $j = 1, \dots, n$, and for $\lambda \in I\!R$, consider the planar systems

$$\dot{y}_j = \begin{pmatrix} \lambda & 0 \\ 0 & B_{jj}(t) \end{pmatrix} y_j \tag{2.16}$$

and

$$\dot{y}_j = \begin{pmatrix} B_{jj}(t) & 0 \\ 0 & \lambda \end{pmatrix} y_j. \tag{2.17}$$

We are ready to define the **integral separation spectrum**.

Definition 2.6. *For (2.15), the integral separation spectrum is given by $\Sigma_{\rm IS} = \bigcup_{j=1}^n \Lambda_j$ where $\Lambda_j = \Lambda_j^+ \cap \Lambda_j^-$ with*

$$\Lambda_j^+ = \{\lambda \in I\!R : (2.16) \ \textit{is not integrally separated}\}$$

and

$$\Lambda_j^- = \{\lambda \in I\!R : (2.17) \ \textit{is not integrally separated}\},$$

for all $j = 1, \dots, n$.

Remark 2.1. A simple rewriting gives

$$\Lambda_j = \{\lambda \in I\!R : (2.16) \ \textit{and} \ (2.17) \ \textit{are not integrally separated}\}, \ j = 1, \dots, n.$$

We then have

THEOREM 2.7. *For (2.15), $\Sigma_{\rm IS} = \Sigma_{\rm ED}$.*

Proof. See [7]. ∎

We next introduce three spectra which are useful for approximating $\Sigma_{\rm ED}$, bypassing the need for uniformity intrinsic in its definition (e.g., "for all t, s such that $t \geq s$").

Let $H > 0$ be given, and, for $j = 1, \dots, n$, consider

$$\alpha_j^H = \inf_t \frac{1}{H} \int_t^{t+H} B_{jj}(s)ds \quad \text{and} \quad \beta_j^H = \sup_t \frac{1}{H} \int_t^{t+H} B_{jj}(s)ds \ . \tag{2.18}$$

For diagonal systems (2.15), in [8] we have used $[\alpha_j^H, \beta_j^H]$, to approximate the j-th spectral interval of Σ_{ED}, $j = 1, \dots, n$, associated to (2.15). The following result, see [8], justifies this approach.

THEOREM 2.8. *Consider (2.15): $\dot{x} = D(t)x$ where $D = \text{diag}(B_{jj}, j = 1, \dots, n)$, and let Λ_j be the j-th interval in Σ_{ED} for this system. For $j = 1, \dots, n$, let α_j^H and β_j^H be given as in (2.18). For any given $H > 0$: $\Lambda_j \subseteq [\alpha_j^H, \beta_j^H]$, for each $j = 1, \dots, n$. Moreover, for $H > 0$ sufficiently large, $[\alpha_j^H, \beta_j^H] \subseteq \Lambda_j$ and hence $[\alpha_j^H, \beta_j^H] = \Lambda_j$, $j = 1, \dots, n$.*

Based upon the construction just outlined, we now define the integral separation spectrum with $H > 0$, $\Sigma_{\text{IS}}^{H>0}$, as

$$\Sigma_{\text{IS}}^{H>0} := \bigcup_{j=1}^{n} [\alpha_j^H, \beta_j^H], \ \alpha_j^H = \inf_t \frac{1}{H} \int_t^{t+H} B_{jj}(s)ds, \ \beta_j^H = \sup_t \frac{1}{H} \int_t^{t+H} B_{jj}(s)ds, \tag{2.19}$$

the integral separation spectrum with $H = 0$, $\Sigma_{\text{IS}}^{H=0}$, as

$$\Sigma_{\text{IS}}^{H=0} := \bigcup_{j=1}^{n} [\alpha_j^0, \beta_j^0], \ \alpha_j^0 = \inf_t B_{jj}(t), \ \beta_j^0 = \sup_t B_{jj}(t), \tag{2.20}$$

and the kinematic eigenvalue spectrum, Σ_{KE}, as

$$\Sigma_{\text{KE}} := \bigcup_{j=1}^{n} [a_j, b_j], \ a_j = \liminf_{t \to \infty} B_{jj}(t), \ b_j = \limsup_{t \to \infty} B_{jj}(t). \tag{2.21}$$

To define the last spectrum associated to a triangular system, we recall the work of Lillo. In [15], Lillo considered a quantity similar to the largest Lyapunov exponent of (2.1). Letting $x(t)$ be a solution trajectory of (2.1), then Lillo defined the number

$$\lambda(A) := \limsup_{t \to \infty} [\sup_{x, t_0} \{ \frac{1}{t} \ln(\|x(t_0 + t)\| / \|x(t_0)\|) \}] \ .$$

By a Gronwall argument, Lillo showed that this quantity $\lambda(A)$ is upper semicontinuous. For upper triangular systems, similar quantities may be defined for the diagonal elements:

$$\gamma_i := \limsup_{t \to \infty} [\sup_{t_0} \frac{1}{t} \int_{t_0}^{t_0 + t} B_{ii}(s)ds] \ . \tag{2.22}$$

The Gronwall argument in [15] does not generalize to these quantities. However, we will show in this paper that the γ_i's in (2.22) are precisely the endpoints of the spectral intervals in Σ_{ED}, and hence they are upper semicontinuous since Σ_{ED} is upper semicontinuous by the Roughness Theorem for exponential dichotomies (e.g., see [6, 29]). We now introduce the *Lillo–spectrum*, Σ_{Lillo}. We let

$$\Sigma_{\text{Lillo}} = \bigcup_{i=1}^{n} [\mu_i, \gamma_i] \, , \tag{2.23}$$

where γ_i is defined by (2.22) and $-\mu_i$ is defined using (2.22) for the adjoint equation.

3. Continuity of Lyapunov Spectra. Here we summarize the relevant results on stability of Lyapunov exponents and its connection to integral separation and present a new result, Theorem 3.4, that shows that an upper triangular integrally separated system has integrally separated diagonal elements.

Definition 3.1. *The characteristic exponents $\lambda_1 \geq \ldots \geq \lambda_n$ of system (2.1) are said to be stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\sup_{t\in I\!R^+} ||E(t)|| < \delta$ implies*

$$|\lambda_i - \hat{\lambda}_i| < \epsilon, i = 1, \ldots, n, \tag{3.1}$$

where the $\hat{\lambda}_i$'s are the (ordered) Lyapunov exponents of the perturbed system $\dot{x} = [A(t) + E(t)]x$.

Definition 3.2. [1, cfr. Definition 5.3.2], [3]. *Write a fundamental matrix solution columnwise $X(t) = [X_1(t), \ldots, X_n(t)]$. Then, X is integrally separated if for $i = 1, \ldots, n-1$, there exist $a > 0$ and $d > 0$ such that*

$$\frac{||X_i(t)||}{||X_i(s)||} \cdot \frac{||X_{i+1}(s)||}{||X_{i+1}(t)||} \geq d e^{a(t-s)} , \tag{3.2}$$

for all $t, s \; : \; t \geq s$.

THEOREM 3.1. [1, Property 5.3.1 and 5.3.3]. *Integrally separated systems have distinct Lyapunov exponents.*

THEOREM 3.2. [1, Theorem 5.4.7], [4]. *If the system (2.1) has distinct characteristic exponents $\lambda_1 > \ldots > \lambda_n$, then they are stable if and only if there exists a Lyapunov transformation $z \leftarrow T^{-1}x$ transforming (2.1) to the diagonal form*

$$\dot{z} = \mathrm{diag}[p_1(t), \ldots, p_n(t)]z, \tag{3.3}$$

where the diagonal elements, the p_i, are integrally separated functions.

