REAL HAMILTONIAN LOGARITHM OF A SYMPLECTIC MATRIX

LUCA DIECI

Abstract. In this note we give sharp conditions under which a real symplectic matrix S has a real Hamiltonian logarithm, and explicitly construct a logarithm. In the classical work of Williamson, see [8], necessary and sufficient conditions were alredy given. Our contribution is to provide contructive arguments based on the canonical form of real symplectic matrices derived by Laub and Meyer in [4].

Notation. By $\Lambda(A) = \{\lambda_i(A),\ i = 1, \dots, n\}$ we denote the *spectrum* of the matrix $A \in \mathbb{R}^{n\times n}$; at times we write A_n to highlight the dimension of A. A matrix $S \in \mathbb{R}^{2n\times 2n}$ is *symplectic* if $S^TES = E$, where $E = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. A matrix $H \in \mathbb{R}^{2n\times 2n}$ is *Hamiltonian* if $H^TE + EH = 0$; equivalently, $H = \left[\begin{smallmatrix} H_1 & H_2 \\ H_3 & H_4 \end{smallmatrix}\right]$, $H_2 = H_2^T$, $H_3 = H_3^T$, $H_4 = -H_1^T$.

1. Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$, we will call *logarithm* of A any matrix X such that $e^X = A$. Assuming that a logarithm X exists, it is a simple verification that if $V^{-1}AV = T$ is any similarity transformation of A, then $V^{-1}XV$ is a logarithm of T. In particular, this applies to the Jordan form of A; as a consequence, by letting $\mu_i \in \Lambda(X)$, $i = 1, \dots, n$, and $\lambda_i \in \Lambda(A)$, then $e^{\mu_i} = \lambda_i$. Now, if there exists a (piecewise analytic) function "log" for which $\mu_i = \log(\lambda_i)$, $i = 1, \dots, n$, then the matrix X will be called a *primary* matrix function, otherwise it will be called *non-primary*. With this distinction, we will still nonetheless write $X = \log(A)$, even though X may be a non-primary function of A (i.e., the notation "log" is possibly not indicating a function on the spectrum of A).

It is a well known fact that any invertible matrix A has a logarithm X. However, X may be a complex matrix even if A is real. For practical and theoretical reasons, it is important to characterize under which conditions on $A \in \mathbb{R}^{n\times n}$ we also have a real logarithm X. The following result is well known (e.g., see [3]).

Theorem 1.1. *Let $A \in \mathbb{R}^{n\times n}$ be given. Then, there exists $X \in \mathbb{R}^{n\times n}$ such that $e^X = A$ if and only if A is invertible and has an even number of Jordan blocks of*

1991 *Mathematics Subject Classification.* 15A24.

Key words and phrases. Hamiltonian, logarithm, symplectic.

This work was supported in part under NSF Grant DMS-9625813.

each size relative to every negative eigenvalue. If A has any negative eigenvalue, X cannot be a primary matrix function.

Next, consider a Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$. It is a well known fact from the theory of differential equations (e.g., see [9]) that e^{tH} is a symplectic matrix for all $t \in \mathbb{R}$. Recall that symplectic matrices are closed under multiplication, inversion and transposition, whereas Hamiltonian matrices are closed under matrix addition and transposition. Also, recall that a symplectic similarity transformation of a Hamiltonian matrix gives a Hamiltonian matrix. Consider now a symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$. The question is whether or not S has a real Hamiltonian logarithm.

In the next section, first we give –or recall, if they are known– a number of results which will allow us to then prove Theorem 2.7, where sharp sufficient conditions under which a real symplectic matrix has a Hamiltonian logarithm are given. This result is predated (by 60 years!) by work of Williamson, see [8], who obtained essentially the same conditions and argued that they were also necessary. However, our proof of sufficiency is different and more constructive than that of Williamson. For one thing, at the time of Williamson's work, results like Theorem 1.1 were apparently not available, so he had to spend some time ensuring that a real logarithm existed in the first place. Moreover, he made use of the normal form of symplectic matrices he had previously derived in [7]. There, explicit formulas were only provided for low dimensional cases, and the general extension to higher dimension was not obvious. As a consequence, for the logarithm, especially in the interesting case of eigenvalue -1, explicit formulas are not easy to obtain from Williamson's work. In contrast, we utilize the explicit form of the canonical reduction of a real symplectic matrix as given by Laub and Meyer in [4], who adopt a more constructive approach. Then, we make available explicit formulas for the logarithm. These formulas are useful for testing computational procedures.

2. Building a Logarithm

In essence, we build a logarithm by putting together logarithms of submatrices obtained by restricting S to its generalized eigenspaces. To this end, we need the canonical form of S. Recall that if μ is an eigenvalue of a symplectic matrix, then so are $1/\mu$, $\bar{\mu}$ and $1/\bar{\mu}$.

