POINT-TO-PERIODIC AND PERIODIC-TO-PERIODIC CONNECTIONS.

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ABSTRACT. In this work we consider computing and continuing connecting orbits in parameter dependent dynamical systems. We give details of algorithms for computing connections between equilibria and periodic orbits, and between periodic orbits. The theoretical foundation for these techniques is given by the seminal work of Beyn [5] where a numerical technique is also proposed. Our algorithms consist of splitting the computation of the connection from that of the periodic orbit(s). To set up appropriate boundary conditions, we follow the algorithmic approach used in [9] for the case of connecting orbits between equilibria, and construct and exploit the smooth block Schur decomposition of the monodromy matrices associated to the periodic orbits. Numerical examples illustrate the performance of the algorithms.

1. INTRODUCTION

Consider a dynamical system of the form

$$\dot{x} = f(x, \lambda), \qquad x(t) \in \mathbb{R}^m, \quad \lambda \in \Lambda \subset \mathbb{R}^p, \tag{1.1}$$

where $f : \mathbb{R}^m \times \mathbb{R}^p \longrightarrow \mathbb{R}^m$ is assumed to be sufficiently smooth, and Λ is compact (often, Λ is a closed subinterval of the real line). Let $M_-(\lambda)$ be either a hyperbolic[1] equilibrium $y_-(\lambda)$ of (1.1), or a hyperbolic[2] periodic orbit $\gamma_-(\lambda)$ of (1.1), and let $M_+(\lambda)$ be a hyperbolic periodic orbit $\gamma_+(\lambda)$ of (1.1), for $\lambda \in \Lambda$:

$$\begin{aligned} M_-(\lambda) &= y_-(\lambda), \qquad \text{or} \qquad M_-(\lambda) = \gamma_-(\lambda), \\ M_+(\lambda) &= \gamma_+(\lambda), \qquad \lambda \in \Lambda\,. \end{aligned} \tag{1.2}$$

The periodic orbit(s) correspond to periodic solution(s) of (1.1), and we will use the notation $y_+(t, \lambda)$ to denote the periodic solution (say, of minimal period τ_+) corresponding to $\gamma_+(\lambda)$, and similarly $y_-(t, \lambda)$ will be the τ_-–periodic solution relative to $\gamma_-(\lambda)$, if $M_-(\lambda) = \gamma_-(\lambda)$. In practice, $M_\pm(\lambda)$ will need to be found.

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[1]No eigenvalue of $f_x(y_-(\lambda))$ is on the imaginary axis

[2]Only one Floquet multiplier is on the unit circle

A solution $x(t,\lambda)$, $t \in \mathbb{R}$ of (1.1) is called a *connecting orbit* from $M_-(\lambda)$ to $M_+(\lambda)$ if

$$\text{dist}\,(x(t,\lambda),\, M_\pm(\lambda)) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty. \tag{1.3}$$

A connecting orbit (if it exists) is determined only up to a time (phase) shift, since if $x(t)$ is a connecting orbit, then also $x(t+\sigma)$ is, for any real σ. So, momentarily, we put forward the following system as the one which has to be satisfied by a connecting orbit:

$$\begin{aligned} &\dot{x} = f(x,\lambda)\,, \quad -\infty < t < \infty\,, \\ &\lim_{t\to-\infty} \text{dist}(x(t,\lambda), M_-(\lambda)) = 0\,, \quad \lim_{t\to+\infty} \text{dist}(x(t,\lambda), M_+(\lambda)) = 0\,, \\ &\psi(x, y_-, y_+, \lambda) = 0\,, \end{aligned} \tag{1.4}$$

where the scalar equation $\psi(x, y_-, y_+, \lambda) = 0$ is the so-called *phase condition* and it is used to fix the time shift; a particular choice will be given below. Depending on whether $M_-(\lambda)$ is an equilibrium or a periodic orbit, we will refer to these as *point-to-periodic* or *periodic-to-periodic* connections, respectively. In the latter case, we will talk about a heteroclinic connection if $\gamma_- \neq \gamma_+$, and a homoclinic connection if $\gamma_- = \gamma_+$.