THEOREM 3.3. [1, Theorem 5.4.8], [4]. *If the system (2.1) has distinct characteristic exponents $\lambda_1 > \ldots > \lambda_n$, then they are stable if and only if there exists a fundamental matrix solution with integrally separated columns as in Definition 3.2.*

Example 3.3. This is an example (see [1, p. 95]) of a system with unstable Lyapunov exponents. Consider the (perturbed) system $\dot{x} = A(t)x$ where

$$A(t) = \begin{pmatrix} \sin(\log(t)) + \cos(\log(t)) - 2a & -e^{-at} \\ 0 & -a \end{pmatrix} \tag{3.4}$$

where $1 < 2a < (2 + e^{-\pi})/2$. Observe that this is a perturbation of a diagonal system with the perturbation decaying to zero as $t \to \infty$. It is shown in [1] that the Lyapunov exponents are unstable (the Lyapunov exponents of the unperturbed, diagonal, system are $1 - 2a$ and $-a$). The Sacker-Sell spectrum is easily seen to be $\Sigma_{\mathrm{ED}} = [-\sqrt{2} - 2a, \sqrt{2} - 2a] \cup \{-a\}$.

Now, if X is an integrally separated fundamental matrix solution for (2.1), then by [1, Property 5.3.2] the fundamental matrix solution $R(t)$, obtained from the orthogonal change of variables $X(t) = Q(t)R(t)$, is integrally separated. Recall that R is triangular, and satisfies the triangular differential system $\dot{R} = B(t)R$.

In [8], we proved that a sufficient condition for integral separation of the matrix solution R is given by integral separation of the diagonal system (2.15). Next, we give a converse of this result.

THEOREM 3.4. *For $t \geq 0$, consider $B(t)$ bounded, continuous, and upper triangular. Assume that $R(t)$ is an integrally separated, triangular fundamental matrix solution for $\dot{R} = B(t)R$. Then the diagonal of B is integrally separated.*

Proof. As in [1, Corollary 5.3.2] and its proof, take the upper triangular Lyapunov transformation L given by $L = [R_1/||R_1||_2, R_2/||R_2||_2, \cdots, R_n/||R_n||_2]$ where $R(t) = [R_1(t), \cdots, R_n(t)]$ is the upper triangular fundamental matrix solution written columnwise. Then $L^{-1}BL - L^{-1}\dot{L} = \text{diag}(P_{11}, \cdots, P_{nn})$ where $P_{jj} = \frac{d}{dt}\ln ||R_j||_2$ for $j = 1, \cdots, n$. Thus, $P_{11}(t) = B_{11}(t)$ and $P_{jj}(t) = B_{jj}(t) - \dot{L}_{jj}(t)/L_{jj}(t)$ for $j = 2, \cdots, n$. Since L is a Lyapunov transformation, there exist $M, m > 0$ such that $M \geq L_{jj}(t) \geq m$, for all $t \geq 0$ and $j = 1, \ldots, n$. Moreover, again by [1, Corollary 5.3.2], the P_{jj} are integrally separated, i.e., there exist $a > 0$ and $d \geq 0$ such that

$$\int_s^t \big(P_{jj}(\tau) - P_{j+1,j+1}(\tau)\big)d\tau \geq a(t-s) - d,\ t \geq s,\ j = 1, ., , n-1. \tag{3.5}$$

Thus, for $t \geq s$ and $j = 1, \ldots, n-1$,

$$\begin{aligned} &\int_s^t \big(B_{jj}(\tau) - B_{j+1,j+1}(\tau)\big)d\tau = \\ &\int_s^t \big(P_{jj}(\tau) - P_{j+1,j+1}(\tau)\big)d\tau - \ln\Big(\frac{L_{j+1,j+1}(t)}{L_{j+1,j+1}(s)}\frac{L_{jj}(s)}{L_{jj}(t)}\Big) \geq a(t-s) - \tilde{d}, \end{aligned} \tag{3.6}$$

where $\tilde{d} = d + 2\ln(M/m)$ since $\ln(\frac{L_{j+1,j+1}(t)}{L_{j+1,j+1}(s)}\frac{L_{jj}(s)}{L_{jj}(t)}) \leq 2\ln(M/m)$.
□

Remark 3.1. Theorem 3.4 and the fact that integral separation is maintained under Lyapunov transformations show that integral separation implies integral separation of the diagonal elements of the transformed upper triangular coefficient matrix. Coupled with [8, Theorem 5.1], Theorem 3.4 shows that integral separation of the diagonal of B is equivalent to having integral separation. This has been previously shown in [24] using a somewhat different approach and a similar result on the diagonalizability for systems corresponding to the case of non-distinct Lyapunov exponents is proven in [25].

For the case of non-distinct Lyapunov exponents we need some definitions before stating the theorem due to Bylov and Izobov [4] and Millionshchikov [18] on stability of Lyapunov exponents.

Definition 3.4. [1]. *Bounded, measurable functions, $l(t)$ and $u(t)$, defined on $\mathbb{R}^+$, are said to be lower and upper functions for (2.1) if for any solution x of (2.1) and any $\epsilon > 0$ there exist positive constants $d_{l,\epsilon}$ and $D_{u,\epsilon}$ such that*

$$d_{l,\epsilon}\exp\Big(\int_s^t (l(\tau) - \epsilon)d\tau\Big) \leq \frac{||x(t)||}{||x(s)||} \leq D_{u,\epsilon}\exp\Big(\int_s^t (u(\tau) + \epsilon)d\tau\Big) \tag{3.7}$$

for $t \geq s \geq 0$ and the quantities $d_{l,\epsilon}, D_{u,\epsilon}$ are independent of t and s.

Finally, for (2.1), we define the following two quantities:

$$\Omega = \inf_u\{\limsup_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds\}, \tag{3.8}$$

where the infimum is taken over all upper functions, called upper central exponent in [1], and

$$\bar{\omega} = \sup_l\{\limsup_{t\to\infty} \frac{1}{t}\int_0^t l(s)ds\}, \tag{3.9}$$

where the supremum is taken over all lower functions.

We are ready to state the stability theorem for Lyapunov exponents in the case of non-distinct Lyapunov exponents.

THEOREM 3.5. [4], [18], [1, Theorem 5.4.9]. *The Lyapunov exponents of $\dot{x} = A(t)x$ are stable if and only if there exists a Lyapunov transformation T that transforms $\dot{x} = A(t)x$ to the block diagonal form*

$$\dot{z} = \text{diag}[P_1(t), \cdots, P_q(t)]z \tag{3.10}$$

where each $P_k(t)$ is upper triangular of dimension n_k. Moreover, for the block systems $\dot{z}_k = P_k(t)z_k$, we have:

(i) all solutions of the block have the same Lyapunov exponents, Λ_k, and $\bar{\omega}_k = \Omega_k = \Lambda_k$;
(ii) for any p_i an arbitrary diagonal element of P_k and p_j an arbitrary diagonal element of P_{k+1}, p_i and p_j are integrally separated.

Example 3.5. In [1, Example 5.1.4], an example is given of a regular system with equal but unstable Lyapunov exponents. The coefficient matrix is $A(t) = \mathrm{diag}(0, \pi \sin(\pi\sqrt{t}))$. For this example $\Sigma_{\mathrm{ED}} = [-\pi, +\pi]$.

Example 3.6. An example of a non-regular system with equal and stable Lyapunov exponents is given by $A(t) = \mathrm{diag}(b(t), b(t) + \cos(t))$ where $b(t) = \sin(\ln(t))$. Stability of the exponents here follows from condition (5.2) of Theorem 5.1 given below.

4. Lack of Integral Separation. We will assume that an orthogonal change of variables has been performed so that we may consider $\dot{x} = B(t)x$, with B upper triangular. We now focus on the case in which the diagonal elements are not necessarily integrally separated. Instead of reducing to a diagonal coefficient matrix function we are only able to reduce to a block diagonal coefficient matrix function with upper triangular blocks. The integral separation of the diagonal elements of the coefficient matrix function determines the size of the blocks. The extremes are: n blocks when each pair of consecutive diagonals is integrally separated, and a single block when none of the diagonal elements are integrally separated.