Williamson in [7], and then Laub & Meyer in [4], gave a complete classification of the canonical form of a real symplectic matrix. The technique in [4] is more constructive, and this is the approach we follow. The following result summarizes the fundamental decomposition theorem.

Fact 2.1. *Let $S \in \mathbb{R}^{2n\times 2n}$ be symplectic. Then, there exists a symplectic $T \in \mathbb{R}^{2n\times 2n}$ such that*

$$V := T^{-1}ST = \begin{bmatrix} A & B \\ C & D \end{bmatrix} , \quad \text{where} \quad A = \mathrm{diag}(A_1, \dots, A_p) ,$$
$$B = \mathrm{diag}(B_1, \dots, B_p) ,\ C = \mathrm{diag}(C_1, \dots, C_p) ,\ D = \mathrm{diag}(D_1, \dots, D_p) , \tag{2.1}$$

and all matrices A_i, B_i, C_i, D_i are square of dimensions n_i, $n_1 + \cdots + n_p = n$. The blocks $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$ are canonical blocks, analogous to the usual Jordan blocks, whose particular form depends on the eigenvalues of S. Each of these canonical blocks is a symplectic matrix. The precise form of these blocks relative to the different possibilities for eigenvalues of S can be obtained from [4]; in any case, each of these blocks is associated to one of these eigenvalues' types:

- *a pair of real reciprocal eigenvalues μ, $1/\mu$;*
- *a quadruplet of complex conjugate eigenvalues not on the unit circle $(\alpha \pm i\beta)^{\pm 1}$;*
- *a pair of complex conjugate eigenvalues on the unit circle $c + is$, $c^2 + s^2 = 1$, $s \neq 0$;*
- *the eigenvalue 1;*
- *the eigenvalue -1.* □

Remark 2.2. Without loss of generality, we assume that in the matrix V of (2.1), the diagonal blocks in A, B, C, D relative to positive eigenvalues appear first, followed by those relative to complex conjugate eigenvalues, then by those relative to the eigenvalues on the negative real axis different than -1, and then by those relative to -1. Such ordering can always be achieved by similarity transformations with symplectic permutations of the type $\left[\begin{smallmatrix} P_{k,l} & 0 \\ 0 & P_{k,l} \end{smallmatrix}\right]$, where $P_{k,l}$ exchanges the k and l blocks in A, B, C, D. So, A in (2.1) is $A = \mathrm{diag}(A_1, \dots, A_\nu, A_{\nu+1}, \dots, A_p)$ and the same for B, C, D, with the canonical blocks $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$ of V, $i = \nu + 1, \dots, p$, being associated to negative eigenvalues.

Lemma 2.3. *With the above notation, suppose that there exist $L^{(i)} = \begin{bmatrix} L_1^{(i)} & L_2^{(i)} \\ L_3^{(i)} & L_4^{(i)} \end{bmatrix}$, real Hamiltonian logarithms of the symplectic matrices $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$, for $i = 1, \dots, \nu$, and $\widehat{L} = \left[\begin{smallmatrix} \widehat{L}_1 & \widehat{L}_2 \\ \widehat{L}_3 & \widehat{L}_4 \end{smallmatrix}\right]$ real Hamiltonian logarithm of the matrix $\begin{bmatrix} A_{\nu+1} & & & B_{\nu+1} & & \\ & \ddots & & & \ddots & \\ & & A_p & & & B_p \\ C_{\nu+1} & & & D_{\nu+1} & & \\ & \ddots & & & \ddots & \\ & & C_p & & & D_p \end{bmatrix}$. Let $\widetilde{L} := \left[\begin{smallmatrix} \widetilde{L}_1 & \widetilde{L}_2 \\ \widetilde{L}_3 & \widetilde{L}_4 \end{smallmatrix}\right]$, where $\widetilde{L}_l = \mathrm{diag}(L_l^{(i)})$, for $i = 1, \dots, \nu$ and $l = 1,\ 2,\ 3,\ 4$. Then, the matrix*

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} , \quad \text{where} \quad L_l = \mathrm{diag}(\widetilde{L}_l, \widehat{L}_l) \ , \ l = 1, 2, 3, 4 , \tag{2.2}$$

is a real Hamiltonian logarithm of V, and hence TLT^{-1} of S.