Computation of connecting orbits between two hyperbolic equilibria (*point-to-point connections*), has received extensive numerical treatment in the last fifteen years (see [4, 6, 9, 12, 13, 15] for a representative list of works on this subject), at the point that reliable software, [14], is available for computation and continuation of point-to-point connections. Point-to-periodic and periodic-to-periodic connections have so far received less numerical attention, though the work of Beyn [5] laid down the theoretical justification for numerical methods almost 10 years ago (see also [20] for more recent theoretical work). Still, there are clear indications of the presently increasing relevance of these connections in understanding the dynamics of (1.1). For example, in [8] homoclinic orbits to a periodic orbit (periodic traveling waves) are observed in a biological setting and in [7] in the water-wave model, in [17] the authors anticipate the existence of homoclinic and heteroclinic periodic-to-periodic connections in a model of semiconductors, and in [18, 21] the authors find homoclinic and heteroclinic connections in a model of celestial mechanics and further elucidate how this may lead to space exploration with prescribed itineraries. No numerical methods are developed in [8] nor in [17]. In [18] and [21], the authors use a technique taylor-made for the particular three-body problem they explore: Essentially, they use a Poincaré map approach, and a Poincaré map approach was also proposed in the early paper [3]. In [5], Beyn proposes a boundary value technique for the case of point-to-periodic connections, and illustrates its performance on a point-to-periodic connection for the Lorenz system. In this numerical study, Beyn uses so–called projection boundary conditions, but in an approximate form (see Remark 3.5 below), and this will generally lead to loss of accuracy. A modification of Beyn's approach (but still with an approximate form of projection boundary conditions) is discussed in [20], where a point-to-periodic connection in the Lorenz system is also computed. In [7], Champneys and Lord compute homoclinic periodic-to-periodic connections also using boundary value techniques. Because of the specialized character of their problem (it is Hamiltonian, and they are only concerned with homoclinic connections), Champneys and Lord make a number of modifications to the problem, such as introducing an artificial parameter, unfolding, so that the Hamiltonian problem is embedded into a larger generic system. Also, they do not extend their method to the general case of periodic-to-periodic connections for (1.4). Nevertheless, our general algorithms for (1.4) have some similarities to the work [7], in that we will end up solving a boundary value problem on a finite interval using projection boundary conditions obtained relying on the monodromy matrix(-ces) associated to the periodic orbit(s). On the other hand, in [7], the authors left open the study of error estimates due to truncation, and also used an inexact form of the projection boundary conditions.

The central issue in all methods to compute connecting orbits is how to replace the limiting process of (1.4) with a condition on a finite interval. In general terms, one truncates the real line to a finite, but sufficiently large, interval $[T_-, T_+]$, $T_- < 0 < T_+$, and imposes boundary conditions at $T_\pm$, replacing the limits in (1.4). An appropriate choice for these boundary conditions is arrived at by realizing that the connecting orbit must leave $M_-(\lambda)$ along its unstable manifold and enter $M_+(\lambda)$ along its stable manifold: By the stable manifold theorem (e.g., see [16]), these are tangent to the unstable subspace $E^u_-(\lambda)$ of $M_-(\lambda)$ and to the stable subspace $E^s_+(\lambda)$ of $M_+(\lambda)$, respectively.

Remark 1.1. Recall that if $M_-(\lambda)$ is an equilibrium, then $E^u_-(\lambda)$ is spanned by the generalized eigenvectors associated to the eigenvalues of $Df_x(y_-(\lambda))$ in the positive half plane. If $M_-(\lambda)$ is the periodic orbit $\gamma_-(\lambda)$, then $E^u_-(\lambda)$ is the span of the generalized eigenvectors associated to the eigenvalues of the monodromy matrix (the multipliers) outside the unit circle. Similarly, $E^s_+(\lambda)$ is spanned by the generalized eigenvectors of the monodromy matrix relative to $\gamma_+(\lambda)$ that are inside the unit circle. **Caution**: As it is well known, the multipliers are uniquely defined for a given periodic orbit, that is they do not depend on the origin of time we choose for the periodic orbit itself. However, the generalized eigenvectors associated to monodromy matrices obtained by linearizing about two periodic solutions with different time shifts (i.e., origins of time) are different in general.