The next lemma allows reduction to a block diagonal structure with upper triangular blocks based upon integral separation of the diagonal elements of B.

LEMMA 4.1. *Consider $\dot{x} = B(t)x$ with B bounded, continuous, and upper triangular. For $i < j$ if B_{ii} and B_{jj} are integrally separated, then there exists a Lyapunov transformation $T = I + e_i e_j^T x$ such that the transformed coefficient matrix function $C = T^{-1}BT - T^{-1}\dot{T}$ is still upper triangular, continuous and bounded, and moreover*

$$C_{ij} = 0, C_{kl} = B_{kl}, \quad \text{for} \quad k \neq i,\ l \neq j\ ,$$

$$C_{kl} = B_{kl} - xB_{jl}\ ,\ \ k = i, l \neq j\ ,$$

and

$$C_{kl} = B_{kl} + xB_{ki}\ ,\ \ k \neq i, l = j\, .$$

In particular, $C_{kl} = B_{kl}$ for $j > l$ or $k > i$.

Proof. Write $T = I + e_i e_j^T x$ where e_i and e_j are the standard unit vectors in $\mathbb{R}^n$, so that $T^{-1} = I - e_i e_j^T x$ and $\dot{T} = e_i e_j^T \dot{x}$. Then to have $0 = C_{ij}(t) = e_i^T[T^{-1}B(t)T - T^{-1}\dot{T}]e_j$ for all t, x must satisfy

$$\left\{ \begin{array}{l} \dot{x} = B_{ii}x - xB_{jj} + B_{ij}, \\ \lim_{T\to\infty} x(T) = 0. \end{array} \right. \tag{4.1}$$

Thus, as in the proof of [8, Theorem 5.1], T, T^{-1} and $\dot{T}$ are bounded if B_{ii} and B_{jj} are integrally separated. Since $C_{kl} = e_k^T[T^{-1}B(t)T - T^{-1}\dot{T}]e_l$, and B is upper triangular, the result follows.
□

THEOREM 4.2. *Consider $\dot{x} = B(t)x$ with B bounded, continuous, and upper triangular. If S denotes the set of tuples (i, j) with $i < j$ such that B_{ii} and B_{jj} are integrally separated, then there exists a Lyapunov transformation T such that $C = T^{-1}BT - T^{-1}\dot{T}$ and $(i, j) \in S$ implies $C_{ij} = 0$ and $(i, j) \notin S$ implies $C_{ij} = B_{ij}$.*

Proof. Apply Lemma 4.1 starting with the second column, then proceeding to the third column, etc., and finally ending with the last column. Within each column start from the element on the

superdiagonal and work up, i.e. for the ith column apply if possible the Lyapunov change of variables first with $(i,j) = (i-1,i)$, then $(i-2,i),\cdots,(1,i)$. Observe by Lemma 4.1 that for $T = I + e_i e_j^T x$, the only elements of the coefficient matrix function that are possibly changed are the elements of the coefficient matrix function in row i or column j. However, when annihilating element (i,j) the elements $C_{kl} = B_{kl}$ for $l < j$ or $k > i$, so in fact the only elements of the coefficient matrix function that are possibly modified are $C_{ij} \equiv 0$, $C_{i-1,j},\cdots,C_{1,j}$, and $C_{i,j+1},\cdots,C_{i,n}$.
□

5. Obtaining Spectra from the Diagonal. We next state and prove two results. The first gives sufficient conditions under which, for a continuous, bounded, upper triangular coefficient matrix function, $B(\cdot)$, the Lyapunov spectrum may be obtained from the diagonal of B. The second result shows that the Sacker-Sell spectrum may be obtained from the diagonal of B.

The following formulas will be useful. The solution of

$$\dot{R}(t) = B(t)R(t), \quad R(0) \text{ nonsingular upper triangular},$$

is

$$R_{ij}(t) = \frac{R_{ii}(t)}{R_{ii}(0)}[R_{ij}(0) + R_{ii}(0)\int_0^t R_{ii}^{-1}(\tau)\sum_{k=i+1}^{j} B_{ik}(\tau)R_{kj}(\tau)d\tau]\,, \quad i<j\,,$$

where $R_{ii}(t) = \exp(\int_0^t B_{ii}(\tau)d\tau)R_{ii}(0)$. For the associated Cauchy problem

$$\dot{R}(t,s) = B(t)R(t,s), \quad R(s,s) = I,$$

the solution is

$$R_{ij}(t,s) = R_{ii}(t,s)\int_s^t R_{ii}^{-1}(\tau,s)\sum_{k=i+1}^{j} B_{ik}(\tau)R_{kj}(\tau,s)d\tau. \tag{5.1}$$

THEOREM 5.1. *For $t \geq 0$, consider $\dot{x} = B(t)x$ with $B(\cdot)$ bounded, continuous, and upper triangular. If either of the following conditions hold*

(i) the diagonal elements of B are integrally separated, or

(ii) the diagonal elements of B are not all integrally separated and for nonintegrally separated diagonal elements, $B_{ii}(t)$ and $B_{jj}(t)$, within an upper triangular block (see Theorem 4.2), for every $\epsilon > 0$ there exists $M_{ij}(\epsilon) > 0$ such that

$$|\int_s^t (B_{ii}(\tau) - B_{jj}(\tau))d\tau| \leq M_{ij}(\epsilon) + \epsilon(t-s), \; t \geq s, \tag{5.2}$$

then the Lyapunov spectrum Σ_{L} is obtained as

$$\Sigma_{\mathrm{L}} := \bigcup_{j=1}^{n}[\lambda_{jj}^i, \lambda_{jj}^s]\,, \quad \lambda_{jj}^i = \liminf_{t\to\infty}\frac{1}{t}\int_0^t B_{jj}(s)ds\,, \quad \lambda_{jj}^s = \limsup_{t\to\infty}\frac{1}{t}\int_0^t B_{jj}(s)ds\,, \tag{5.3}$$

and Σ_{L} is stable.

Remark 5.1. If we consider the diagonal elements of an upper triangular block of B without integral separation between the diagonal elements and set $D(t) = \mathrm{diag}(B_{ii}(t),\cdots,B_{jj}(t))$, the diagonal of one of these blocks, then condition (5.2) implies that the Sacker-Sell spectrum of $D(t) - B_{kk}(t)\cdot I$ is $\{0\}$ for any $k \in \{i,\cdots,j\}$. In particular, this implies that all Lyapunov exponents of the diagonal system are the same, and are given by $\limsup_{t\to\infty}\frac{1}{t}\int_0^t B_{kk}(r)dr$ for any $k \in \{i,\cdots,j\}$.

Remark 5.2. The condition (5.2) is stronger than just lack of integral separation: (5.2) is a uniform bound on the integral of the difference of the diagonal elements, whereas the condition for lack of integral separation of consecutive diagonal elements requires that (see [1]): *For any* $\delta > 0$ *there exists an infinite sequence of intervals* $\{[\tau_k, \theta_k]\}$ *such that* $d_k = \theta_k - \tau_k \to \infty$ *and* $\tau_k \to \infty$ *both monotonically as* $k \to \infty$, *and*

$$\int_{\tau_k}^{\theta_k} (B_{ll}(\tau) - B_{l+1,l+1}(\tau))d\tau < \delta d_k. \tag{5.4}$$

The following theorem is used to prove Theorem 5.1. It shows that given condition (5.2) we can bound the off diagonal elements of the fundamental matrix solution in a uniform way by the diagonal of the fundamental matrix solution.