Proof. To show that $e^L = V$ consider the block permutation matrix P associated to the indexing $[1, p+1, 2, p+2, \dots, \nu, \nu+p, \nu+1, \dots, p, \nu+p+1, \dots, 2p]$ (here, identity blocks in P are of the same dimensions as corresponding blocks of V). Then, it is obvious that e^{PLP^T} is a logarithm of the block diagonal matrix PVP^T. To show that L is Hamiltonian, first observe that $\widetilde{L}^T E + E\widetilde{L}$ is of the form

$$\begin{bmatrix} -\operatorname{diag}((L_3^{(i)})^T) & \operatorname{diag}((L_1^{(i)})^T) \\ -\operatorname{diag}((L_4^{(i)})^T) & \operatorname{diag}((L_2^{(i)})^T) \end{bmatrix} + \begin{bmatrix} \operatorname{diag}(L_3^{(i)}) & \operatorname{diag}(L_4^{(i)}) \\ -\operatorname{diag}(L_1^{(i)}) & -\operatorname{diag}(L_2^{(i)}) \end{bmatrix},$$

and hence $\widetilde{L}^T E + E\widetilde{L} = 0$ because the $L^{(i)}$ are Hamiltonian. A similar verification then gives that $L^T E + EL = 0$, since also $\widehat{L}$ is Hamiltonian. □

Because of Lemma 2.3, we can restrict ourselves to finding real Hamiltonian logarithms of the form $\widetilde{L}$ and $\widehat{L}$. Now, to obtain $\widetilde{L}$ of the form described in Lemma 2.3, does not present conceptual difficulty. In fact, the matrices $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$, $i = 1, \dots, \nu$, are symplectic matrices with no negative eigenvalues, and hence the $L^{(i)}$ can be taken real Hamiltonian logarithms as in [9]. Thus, we can concentrate on finding logarithms of the form $\widehat{L}$. To this end, we will use the specific structure of the canonical blocks relative to negative eigenvalues.

Lemma 2.4. *Let $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$ be a canonical block of size $2k$ relative to a negative eigenvalue ρ.*

(a) *If $\rho < 0$, $\rho \neq -1$, then the canonical block is* $\begin{bmatrix} J & 0 \\ 0 & J^{-T} \end{bmatrix}$, *where*

$$J = \begin{bmatrix} \rho & 0 & \dots & \dots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & \rho \end{bmatrix}, \quad \textit{hence} \quad J^{-T} = \begin{bmatrix} \rho^{-1} & -\rho^{-2} & \dots & (-1)^{k+1}\rho^{-k} \\ 0 & \rho^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\rho^{-2} \\ 0 & \dots & 0 & \rho^{-1} \end{bmatrix}.$$

(b) *If $\rho = -1$, we can write the canonical block in one of the following three forms:*

(i) *diagonal:* $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} -I & 0 \\ 0 & -I \end{smallmatrix}\right] = -I_{2k}$;

(ii) *block diagonal:* $\begin{bmatrix} K & 0 \\ 0 & K^{-T} \end{bmatrix}$, *where*

$$K = -\begin{bmatrix} 1 & 0 & \dots & 0 \\ 2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 2 & \dots & 2 & 1 \end{bmatrix}, \quad \textit{hence} \quad K^{-T} = -\begin{bmatrix} 1 & -2 & \dots & (-1)^{k+1}2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2 \\ 0 & \dots & 0 & 1 \end{bmatrix};$$

(iii) *block triangular:* $\begin{bmatrix} K & 0 \\ W & K^{-T} \end{bmatrix}$, *where K and K^{-T} are as in* (ii) *and*

$$W = -\begin{bmatrix} -2 & \dots & -2 \\ 2 & \dots & 2 \\ \dots & \dots & \dots \\ (-1)^k 2 & \dots & (-1)^k 2 \end{bmatrix}.$$

Proof. The statement for (a) is verbatim in [4] (see formula (10) there). The statements relative to cases (b)(ii)-(iii) can also be obtained from [4], by easily adapting to the case of eigenvalue -1 the canonical forms relative to the eigenvalue 1 (in particular, see formulas (12) and (13) there). The case (b)-(i) is given for completeness, since we treat it differently from case (b)-(ii). $\square$

We now consider the cases (a) and (b) of Lemma 2.4. Let us first realize in which sense the canonical blocks of which in Lemma 2.4 correspond to standard Jordan blocks. In the sequel, we make use of the following result, which we recall for completeness (it is Theorem 6.2.25 in [3]). For notational convenience, we write Jordan blocks as lower triangular blocks, rather than in the more standard way as upper triangular blocks; clearly, these rewriting are equivalent.

Lemma 2.5. *Given a $m \times m$ Jordan block* $J = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & & \vdots \\ 0 & 1 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & \lambda \end{bmatrix}$, *and a function f $(m-1)$ times differentiable at λ. Then, the value of the primary matrix function f evaluated at J is given by*

$$f(J) = \begin{bmatrix} f(\lambda) & 0 & 0 & \dots & 0 \\ f'(\lambda) & f(\lambda) & 0 & & \vdots \\ \frac{1}{2}f''(\lambda) & f'(\lambda) & f(\lambda) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \frac{1}{(m-1)!}f^{(m-1)}(\lambda) & \dots & \frac{1}{2}f''(\lambda) & f'(\lambda) & f(\lambda) \end{bmatrix}.$$

Moreover, if $f'(\lambda) \neq 0$, then the Jordan form of $f(J)$ is the single Jordan block $J(f(\lambda))$, that is $\begin{bmatrix} f(\lambda) & 0 & 0 & \dots & 0 \\ 1 & f(\lambda) & 0 & & \vdots \\ 0 & 1 & f(\lambda) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & f(\lambda) \end{bmatrix}$.

With the notation of Lemma 2.4, quite clearly the canonical block in (a) corresponds to two Jordan blocks, relative to ρ and $1/\rho$ (we can view J^{-1} as the result of computing the function $1/\rho$, and use Lemma 2.5). Of course, case (b)-(i) corresponds to $2k$ simple Jordan blocks. For cases (b)-(ii) and (b)-(iii), a little algebra gives

Lemma 2.6. *The real Jordan form of the $m \times m$ matrix* $K = -\begin{bmatrix} 1 & 0 & \dots & 0 \\ 2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 2 & \dots & 2 & 1 \end{bmatrix}$ *is given by the single Jordan block*

$$J = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & & \vdots \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{bmatrix}. \tag{2.3}$$

As a consequence:

- *the canonical block in* (b)-(ii) *corresponds to two identical Jordan blocks J as in* (2.3) *of size k;*
- *the canonical block in* (b)-(iii) *corresponds to a unique Jordan block J as in* (2.3) *of size $2k$.*

Proof. The claim about the Jordan form of K is verified upon looking at K as the function of J, $K = p(J)$ and using Lemma 2.5. Here, p is the unique osculatory polynomial satisfying: $p(-1) = -1$, $p'(-1) = -2$, $p''(-1) = -2 \cdot 2!$, …, $p^{(m-1)}(-1) = -2 \cdot (m-1)!$. Next, consider the block in (b)-(ii): $-\left[\begin{smallmatrix} K & 0 \\ 0 & K^{-T} \end{smallmatrix}\right]$. Take $P_1 = \mathrm{diag}(-1, 1, \dots, (-1)^k)$ and realize that $P_1 K^{-T} P_1 = K^T$, then apply the permutation $P_2 = (k, k-1, \dots, 2, 1)$ to get $P_2 K^T P_2 = K$. Thus, with $P := \left[\begin{smallmatrix} I & 0 \\ 0 & P_1P_2 \end{smallmatrix}\right]$, we get $P^{-1} \left[\begin{smallmatrix} K & 0 \\ 0 & K^{-T} \end{smallmatrix}\right] P = \left[\begin{smallmatrix} K & 0 \\ 0 & K \end{smallmatrix}\right]$. Finally, consider the block in (b)-(iii). With same notation, we now have that $-P^{-1} \left[\begin{smallmatrix} K & 0 \\ W & K^{-T} \end{smallmatrix}\right] P = -K_{2k}$, and hence it corresponds to J in (2.3) of size $2k$. □

We are now ready to state

Theorem 2.7. *Let a symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$ be given.*

1. *Suppose $-1 \notin \Lambda(S)$. Then there exists a Hamiltonian $H \in \mathbb{R}^{2n\times 2n}$ such that $e^H = S$ if and only if S has an even number of canonical blocks of type* (a) *of each size relative to every negative eigenvalue.*
2. *Suppose now that $-1 \in \Lambda(S)$, and that relative to the other negative eigenvalues the conditions in* (1) *are satisfied. Then S has a real Hamiltonian logarithm if, relative to -1, there are only blocks of type* (b)-(i)*, blocks of type* (b)–(ii) *with k odd, and an even number of blocks of each size of type* (b)–(iii) *or* (b)–(ii) *with k even.*

Remark 2.8. In case S has no negative eigenvalue, the result can be found in [9] and in [6]. The difficulty in the case of negative eigenvalues is that H cannot possibly be a primary matrix function (see Theorem 1.1). As a consequence, the techniques used in the above works cannot be used. In particular, the technique of [9] is based on a contour integral representation for piecewise analytic functions, which of course cannot be directly used. The technique of [6] is more algebraic, but essential use is nonetheless made of having a primary matrix function. The statement of Theorem 2.7 should be compared with Theorem 3 in [8]. Despite differences in notation, the conditions appear to be the same. Williamson's result is truly remarkable, as it predates by almost 30 years the "simpler" Theorem 1.1. In fact, as far as we could determine, Theorem 1.1 was first proved by Culver in 1966, see [1].

Remark 2.9. Theorem 2.7 may appear surprising, since the assumptions needed are almost the same needed to guarantee the existence of a real logarithm of S, recall Theorem 1.1. In fact, with the exception of blocks of type (b)-(ii) with k even, the other assumptions amount to requiring an even number of Jordan blocks of each size for the negative eigenvalues (see Lemma 2.6 and the remarks after Lemma 2.5). However, the reduction to Jordan form of a matrix is not a symplectic similarity transformation, and thus the construction used to infer the existence of a real logarithm of S, as in [3], does not deliver the existence of a real Hamiltonian logarithm, regardless of the case (b)-(ii).