THEOREM 5.2. *Consider* $\dot{x} = B(t)x$ *where* $B(\cdot)$ *is bounded, continuous, and upper triangular. Assume that for any* $\epsilon > 0$ *and* $i < j$ *there exists* $M_{ij}(\epsilon) > 0$ *such that (5.2) is satisfied. Then for any* $\epsilon > 0$ *and* $i < j$, *there exists* $\bar{K}_{ij} > 0$ *such that for* $t \geq s$,

$$|R_{ij}(t,s)| \leq \bar{K}_{ij} \exp(\int_s^t B_{ii}(\tau)d\tau) \sum_{k=1}^{j-i} \frac{1}{k!} [\frac{E(\epsilon(t-s))}{\epsilon}]^k, \quad E(x) = e^x - 1\,. \tag{5.5}$$

Remark 5.3. A similar result holds for the off diagonal elements of the adjoint. That is, for $i < j$, and given k, there exists $\hat{K}_{ij} > 0$ such that for $t \geq s$,

$$|R^{-1}_{ij}(t,s)| \leq \hat{K}_{ij} \exp(-\int_s^t B_{ii}(\tau)d\tau) \sum_{k=1}^{j-i} \frac{1}{k!} [\frac{E(\epsilon(t-s))}{\epsilon}]^k, \quad E(x) = e^x - 1\,. \tag{5.6}$$

Proof. The proof is by induction on $j - i$. Hereafter, define M so that $|B_{ij}(t)| \leq M$, for all t and all i, j.

For $j - i = 1$, using (5.1) and the bound (5.2), we have

$$\begin{aligned} |R_{ij}(t,s)| &\leq \int_s^t R_{ii}(t,\tau) \cdot |B_{ij}(\tau)| \cdot R_{jj}(s,\tau)d\tau \\ &\leq \int_s^t e^{\int_\tau^t B_{ii}(r)dr} \cdot M \cdot e^{\int_s^\tau B_{jj}(r)dr} d\tau \leq M e^{M_{ij}(\epsilon)} e^{\int_s^t B_{ii}(r)dr} \int_s^t e^{\epsilon(\tau - s)} d\tau \\ &= \bar{K}_{ij} e^{\int_s^t B_{ii}(r)dr} \frac{1}{\epsilon}(e^{\epsilon(t-s)} - 1) = \bar{K}_{ij} e^{\int_s^t B_{ii}(r)dr} \frac{1}{\epsilon} E(\epsilon(t-s)) \end{aligned} \tag{5.7}$$

where $\bar{K}_{ij} = M e^{M_{ij}(\epsilon)}$. Next, assume that (5.5) holds for $j - i < m$ and we prove that it holds for $j - i = m$. We have

$$\begin{aligned} |R_{ij}(t,s)| &\leq e^{\int_s^t B_{ii}(r)dr} \int_s^t e^{-\int_s^\tau B_{ii}(r)dr} \sum_{k=i+1}^{j} |B_{ik}(\tau) R_{kj}(\tau,s)| d\tau \\ &\leq M e^{\int_s^t B_{ii}(r)dr} \int_s^t e^{-\int_s^\tau B_{ii}(r)dr} \sum_{k=i+1}^{j} |R_{kj}(\tau,s)| d\tau \\ &\leq M e^{\int_s^t B_{ii}(r)dr} \int_s^t e^{-\int_s^\tau B_{ii}(r)dr} (\sum_{k=i+1}^{j-1} \bar{K}_{kj} e^{\int_s^\tau B_{kk}(r)dr} \{\sum_{l=1}^{j-k} \frac{(E(\epsilon(\tau-s)))^l}{l!\epsilon^l}\} + e^{\int_s^\tau B_{jj}(r)dr}) d\tau \\ &\leq M e^{\int_s^t B_{ii}(r)dr} \int_s^t (\sum_{k=i+1}^{j-1} \bar{K}_{kj} e^{M_{kj}(\epsilon)} e^{\epsilon(\tau-s)} \{\sum_{l=1}^{j-k} \frac{(E(\epsilon(\tau-s)))^l}{l!\epsilon^l}\}) + e^{M_{ij}(\epsilon)} e^{\epsilon(\tau-s)} d\tau \\ &\leq \bar{K}_{ij} e^{\int_s^t B_{ii}(r)dr} \int_s^t e^{\epsilon(\tau-s)} \sum_{k=0}^{j-i-1} \frac{(E(\epsilon(\tau-s)))^k}{k!\epsilon^k} d\tau\,. \end{aligned} \tag{5.8}$$

Using the fact that

$$\int_0^x e^u(e^u-1)^p du = \frac{(e^x-1)^{p+1}}{p+1} \ ,$$

we obtain the result.
□

Proof. (of Theorem 5.1) If the diagonal elements of B are integrally separated, then the proof follows from [8, Theorem 5.1 and Corollary 5.1].

If the diagonal elements of B are not integrally separated, then the proof will follow from Theorem 4.2 (reduction to block diagonal with triangular blocks) and Theorem 5.2 (control on the off diagonal elements).

First, apply Theorem 4.2 to reduce $B(t)$ to block diagonal form with upper triangular blocks based upon the integral separation in the diagonal elements of B. Next consider an arbitrary upper triangular block of B, call it B for simplicity, of dimension n. We are now assuming that the diagonals of the block B are not integrally separated, but that (5.2) holds. The idea of the proof is to verify hypothesis (i) of Theorem 3.5 since we assume integral separation across blocks. This means that we must show that within a block all solutions of the block have the same characteristic exponent Λ and $\bar{\omega} = \Lambda = \Omega$.

We show that when (5.2) is satisfied $\bar{\omega}$ may be obtained using the lower function $l = B_{kk}$ and Ω may also be obtained using the upper function $u = B_{kk}$ so that $\bar{\omega} = \Omega$ where k may be arbitrarily chosen from $\{1, \cdots, n\}$.

Consider the Cauchy matrix $R(t,s) = R(t)R^{-1}(s)$, and let $x(t)$ be any solution of the system: $\dot{x} = B(t)x$. By [1, Lemma 5.1.1], we have

$$||R(t,s)|| = \max_x \frac{||x(t)||}{||x(s)||} \text{ and } 1/||R^{-1}(t,s)|| = \min_x \frac{||x(t)||}{||x(s)||}. \tag{5.9}$$

To show (3.7) with $l(t) = B_{kk}(t)$ and $u(t) = B_{kk}(t)$ it then suffices to show that for every $\epsilon > 0$ and $i \le j$, there exist positive constants $d_{B_{kk},\epsilon}$ and $D_{B_{kk},\epsilon}$ such that

$$d_{B_{kk},\epsilon} \exp(\int_s^t (B_{kk}(\tau) - \epsilon) d\tau) \le 1/|R^{-1}_{ij}(t,s)| \ , \tag{5.10}$$

and

$$D_{B_{kk},\epsilon} \exp(\int_s^t (B_{kk}(\tau) + \epsilon) d\tau) \ge |R_{ij}(t,s)| \ . \tag{5.11}$$

Clearly (5.10) and (5.11) hold when $i = j$ using (5.2). To prove (5.10) and (5.11) for $i < j$ we employ Theorem 5.2 which shows that for any $\delta > 0$ and $i < j$, there exists $\bar{K}_{ij} > 0$ such that

$$|R_{ij}(t,s)| \le \bar{K}_{ij} \exp(\int_s^t B_{ii}(\tau) d\tau) \sum_{l=1}^{j-i} \frac{1}{l!} \big[\frac{E(\delta(t-s))}{\delta}\big]^l \ .$$

Then, because of (5.2), for any $\epsilon > 0$ there exists $\delta > 0$, such that

$$|R_{ij}(t,s)| \le \exp(M_{ik}(\epsilon/2) + \int_s^t (B_{kk}(\tau) + \epsilon/2) d\tau) \bar{K}_{ij} \sum_{l=1}^{j-i} \frac{1}{l!} \big[\frac{E(\delta(t-s))}{\delta}\big]^l \ , \tag{5.12}$$

and there exists $D_{B_{kk},\epsilon} \ge 1$ such that

$$\max_{i<j} \{ e^{M_{ik}(\epsilon/2)} \bar{K}_{ij} \sum_{l=1}^{j-i} \frac{1}{l!} \big[\frac{E(\delta(t-s))}{\delta}\big]^l \} \le D_{B_{kk},\epsilon} \exp(\epsilon(t-s)/2)$$

and (5.11) follows.