We prove Theorem 2.7 constructively, by putting together logarithms of appropriate canonical blocks for which we give explicit formulas.

Lemma 2.10. *With the notation of Lemma 2.4, let two canonical blocks of type* (a)*, both of size $2k$, relative to the same eigenvalue $\rho < 0$, be given. That is, we have the symplectic matrix $S_\rho = \begin{bmatrix} \left[\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right] & 0 \\ 0 & \left[\begin{smallmatrix} J^{-T} & 0 \\ 0 & J^{-T} \end{smallmatrix}\right] \end{bmatrix}$. Let $\Pi = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$ be the symplectic permutation matrix with P being the permutation matrix associated to the indices $(1, k+1, 2, k+2, \dots, k, 2k)$. Then, S_ρ has the real Hamiltonian logarithm*

$$\log(S_\rho) \;=\; \Pi^T \begin{bmatrix} L_\rho & 0 \\ 0 & -L_\rho^T \end{bmatrix} \Pi\,, \quad \textit{where}$$

$$(2.4) \qquad L_\rho \;=\; \begin{bmatrix} D & 0 & 0 & \dots & 0 \\ \frac{1}{\rho}I & D & 0 & \dots & 0 \\ \frac{-1}{2\rho^2}I & \frac{1}{\rho}I & D & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{(-1)^k}{k\rho^k}I & \dots & \frac{-1}{2\rho^2}I & \frac{1}{\rho}I & D \end{bmatrix}\,, \quad \textit{and} \quad D \;=\; \begin{bmatrix} \log|\rho| & \pi \\ -\pi & \log|\rho| \end{bmatrix}\,.$$

Proof. We begin by noticing that if $P^T e^{L_\rho} P = \left[\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right]$, then $P^T e^{-L_\rho^T} P = \left[\begin{smallmatrix} J^{-T} & 0 \\ 0 & J^{-T} \end{smallmatrix}\right]$. That $\log(S_\rho)$ in (2.4) is Hamiltonian is evident. Now, we have

$$P \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} P^T = \begin{bmatrix} \rho I & 0 & 0 & 0 & \dots & 0 \\ I & \rho I & 0 & 0 & \dots & 0 \\ 0 & I & \rho I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & \rho I & 0 \\ 0 & \dots & 0 & 0 & I & \rho I \end{bmatrix}\,,$$

where all identity blocks are 2×2, and it is simple to verify that $e^{L_\rho} = P \left[\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right] P^T$, using Lemma 2.5 and the fact that $V^{-1}DV = \begin{bmatrix} i\pi+\log|\rho| & 0 \\ 0 & i\pi+\log|\rho| \end{bmatrix}$ with $V = \frac{1}{\log|\rho|} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$. ∎

Remark 2.11. There are infinitely many choices for a real logarithm of the block $\left[\begin{smallmatrix} \rho & 0 \\ 0 & \rho \end{smallmatrix}\right]$, since we can replace π in (2.4) by $(2m+1)\pi$, $m = 0, \pm 1, \pm 2, \dots$. This same remark applies later on as well.

Next we consider the case (b)-(i), and then the cases (b)-(ii) and (b)-(iii).

Lemma 2.12. *Consider the symplectic matrix $-I_{2k}$. Then $-I_{2k}$ has a real Hamiltonian logarithm given by*

$$(2.5) \qquad L \;=\; \begin{bmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{bmatrix}\,, \quad \textit{where} \quad \Sigma = \begin{bmatrix} 0 & \dots & 0 & 0 & \pi \\ 0 & \dots & 0 & \pi & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \pi & 0 & \dots & 0 \\ \pi & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}\,.$$

Proof. That the matrix in (2.5) is Hamiltonian is evident. That it gives a logarithm is appreciated upon using the permutation $(k, k+1, k-1, k+2, \dots, 1, 2k)$ to obtain $e^L = -I$. ∎

Remark 2.13. Of course, the choice in (2.5) is by no means the only possible one. Besides the usual freedom in replacing π by an odd multiple of π, at times some rewritings of (2.5) are more insightful. For example, for k even, it is perhaps more intuitive to give a logarithm as

$$\mathrm{diag}(\underbrace{D,\dots,D}_{k \text{ times}}), \quad \text{with} \quad D = \left[\begin{smallmatrix} 0 & \pi \\ -\pi & 0 \end{smallmatrix}\right],$$

whereas for k odd it will turn out to be useful to consider as a logarithm

$$L = \begin{bmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{bmatrix}, \quad \text{where} \quad \Sigma = \left[\begin{smallmatrix} 0 & \dots & 0 & 0 & \pi \\ 0 & \dots & 0 & -\pi & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & -\pi & 0 & \dots & 0 \\ \pi & 0 & 0 & \dots & 0 \end{smallmatrix}\right] \in \mathbb{R}^{k\times k}. \tag{2.6}$$

Lemma 2.14. *Consider the matrix $K \in \mathbb{R}^{k\times k}$ of case* (b)-(ii) *(see Lemma 2.4): $K = \left[\begin{smallmatrix} -1 & 0 & \dots & 0 \\ -2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -2 & \dots & -2 & -1 \end{smallmatrix}\right]$. Then, it has the (complex) logarithm*

$$C := \log(K) = \begin{bmatrix} i\pi & 0 & \dots \\ 2 & i\pi & 0 & \dots \\ 0 & 2 & i\pi & 0 & \dots \\ \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ \frac{2}{7} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{2}{k-1} & \dots & \frac{2}{7} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi \end{bmatrix}, \quad \text{if } k \text{ even},$$

$$C := \log(K) = \begin{bmatrix} i\pi & 0 & \dots \\ 2 & i\pi & 0 & \dots \\ 0 & 2 & i\pi & 0 & \dots \\ \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{2}{k-2} & 0 & \dots & \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi & 0 \\ 0 & \frac{2}{k-2} & 0 & \dots & \frac{2}{5} & 0 & \frac{2}{3} & 0 & 2 & i\pi \end{bmatrix}, \quad \text{if } k \text{ odd}. \tag{2.7}$$

Proof. Let $N := C - i\pi I$, so that $e^C = -e^N$. Let $G := \left[\begin{smallmatrix} 1 & 0 & \dots \\ 2 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & 2 & 1 \end{smallmatrix}\right]$. We now show that G has a logarithm given by N, so that $e^N = G$, and the result will follow. Since all eigenvalues of G are equal to 1, a logarithm for it can be obtained from the series

$$\log(G) = 2\sum_{j=0}^{\infty} \frac{1}{2j+1}\left[(G-I)(G+I)^{-1}\right]^{2j+1},$$

which converges for all matrices G with eigenvalues with positive real parts (e.g., see [2]). Now, observe that

$$(G-I)(G+I)^{-1} = \left[\begin{smallmatrix} 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right]^{-1} =: F,$$

where F is the (forward shift) matrix $\begin{bmatrix} 0 & \dots & & & \\ 1 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$. Therefore, since $F^{k+j} = 0$, $j = 0, 1, \dots$, and F^j is just the matrix of all 0's except the j-th subdiagonal of 1's, we have that the above series for $\log(G)$ gives $\log(G) = 2\sum_{j=0}^{\lfloor k/2 \rfloor -1} \frac{1}{2j+1} F^{2j+1}$, which is N. □

Next, consider the case of two blocks of type (b)–(ii).

Lemma 2.15. *Consider two canonical blocks of type* (b)-(ii) *as in Lemma 2.4, each of size $2k$. That is, we have the symplectic matrix* $\begin{bmatrix} \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} K^{-T} & 0 \\ 0 & K^{-T} \end{bmatrix} \end{bmatrix}$ *with $K \in \mathbb{R}^{k\times k}$. Then, this matrix has the real Hamiltonian logarithm*

$$L := \begin{bmatrix} P^T & 0 \\ 0 & P^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -R^T \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, \tag{2.8}$$

where P is the permutation associated to the indices $[1, k+1, 2, k+2, \dots, k, 2k]$,

$$R = \begin{bmatrix} D & 0 & \dots & & & & & \\ 2I & D & 0 & \dots & & & & \\ 0 & 2I & D & 0 & \dots & & & \\ \frac{2}{3}I & 0 & 2I & D & 0 & \dots & & \\ 0 & \frac{2}{3}I & 0 & 2I & D & 0 & \dots & \\ \frac{2}{5}I & 0 & \frac{2}{3}I & 0 & 2I & D & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{2}{k-1}I & \dots & \frac{2}{5}I & 0 & \frac{2}{3}I & 0 & 2I & D \end{bmatrix}, \quad \text{if } k \text{ even },$$

$$R = \begin{bmatrix} D & 0 & \dots & & & & & \\ 2I & D & 0 & \dots & & & & \\ 0 & 2I & D & 0 & \dots & & & \\ \frac{2}{3}I & 0 & 2I & D & 0 & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{2}{k-2}I & 0 & \dots & \frac{2}{3}I & 0 & 2I & D & 0 \\ 0 & \frac{2}{k-2}I & 0 & \dots & \frac{2}{3}I & 0 & 2I & D \end{bmatrix}, \quad \text{if } k \text{ odd }, \tag{2.9}$$

and $D = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}$.

Proof. That L is Hamiltonian is evident, since $\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$ is symplectic. To show that it is a logarithm, it suffices to show that R is a logarithm of $P\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}P^T$. But this follows at once from Lemma 2.14, using C there to obtain $P\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}P^T$ and then replacing the diagonal blocks $\begin{bmatrix} i\pi & 0 \\ 0 & i\pi \end{bmatrix}$ by D. □

Next, consider a block of type (b)–(ii) of size $2k$ with k odd.