Similarly, since $R_{ij}^{-1}(t,s) = R_{ij}(s,t)$, for any $\delta > 0$ and $i < j$, there exists $\hat{K}_{ij} > 0$ such that

$$|R_{ij}^{-1}(t,s)| \leq \hat{K}_{ij} \exp(-\int_s^t B_{ii}(\tau)d\tau) \sum_{l=1}^{j-i} \frac{1}{l!} \big[\frac{E(\delta(t-s))}{\delta}\big]^l .$$

Again, using (5.2), for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R_{ij}^{-1}(t,s)| \leq \exp(M_{ik}(\epsilon/2) - \int_s^t (B_{kk}(\tau) - \epsilon/2)d\tau)\hat{K}_{ij} \sum_{l=1}^{j-i} \frac{1}{l!} \big[\frac{E(\delta(t-s))}{\delta}\big]^l , \tag{5.13}$$

and $d_{B_{kk},\epsilon} \leq 1$ such that

$$\max_{i<j}\{e^{M_{ik}(\epsilon/2)}\hat{K}_{ij} \sum_{l=1}^{j-i} \frac{1}{l!} \big[\frac{E(\delta(t-s))}{\delta}\big]^l\} \leq \frac{1}{d_{B_{kk},\epsilon}} \exp(\epsilon(t-s)/2)$$

and (5.10) follows.

To complete the proof, observe that (see Remark 5.1) all Lyapunov exponents of the diagonal system are identical, and Theorem 5.2 guarantees that the Lyapunov exponents must come from the diagonal. Therefore, we must have

$$\Lambda = \limsup_{t\to\infty} \frac{1}{t} \int_0^t B_{kk}(r)dr$$

and the proof is complete.

□

Remark 5.4. A similar result, working with adjoint, gives the stability of the lower exponent of the block and hence the stability of the spectral interval.

The condition (5.2) was motivated by a condition used by Lillo in [14] to ensure that the Lyapunov exponents of regular systems are stable.

THEOREM 5.3. [14] *Consider $\dot{x} = B(t)x$ with B bounded, continuous, upper triangular, and assume that this system is regular. Let $\{\lambda_i\}_{i=1}^n$ be the Lyapunov exponents of the system. If for all $\epsilon > 0$ and $i = 1, \cdots, n$, there exists $M_i(\epsilon)$ such that*

$$|\int_s^t (B_{ii}(\tau) - \lambda_i)d\tau| \leq M_i(\epsilon) + \epsilon(t-s),\ t \geq s, \tag{5.14}$$

then the Lyapunov exponents are stable.

Proof. A proof is given in [14], but an alternative proof may be given using Theorem 3.5. The idea of the proof is similar to the proof of Theorem 5.1. First reduce to block diagonal based on the integral separation of the diagonal. Then consider an arbitrary block and recall that necessarily to have stable Lyapunov exponents all solutions of the block have identical Lyapunov exponents, call it λ. Since the system is regular the time averages of the diagonal entries (which have limits) are the Lyapunov exponents.

For an n-dimensional block the assumption (5.14) gives us the bound (5.5) with B_{ii} replaced by λ on $R_{ij}(t,s)$ for $i < j$ and $t \geq s$. The proof now proceeds like the proof of Theorem 5.1 taking lower and upper functions $l(t) = \lambda$ and $u(t) = \lambda$.

□

We now elucidate the relation between condition (5.14) and integral separation, since both imply stability.

THEOREM 5.4. *Consider $\dot{x} = B(t)x$ with B bounded, continuous, upper triangular, and with distinct upper/lower Lyapunov exponents. Then, (5.14) implies integral separation, and hence stability of the exponents.*

Proof. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ be the ordered upper Lyapunov exponents of the system. [Similarly we can consider lower exponents by working with the adjoint].

For $i = 1, \ldots, n-1$, assumption (5.14) gives

$$\int_s^t (B_{ii}(\tau) - \lambda_i)d\tau \geq -M_i(\epsilon) - \epsilon(t-s),\ t \geq s,$$

and

$$-\int_s^t (B_{i+1,i+1}(\tau) - \lambda_{i+1})d\tau \geq -M_{i+1}(\epsilon) - \epsilon(t-s),\ t \geq s\ .$$

Adding these two inequalities, we obtain

$$\int_s^t (B_{ii}(\tau) - B_{i+1,i+1}(\tau))d\tau \geq (\lambda_i - \lambda_{i+1} - 2\epsilon)(t-s) - (M_i(\epsilon) + M_{i+1}(\epsilon)),\ t \geq s\ .$$

Since ϵ is arbitrary, the diagonal of B is integrally separated, hence the system is integrally separated and the Lyapunov exponents are stable.
□

The following theorem states that regardless of whether or not the diagonal of B is integrally separated, the Sacker-Sell spectrum may be obtained from the diagonal of B.

THEOREM 5.5. *For $t \geq 0$, consider $\dot{x} = B(t)x$ with $B(\cdot)$ bounded, continuous, and upper triangular. Then the Sacker-Sell spectrum $\Sigma_{\rm ED}$ of this system is the same as the Sacker-Sell spectrum of the diagonal system $\dot{x} = {\rm diag}(B(t))x$.*

To prove Theorem 5.5, we first provide bounds on the off diagonal elements of the fundamental matrix solution in terms of the maximum growth rate in the diagonal elements. We employ the notation R^λ to indicate the fundamental matrix solution of the shifted system: $\dot{R}^\lambda = [B - \lambda I]R^\lambda$, that is $R^\lambda(t) = e^{-\lambda t}R(t)$.

THEOREM 5.6. *Consider $\dot{x} = B(t)x$ where $B : I\!R \to I\!R^{n\times n}$ is bounded, continuous, and upper triangular. Assume that $D(t) = {\rm diag}(B_{11}(t), \cdots, B_{nn}(t))$ has $\Sigma_{\rm ED}(D) = \bigcup_{k=1}^n [\alpha_k^D, \beta_k^D]$. For $\lambda > \max_k\{\beta_k^D\}$ there exist $K \geq 1$ and $\alpha > 0$ such that $R_{kk}^\lambda(t,s) := e^{-\lambda(t-s)}R_{kk}(t,s) \leq Ke^{-\alpha(t-s)}$ for $t \geq s$. Then for $i < j$ there exists $\bar{K}_{ij} > 0$ such that for $t \geq s$,*

$$|R_{ij}^\lambda(t,s)| \leq \bar{K}_{ij}e^{-\alpha(t-s)}\sum_{k=1}^{j-i}(t-s)^k. \tag{5.15}$$

Remark 5.5. A similar result holds for the off diagonal elements of the fundamental matrix solution of the adjoint equation so that for $\lambda > \max_k\{-\alpha_k^D\}$ there exists $K \geq 1$ and $\beta > 0$ such that $(R_{kk}^\lambda(t,s))^{-1} := e^{\lambda(t-s)}R_{kk}^{-1}(t,s) \leq Ke^{-\beta(t-s)}$ for $t \geq s$. Then for $i < j$ there exists $\hat{K}_{ij} > 0$ such that for $t \geq s$,

$$|(R^{-1})_{ij}^\lambda(t,s)| \leq \hat{K}_{ij}e^{-\beta(t-s)}\sum_{k=1}^{j-i}(t-s)^k. \tag{5.16}$$

Proof. The proof is by induction on $j - i$. Recall that

$$R_{ij}^\lambda(t,s) = \int_s^t R_{ii}^\lambda(t,\tau)\sum_{k=i+1}^{j} B_{ik}(\tau)R_{kj}^\lambda(\tau,s)d\tau. \tag{5.17}$$

So, for $j-i=1$ we have

$$|R_{ij}^{\lambda}(t,s)| \leq MK^2 \int_s^t e^{-\alpha(t-\tau)} \cdot e^{-\alpha(\tau-s)} d\tau = K^2 M(t-s)e^{-\alpha(t-s)} =: \bar{K}_{ij}(t-s)e^{-\alpha(t-s)}\,. \tag{5.18}$$

Next, we assume that (5.15) holds for $j-i<m$ and prove that it holds for $j-i=m$. We have

$$\begin{aligned} |R_{ij}^{\lambda}(t,s)| &\leq MK \int_s^t e^{-\alpha(t-\tau)} \sum_{k=i+1}^{j} |R_{kj}^{\lambda}(\tau,s)| d\tau \\ &\leq MKe^{-\alpha(t-s)} \int_s^t \sum_{k=i+1}^{j} \bar{K}_{kj} \sum_{l=1}^{j-k} (\tau-s)^l d\tau \\ &\leq \bar{K}_{ij} e^{-\alpha(t-s)} \sum_{k=1}^{j-i} (t-s)^k. \end{aligned} \tag{5.19}$$

□

THEOREM 5.7. *Given a bounded, continuous, upper triangular coefficient matrix function $B(\cdot)$ in which the diagonal elements are not integrally separated, the system $\dot{x} = B(t)x$ has only one Sacker-Sell spectral interval.*

Proof. Suppose $[\alpha, \beta]$ is the smallest closed interval that contains the Sacker-Sell spectrum of the system. Assume that there exists $\lambda \in (\alpha, \beta)$ such that $\dot{x} = [B(t) - \lambda I]x$ has exponential dichotomy. To complete the proof we derive a contradiction. The argument follows the proof of [8, Theorem 6.3] where it is shown that the existence of λ for which the shifted system has exponential dichotomy implies that there is integral separation between columns of the fundamental matrix solution. By Theorem 3.4 and [8, Theorem 5.1] which together show the equivalence of diagonal integral separation and integral separation we would have that there is integral separation between some of the diagonal elements of B, a contradiction.
□

Proof. (of Theorem 5.5) If the diagonal elements of B are integrally separated, then the proof follows from [8, Theorem 5.1] since in this case there exists a Lyapunov transformation that transforms the coefficient matrix B to the diagonal of B.

If the diagonal elements of B are not integrally separated, then the proof follows from Theorem 4.2 (block diagonal with triangular blocks), Theorem 5.6 (uniform control of off diagonal terms), openness of the resolvent, and Theorem 5.7.

First, use Theorem 4.2 to transform $B(\cdot)$ to a block diagonal form with upper triangular blocks of the form described in Theorem 3.5 where the diagonal elements within a block are not integrally separated but diagonal elements from consecutive blocks are integrally separated. Next, focus on an arbitrary block (call it B and assume it has dimension n), choose λ, K, α as in the statement of Theorem 5.6, and let $\epsilon = \alpha/2$. Then for $1 \leq i < j \leq n$, there exists $L_{ij} > 0$ such that

$$\bar{K}_{ij} \sum_{k=1}^{j-i} (t-s)^k \leq L_{ij} e^{\epsilon(t-s)}, \ \ t \geq s. \tag{5.20}$$

So by (5.15) of Theorem 5.6,

$$|R_{ij}^{\lambda}(t,s)| \leq L_{ij} e^{-\frac{\alpha}{2}(t-s)}. \tag{5.21}$$

Since this holds for any $i<j$ and for any $\lambda > \beta^D := \max_k\{\beta_k^D\}$, the maximum over all right endpoints of the Sacker-Sell spectral intervals of the diagonal of B, and the resolvent is open, the right endpoint of the Sacker-Sell interval for the upper triangular system is no greater than β^D. A

similar argument with the adjoint equation shows that the left endpoint of the Sacker-Sell interval of the upper triangular system is not less than $\alpha^D := \min_k\{\alpha_k^D\}$, the minimum of the left endpoints of the Sacker-Sell spectral intervals of the diagonal of B. Thus, using Theorem 5.7, the Sacker-Sell spectrum for the block is $[\alpha^D, \beta^D]$.
□

COROLLARY 5.1. *Consider a bounded, continuous, block diagonal coefficient matrix function $B(t) = \mathrm{diag}(B_1(t), \cdots, B_p(t))$ where each block $B_k(t)$ is upper triangular. Assume that for any b_i an arbitrary diagonal element of B_k and b_j an arbitrary diagonal element of B_{k+1}, b_i and b_j are integrally separated, but that within each block the diagonal elements are not integrally separated. Then the Sacker-Sell spectrum consists of at most p intervals.*

6. Relationships Between Stability Spectra. In this section we establish relationships between the spectra introduced in Section 2. As usual, we can restrict consideration to triangular systems: $\dot{x} = B(t)x$, $t \geq 0$, with $B(t)$ upper triangular, bounded, and continuous, for $t \geq 0$. We begin by showing the equivalence between Σ_{ED} and Σ_{Lillo}.

THEOREM 6.1. *For triangular systems, Lillo type exponents (2.22) are the endpoints of the Sacker-Sell intervals. That is, for $i = 1, \cdots, n$, we have*

$$[\alpha_i, \beta_i] = [\mu_i, \gamma_i]\,,$$

where $[\alpha_i, \beta_i]$ are the Sacker–Sell intervals associated to the triangular system (see Theorem 5.5), and $[\mu_i, \gamma_i]$ are defined in (2.23).

Proof. We show that $\beta_i = \gamma_i$, the proof that $\alpha_i = \mu_i$ is similar. Recall that the endpoints are defined as $\beta_i = \sup_t \frac{1}{H}\int_t^{t+H} B_{ii}(s)ds$ for H sufficiently large and $\gamma_i = \inf_\tau g_i(\tau) = \lim_{\tau\to\infty} g_i(\tau)$ where $g_i(\tau) = \sup_{t\geq\tau}\sup_{t_0}\frac{1}{t}\int_{t_0}^{t_0+t} B_{ii}(s)ds$.

Then for all $\epsilon > 0$ there exists $\tilde{H}(\epsilon)$ such that

$$\beta_i + \epsilon \geq \sup_{H\geq\tilde{H}(\epsilon)}\sup_{t_0}\frac{1}{H}\int_{t_0}^{t_0+H} B_{ii}(s)ds \geq \lim_{\tau\to\infty}\{\sup_{H\geq\tau}\sup_{t_0}\frac{1}{H}\int_{t_0}^{t_0+H} B_{ii}(s)ds\} = \gamma_i. \tag{6.1}$$

Conversely, for all $\epsilon > 0$ there exists $T(\epsilon)$ such that $\tau \geq T(\epsilon)$ implies

$$\gamma_i + \epsilon \geq \sup_{H\geq\tau}\sup_{t_0}\frac{1}{H}\int_{t_0}^{t_0+H} B_{ii}(s)ds \geq \beta_i. \tag{6.2}$$