Lemma 2.16. *Consider a block of size $2k$ of type* (b)–(ii)*, with k odd, $U := \begin{bmatrix} K & 0 \\ 0 & K^{-T} \end{bmatrix}$, with K given in Lemma 2.4. Then, U has a real Hamiltonian logarithm given by*

$$L := \begin{bmatrix} L_G & 0 \\ 0 & -L_G^T \end{bmatrix} + \begin{bmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{bmatrix}, \tag{2.10}$$

where L_G is the logarithm of G given in the proof of Lemma 2.14, and Σ is given in (2.6).

Proof. Clearly, the matrix L in (2.10) is Hamiltonian. Observe that $U = -I_{2k}\begin{bmatrix} G & 0 \\ 0 & G^{-T}\end{bmatrix}$, $\begin{bmatrix} L_G & 0 \\ 0 & -L_G^T\end{bmatrix}$ is a Hamiltonian logarithm of $\begin{bmatrix} G & 0 \\ 0 & G^{-T}\end{bmatrix}$, and $\begin{bmatrix} 0 & \Sigma \\ -\Sigma & 0\end{bmatrix}$ is a Hamiltonian logarithm of $-I_{2k}$. Moreover, it is simple to verify that these two logarithms commute. To complete the proof, it is enough to recall that if two matrices A, B commute, $AB = BA$, then $e^{A+B} = e^A e^B$. □

Next, consider blocks of type (b)–(iii).

Lemma 2.17. *With the notation of Lemma 2.4, let two canonical blocks of type* (b)-(iii)*, each of size* $2k$*, be given. That is, we have the symplectic matrix* $\begin{bmatrix} \begin{bmatrix} K & 0 \\ 0 & K\end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \\ \begin{bmatrix} W & 0 \\ 0 & W\end{bmatrix} & \begin{bmatrix} K^{-T} & 0 \\ 0 & K^{-T}\end{bmatrix}\end{bmatrix}$ *with* $K, W \in \mathbb{R}^{k\times k}$*. Then, this matrix has the real Hamiltonian logarithm*

$$L := \begin{bmatrix} P^T & 0 \\ 0 & P^T\end{bmatrix}\begin{bmatrix} R & 0 \\ X & -R^T\end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & P\end{bmatrix}, \tag{2.11}$$

where P *and* R *are as in Lemma 2.15, and* X *is the symmetric matrix (all blocks are* 2×2*)*

$$X = \begin{bmatrix} -\frac{2}{2k-1}I & 0 & -\frac{2}{2k-3}I & 0 & -\frac{2}{2k-5}I & \dots & 0 & -\frac{2}{k+1}I & 0 \\ 0 & \frac{2}{2k-3}I & 0 & \frac{2}{2k-5}I & 0 & \dots & \frac{2}{k+1}I & 0 & \frac{2}{k-1}I \\ -\frac{2}{2k-3}I & 0 & -\frac{2}{2k-5}I & 0 & -\frac{2}{2k-7}I & \dots & 0 & -\frac{2}{k-1}I & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{2}{k-1}I & 0 & \frac{2}{k-3}I & 0 & \dots & \frac{2}{3}I & 0 & 2I \end{bmatrix}, \quad \text{if } k \text{ even},$$

$$X = \begin{bmatrix} -\frac{2}{2k-1}I & 0 & -\frac{2}{2k-3}I & 0 & \dots & 0 & -\frac{2}{k}I \\ 0 & \frac{2}{2k-3}I & 0 & \frac{2}{2k-5}I & \dots & \frac{2}{k}I & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{2}{k}I & 0 & -\frac{2}{k-2}I & 0 & \dots & 0 & -2I \end{bmatrix}, \quad \text{if } k \text{ odd}. \tag{2.12}$$

Proof. Clearly L is Hamiltonian. To show that it is a logarithm, with X as given, realize that

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & (P_1P_2)^{-1} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & (P_1P_2)^{-1}\end{bmatrix}\begin{bmatrix} R & 0 \\ X & R^{-T}\end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & (P_1P_2) & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & (P_1P_2)\end{bmatrix} = \begin{bmatrix} R_{2k} & 0 \\ 0 & R_{2k}\end{bmatrix} \tag{2.13}$$

is a logarithm of $\begin{bmatrix} K_{2k} & 0 \\ 0 & K_{2k}\end{bmatrix}$, where P_1 and P_2 are defined in the proof of Lemma 2.6. □

Proof of Theorem 2.7. With the notation of Lemma 2.3, recall that we needed to find $\widehat{L} = \begin{bmatrix} \widehat{L}_1 & \widehat{L}_2 \\ \widehat{L}_3 & \widehat{L}_4\end{bmatrix}$ real Hamiltonian logarithm of $\widehat{V} := \begin{bmatrix} A_{\nu+1} & & & B_{\nu+1} & & \\ & \ddots & & & \ddots & \\ & & A_p & & & B_p \\ C_{\nu+1} & & & D_{\nu+1} & & \\ & \ddots & & & \ddots & \\ & & C_p & & & D_p \end{bmatrix}$. Without loss of generality, we can assume that in $\widehat{V}$, after an even number of blocks of each size of type (a), there appear first those relative to -1 of type (b)-(i), then those of type (b)-(ii) of each size, and finally an even number of each size of type (b)-(iii).