□

The following theorem states that if the Lyapunov exponents are stable, then they may be computed from the diagonals of the upper triangular coefficient matrix function B. The proof follows almost directly from Theorems 3.3 and 3.5. First consider the reduction to block diagonal given by Theorem 4.2 with m upper triangular blocks, $\{B^{(1)}, ..., B^{(m)}\}$ of dimensions $\{n_1, ..., n_m\}$ and define for $i = 1, ..., m$,

$$a^{(i)} = \liminf_{t\to\infty}\frac{1}{t}\int_0^t B^{(i)}_{n_i,n_i}(s)ds,\;\; b^{(i)} = \limsup_{t\to\infty}\frac{1}{t}\int_0^t B^{(i)}_{11}(s)ds. \tag{6.3}$$

With the above notation, we have:

THEOREM 6.2. *If the Lyapunov exponents are stable, then $\Sigma_{\mathrm{L}} = \bigcup_{i=1}^m [a^{(i)}, b^{(i)}]$.*

Proof. Suppose now that the Lyapunov exponents are stable. Then either the system is integrally separated and Theorem 3.3 holds or the system is not integrally separated, but Theorem 3.5 holds. If the system is integrally separated, then by Theorem 3.4 and [8, Theorem 5.1], we can transform the upper triangular system to the diagonal system, $\mathrm{diag}(B)$, by a Lyapunov transformation and $n_i = 1$ for $i = 1, .., n \equiv m$ in (6.3). If the system is not integrally separated, then by Theorem 4.2 we may transform $B(\cdot)$ to block diagonal with upper triangular blocks leaving the diagonals unchanged. Then since Theorem 3.5 holds every solution in a block must have the same Lyapunov exponent. Consider the Cauchy problem for the block and the associated adjoint system

$$\begin{aligned} \dot{x}^{(i)} &= B^{(i)}(t)x^{(i)}, \quad x^{(i)}(0) = x_0^{(i)}, \\ \dot{y}^{(i)} &= -(B^{(i)})^T(t)y^{(i)}, \quad y^{(i)}(0) = y_0^{(i)}, \end{aligned} \tag{6.4}$$

and choose $x_0^{(i)} = e_1$ for the original equation and $y_0^{(i)} = e_{n_i}$ for the adjoint to obtain the result from (6.3).
□

The following theorem provides further relationships between spectra defined in section 2.

THEOREM 6.3. *For a time varying linear system, with bounded and continuous coefficient matrix on the half line, we have the following relationships between spectra:*

$$\Sigma_{\text{CL}} \subseteq \Sigma_{\text{L}} \subseteq \Sigma_{\text{Lillo}} \equiv \Sigma_{\text{ED}}, \tag{6.5}$$

$$\Sigma_{\text{ED}} \subseteq \Sigma_{\text{KE}} \subseteq \Sigma_{\text{IS}}^{H=0}, \tag{6.6}$$

$$\Sigma_{\text{ED}} \subseteq \Sigma_{\text{IS}}^{H>0} \subseteq \Sigma_{\text{IS}}^{H=0}. \tag{6.7}$$

Proof. We begin by making an orthogonal change of variables to an upper triangular coefficient matrix function, $B(\cdot)$.

We first focus on the inclusions and equivalence in (6.5). Clearly $\Sigma_{\text{CL}} \subseteq \Sigma_{\text{L}}$ since Σ_{L} is a function of the entire upper triangular fundamental matrix while Σ_{CL} is defined in terms of the diagonal of the upper triangular fundamental matrix solution. That $\Sigma_{\text{L}} \subseteq \Sigma_{\text{ED}}$ is proven in [29], see also [8, Theorem 6.2]. To prove that $\Sigma_{\text{Lillo}} \equiv \Sigma_{\text{ED}}$, by Theorem 5.5 we may restrict attention to the diagonal of $B(\cdot)$, so the result follows from Theorem 6.1.

That $\Sigma_{\text{KE}} \subseteq \Sigma_{\text{IS}}^{H=0}$ follows directly from the definitions (2.21) and (2.20). The inclusion $\Sigma_{\text{ED}} \subseteq \Sigma_{\text{KE}}$ in (6.6) follows from [8, Theorem 8.4] since by Theorem 5.5 the Sacker-Sell spectrum may be computed by restricting to the diagonal of $B(\cdot)$ and we may then employ the Lillo type definition to characterize Σ_{ED}. In particular,

$$\limsup_{t\to\infty}\{\sup_{t_0} \frac{1}{t}\int_{t_0}^{t_0+t} B_{ii}(s)ds\} \leq \limsup_{t\to\infty} B_{ii}(t). \tag{6.8}$$

That $\Sigma_{\text{ED}} \subseteq \Sigma_{\text{IS}}^{H>0}$ follows similarly from Theorem 5.5 and [8, Theorem 8.4]. The final inclusion in (6.7) follows since $\sup_t a(t) \geq \sup_t \frac{1}{H}\int_t^{t+H} a(\tau)d\tau$.
□

7. Examples. An important implication of our results is that Σ_{L} and Σ_{ED} can be approximated by working with scalar differential equations. In this section, we give two examples of scalar problems for which we have interesting spectra Σ_{L} and Σ_{ED} including a scalar problem where in a limit Σ_{L} and Σ_{ED} are equal and continuous. By appropriately coupling together such scalar differential equations, linear systems with challenging spectra to approximate may be obtained.

Example 7.1. This is a scalar problem with continuous Σ_{ED}. For $\dot{x} = a(t)x$ define

$$a(t) = \sum_{j=1}^{\infty} \chi_{[a_j,b_j)}(t) \cdot sin(\omega_j(t-a_j)) \tag{7.1}$$

where $a_1 = 0$, $b_1 = 2\pi/\omega_1$, and $a_j = b_{j-1}$, $b_j = a_j + 2\pi/\omega_j$ for $j = 2, 3, \cdots$, and $\chi_{[a_j,b_j)}(t)$ is the characteristic function on $[a_j, b_j)$, and ω_j are nonnegative numbers satisfying $\omega_j \geq \omega_{j+1}$ for $j = 1, 2, \cdots$. Observe that the function $a(\cdot)$ is continuous.

Claim . If $\omega_j \to 0$ as $j \to \infty$, then Σ_{ED} is $[-1, +1]$.

Proof. We will show that for any $H > 0$ and any $\epsilon > 0$ there exists N such that $j \geq N$ implies

$$|1 - \frac{1}{H}\int_{m_j}^{m_j+H} a(s)ds| = |1 - \frac{1}{\omega_j H}[-cos(\pi/2 + \omega_j H)]| \leq \epsilon \tag{7.2}$$

where $m_j = a_j + \pi/(2\omega_j)$. Now, given $H > 0$ and ϵ such that $\pi^2/2 \geq \epsilon > 0$ choose N such that for $j \geq N$, $\omega_j H \leq \sqrt{2\epsilon}$. Then $m_j + H \leq b_j$ since $\omega_j H \leq \pi$. To prove the inequality in (7.2) consider for $h = \omega_j H$, $\frac{1}{h}(-\cos(\pi/2 + h)) = \frac{1}{h}(h + \cos(\xi)h^2/2)$ for $\xi \in [\pi/2, \pi/2 + h]$. Using a trig identity we have $|\cos(\xi)| \leq |\cos(\pi/2+h)| = \sin(h) \leq h$ since $0 \leq h \leq \pi$. Thus, (7.2) follows since $h^2/2 \leq \epsilon$. ☐

It is interesting to study how the value of N in the above proof depends on the rate at which $\omega_j \to 0$. To exemplify, consider $\omega_j = (1/j)^\alpha$ for $\alpha > 0$. Then a direct calculation following the above proof shows that $N \equiv N(H, \epsilon) = \lceil (H/\sqrt{2\epsilon})^{1/\alpha} \rceil$ and the corresponding value of T beyond which (7.2) holds is $T = m_N$ so that $T = \pi(2\sum_{j=1}^{N-1} \frac{1}{\omega_j} + \frac{1}{2}\frac{1}{\omega_N})$. We have $T \equiv T_\alpha \approx \frac{2\pi}{1+\alpha}N^{1+\alpha} = \frac{2\pi}{1+\alpha}\lceil (H/\sqrt{2\epsilon})^{1/\alpha} \rceil^{1+\alpha}$.

Next, we examine Σ_{L}. Since $\int_{a_j}^{b_j} a(s)ds = 0$ for all j we have

$$\lambda^+ = \limsup_{t\to\infty} \frac{1}{t}\int_0^t a(s)ds = \limsup_{j\to\infty, t\in(a_j,b_j)} \frac{1}{\omega_j t}(1 - \cos(\omega_j(t - a_j)). \tag{7.3}$$

Next write $t = a_j + \tau$ for $0 < \tau < (b_j - a_j)$ so that the right hand side of (7.3) becomes

$$\frac{1}{\omega_j t}(1 - \cos(\omega_j(t - a_j)) = \frac{1}{\omega_j a_j + x}(1 - \cos(x)) \tag{7.4}$$

where $x = \omega_j \tau$ and $0 < x < 2\pi$.

At this point, we will see that Σ_{L} depends on the rate at which $\omega_j \to 0$. Two situations are of interest: (a) $\omega_j = (1/j)^\alpha$ for $\alpha > 0$, and (b) $\omega_j = e^{-\beta j}$ for $\beta > 0$.

(a) We have $a_j = 2\pi(1/\omega_1 + \cdots + 1/\omega_{j-1})$ so that for $\omega_j = (1/j)^\alpha$,

$$\begin{aligned} \omega_j a_j &= 2\pi((1/j)^\alpha + \cdots ((j-1)/j)^\alpha) > 2\pi\int_1^{j-1}(z/j)^\alpha dz \\ &= \frac{2\pi j}{1+\alpha}(((j-1)/j)^{\alpha+1} - (1/j)^{\alpha+1}) \to \infty \end{aligned} \tag{7.5}$$

as $j \to \infty$. Since $(1 - \cos(x))$ is bounded we have $\lambda^+ = \lambda^- = 0$ for all $\alpha > 0$ so that the differential equation is regular.

(b) For $\omega_j = e^{-\beta j}$ we have

$$\omega_j a_j = 2\pi(e^{-\beta} + \cdots + e^{-\beta(j-1)}) = 2\pi e^{-\beta}\left(\frac{1 - e^{-\beta(j-1)}}{1 - e^{-\beta}}\right) =: \gamma_j. \tag{7.6}$$

Let $\gamma = \lim_{j\to\infty} \gamma_j$. Thus, to determine the Lyapunov spectrum we seek maxima and minima of the function $f(x) = (1 - \cos(x))/(\gamma + x)$ for $0 \leq x \leq 2\pi$. We have $f'(x) = g(x)/(\gamma + x)^2$ where $g(x) = \sin(x)(\gamma + x) + \cos(x) - 1$. Since $g(0) = g(2\pi) = 0$ and g is increasing for $x \in (0, \pi/2)$ and $x \in (3\pi/2, 2\pi)$ and decreasing for $x \in (\pi/2, 3\pi/2)$, g will have a single root for $x \in (0, 2\pi)$ that lies in $(\pi/2, 3\pi/2)$. This root x^* corresponds to a maximum and since $f(x) \geq 0$ for $x \geq 0$ and $f(0) = 0$, we have $\lambda^- = 0$. A good approximation to x^* especially for $\gamma \ll 1$ is $3\pi/4$ which gives a lower bound for λ^+ of $(1 + \sqrt{2})/(\gamma + 3\pi/4)$ which is greater than $1/\sqrt{2}$ for $\gamma \ll 1$.

Example 7.2. This example is similar to the previous one, but now we construct a scalar problem for which $\Sigma_{\rm ED} = \Sigma_{\rm L} = [-1, 1]$ in a limit.

Similar to the definition of (7.1) define $a_j = e_{j-1}$, $b_j = a_j + 2\pi/\omega_{2j-1}$, $c_j = b_j + \pi$, $d_j = c_j + 2\pi/\omega_{2j}$, and $e_j = d_j + \pi$ for $j = 1, 2, \cdots$, with $e_0 = 0$. Then define $a(t)$ as

$$a(t) = \begin{cases} \cos(\omega_{2j-1}(t - a_j)) & , \quad a_j \le t < b_j, \\ \cos(t - b_j) & , \quad b_j \le t < c_j, \\ -\cos(\omega_{2j}(t - c_j)) & , \quad c_j \le t < d_j, \\ -\cos(t - d_j) & , \quad d_j \le t < e_j. \end{cases} \tag{7.7}$$

Next observe that $\int_{a_j}^{b_j} a(s)ds = \int_{b_j}^{c_j} a(s)ds = \int_{c_j}^{d_j} a(s)ds = \int_{d_j}^{e_j} a(s)ds = 0$. So for $t \in (a_j, b_j)$ we have

$$\frac{1}{t}\int_0^t a(s)ds = \frac{\sin(x)}{\omega_{2j-1}a_j + x} \tag{7.8}$$

where $x = \omega_{2j-1}\tau$, $\tau = t - a_j$, and $0 < x < 2\pi$. For $t \in (c_j, d_j)$ we have

$$\frac{1}{t}\int_0^t a(s)ds = -\frac{\sin(x)}{\omega_{2j}c_j + x} \tag{7.9}$$

where $x = \omega_{2j}\tau$, $\tau = t - c_j$, and $0 < x < 2\pi$.

We have $a_j = 2\pi(1/\omega_1 + \cdots + 1/\omega_{2j-2}) + 2(j-1)\pi$ and $c_j = 2\pi(1/\omega_1 + \cdots + 1/\omega_{2j-1}) + (2j-1)\pi$. Then for $\omega_j = e^{-\beta j}$ with $\beta > 0$ we obtain

$$\begin{aligned} \gamma_j &:= \omega_{2j-1}a_j = \pi\left(2e^{-\beta}\frac{1 - e^{-\beta(2j-2)}}{1 - e^{-\beta}} + (2j-2)e^{-\beta(2j-1)}\right), \\ \kappa_j &:= \omega_{2j}c_j = \pi\left(2e^{-\beta}\frac{1 - e^{-\beta(2j-1)}}{1 - e^{-\beta}} + (2j-1)e^{-\beta(2j)}\right). \end{aligned} \tag{7.10}$$

Let $\gamma = \lim_{j\to\infty}\gamma_j$ and $\kappa = \lim_{j\to\infty}\kappa_j$. Then $\lambda^+ = \sup_{x\in(0,2\pi)} \frac{\sin(x)}{\gamma + x} \to +1$ as $\gamma \to 0$ and $\lambda^- = \inf_{x\in(0,2\pi)} -\frac{\sin(x)}{\kappa + x} \to -1$ as $\kappa \to 0$.

8. Conclusions. In this paper, we studied spectra of linear nonautonomous systems, in particular $\Sigma_{\rm L}$ (Lyapunov spectrum) and $\Sigma_{\rm ED}$ (Sacker–Sell spectrum). Our interest has been fundamentally motivated by ways to numerically approximate these spectra. Since one of the safest and soundest numerical approaches rests on transformation (implicitly or explicitly) of the system to upper triangular form, in this work we have focused on how to retrieve the spectra for upper triangular systems. Our main results show that –for triangular systems $\dot{x} = B(t)x$ with bounded B– $\Sigma_{\rm ED}$ can always be recovered from the diagonal of B, and $\Sigma_{\rm L}$ can also be retrieved from the diagonal of B, as long as $\Sigma_{\rm L}$ is stable. These results fully justify using triangularization techniques to approximate the spectra. The form of the diagonalizing or block diagonalizing transformation obtained in Lemma 4.1 and Theorem 4.2 is useful in quantifying the perturbation theory for Lyapunov exponents obtained in [9, 10].

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