(This we can assume, by the same reasoning as in Remark 2.2). Then, we group these blocks by pairing those of type (a), those of type (b)–(iii), and those of type (b)–(ii), possibly leaving one block of type (b)–(ii) with k odd alone (if we have an odd number of blocks of same size of type (b)–(ii) with k odd). Now we use Lemmata 2.10, 2.12, 2.15, 2.16 and 2.17, to obtain the logs of each of these groups, and hence we obtain $\widehat{L}$ by stacking together these logarithms in the same way as the canonical blocks appeared. Since all logarithms obtained from the above Lemmata are Hamiltonian, so is $\widehat{L}$, by a similar argument to that used in Lemma 2.3. □

As noticed in Remark 2.9, the assumptions we have for Theorem 2.7 are almost the same as those of Theorem 1.1. It is natural to ask whether in Theorem 2.7 one could weaken the assumption relative to blocks of type (b)–(ii), to that of any number of blocks of this type also for k even. Now, we already know from Williamson's arguments ([8]) that we cannot weaken the assumptions of Theorem 2.7. Still, in line with our constructive approach, here we show that the conditions are sharp by means of an example. To be precise, we show that, in case $k = 2$, a block of type (b)–(ii) does not have a Hamiltonian logarithm. To do this, we will use the following result, a proof of which can be found in [5, Theorem 2, p. 69].

Lemma 2.18. *Let $U \in \mathbb{R}^{2k\times 2k}$ be a symplectic matrix with all eigenvalues equal to -1. Then U has a real Hamiltonian logarithm if and only if it has a real symplectic square root.*

Consider now

$$U = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

which is a block of type (b)–(ii) of size 4; to be precise, U is obtained after a symplectic transformation: $U = \left[\begin{smallmatrix} V & 0 \\ 0 & V^{-T} \end{smallmatrix}\right] \left[\begin{smallmatrix} K & 0 \\ 0 & K^{-T} \end{smallmatrix}\right] \left[\begin{smallmatrix} V^{-1} & 0 \\ 0 & V^{T} \end{smallmatrix}\right]$, with $V = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. We want to show that U has no real symplectic square root. To this end, we computed all possible real square roots of U. These are given by the four parameters family

$$U^{1/2} = \begin{bmatrix} -a & 0 & 0 & -\frac{a^2+1}{c} \\ b & -a & \frac{a^2+1}{c} & -\frac{-2abc+a^2d+c+d}{c^2} \\ d & -c & a & b \\ c & 0 & 0 & a \end{bmatrix}, \tag{2.14}$$

where a, b, c, d real and $c \neq 0$. It is now immediate to verify that for no choice of these parameters we can obtain $(U^{1/2})^T E U^{1/2} = E$, and so Theorem 2.7 is sharp.

Acknowledgment. I am grateful to Ken Meyer, who, upon reading an intial version of this note, pointed me to [8], and to Alessandra Papini for her careful reading of the manuscript.

References

[1] W.J. Culver. On the existence and uniqueness of the real logarithm of matrix. *Proc. Amer. Math. Soc.*, 17:1146–1151, 1966.

[2] L. Dieci, B. Morini, and A. Papini. Computational techniques for real logarithms of matrices. *SIAM J. Matrix Anal. & Applic.s*, 17:570–593, 1996.

[3] R.A. Horn and C.R. Johnson. *Topics in Matrix analysis*. Cambridge University Press, New York, 1991.

[4] A. J. Laub and K. R. Meyer. Canonical forms for symplectic and Hamiltonian matrices. *J. Celestial Mechanics*, 9:213–238, 1974.

[5] K. R. Meyer and G. R. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, volume 90. Applied Mathematical Sciences, Springer-Verlag, 1991.

[6] Y. Sibuya. Note on real matrices and linear dynamical systems with periodic coefficients. *J. Mathematical Analysis and Applications*, 1:363–372, 1960.

[7] J. Williamson. On the normal forms of linear canonical transformations in dynamics. *American Journal of Mathematics*, 59:599–617, 1937.

[8] J. Williamson. The exponential representation of canonical matrices. *American Journal of Mathematics*, 61:897–911, 1939.

[9] V.A. Yakubovich and V. M. Starzhinskii. *Linear Differential Equations with Periodic Coefficients*, volume 1&2. John-Wiley, New York, 1975.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 U.S.A.

E-mail address: dieci@math.gatech.edu