In line with this discussion, we will impose so-called *projection boundary conditions* (see [4]) and will consider the following boundary value problem

$$
\begin{aligned}
&\dot{x} = f(x,\lambda), \quad T_- \le t \le T_+ \\
&L_-(\lambda)(x(T_-) - y_-(s(T_-))) = 0\ , \quad L_+(\lambda)(x(T_+) - y_+(s(T_+))) = 0\,, \\
&\psi(x, y_-, y_+, \lambda) = 0\,,
\end{aligned}
\tag{1.5}
$$

where $L_- : \mathbb{R}^p \to \mathbb{R}^{m^c_- + m^s_-, m}$ and $L_+ : \mathbb{R}^p \to \mathbb{R}^{m^u_+ + 1, m}$ are smooth functions of λ, and span $E^{cs}_-(\lambda)$ and $E^{cu}_+(\lambda)$, respectively. Here, $E^{cu}_+(\lambda)$ is the center-unstable subspace of $\gamma_+(\lambda)$, and similarly $E^{cs}_-(\lambda)$. Also, in (1.5), ψ corresponds to the truncated version of the phase condition in (1.4). In (1.5), $y_\pm(s(T_\pm))$ is **a** point on $M_\pm(\lambda)$. Naturally, in case $M_-(\lambda)$ is an equilibrium then $y_-(s(T_-))$ is that very equilibrium. Otherwise, $y_\pm(s(T_\pm))$ is **some** point on the periodic orbits $\gamma_\pm(\lambda)$; the notation $s(T_\pm)$ clarifies that the point(s) on $\gamma_\pm$ in general depend on the value(s) of $T_\pm$. It must be understood that –say– $E^{cu}_+(\lambda)$ is associated to the generalized eigenvectors of the monodromy matrix obtained from linearization about the periodic solution with origin of time at $s(T_+)$.

In Section 2 we recall some theoretical results from [5, 20] on the well posedness of (1.4), and give error estimates due to the truncation in (1.5). In Section 3, we give details of our algorithms and how we implemented them, and in Section 4 we illustrate their performance on some numerical computations of point-to-periodic and periodic-to-periodic connections.

2. The problem to solve

We first introduce some notation and review some results from [5] which are needed for later development.

For each given λ, we will write m^u_-, m^c_-, and m^s_-, for the dimensions of the unstable, center, and stable manifolds of $M_-(\lambda)$, and analogously we will write m^u_+, m^c_+, and m^s_+ relatively to $M_+(\lambda)$. Here, $m^c_\pm$ are the dimensions of $M_\pm(\lambda)$, so, according to (1.2), and because of hyperbolicity of $M_\pm(\lambda)$, we will have that $m^c_- = 0$ if $M_-(\lambda) = y_-(\lambda)$, and $m^c_- = 1$ if $M_-(\lambda) = \gamma_-(\lambda)$. We always have $m^c_+ = 1$, and of course we always have

$$m = m^u_- + m^c_- + m^s_- = m^u_+ + m^c_+ + m^s_+ \,. \tag{2.1}$$

We let $W^{cu}_-(\lambda)$, respectively $W^{cs}_+(\lambda)$, be the (center-) unstable manifold of $M_-(\lambda)$, respectively the (center-) stable manifold of $M_+(\lambda)$. Further, if we rewrite (1.1) as the enlarged system

$$\dot z = g(z), \qquad g(x,\lambda) = \begin{bmatrix} f(x,\lambda) \\ 0 \end{bmatrix} \qquad \text{where} \qquad z = (x,\ \lambda)\ , \tag{2.2}$$

then we can introduce the manifolds foliated by λ: $M_\pm = \bigcup_{\lambda\in\Lambda}(M_\pm(\lambda) \times \{\lambda\})$, and analogously we will write W^{cu}_- and W^{cs}_+ for their (center)-unstable and (center)-stable manifolds, respectively. Now W^{cu}_- has dimension $m^c_- + m^u_- + p$ and W^{cs}_+ has dimension $1 + m^s_+ + p$. Suppose that there is a connecting orbit γ, a solution z of (2.2) connecting M_- and M_+, then, we must have $\gamma \subset W^{cu}_- \cap W^{cs}_+$. We expect the connecting orbit γ to be isolated if for the tangent spaces at $z(t)$ we have

$$T_{z(t)}W^{cu}_- \cap T_{z(t)}W^{cs}_+ = T_{z(t)}\gamma = \text{ span } \{\dot z(t)\}, \qquad \forall\, t \in \mathbb{R}, \tag{2.3}$$

and to persist if the intersection is *transversal*, i.e.

$$T_{z(t)}W_-^{cu} + T_{z(t)}W_+^{cs} = \mathbb{R}^{m+p}\ , \qquad \forall\ t \in \mathbb{R}. \tag{2.4}$$

Using (2.1), and assuming (2.3) and (2.4), one obtains the following fundamental relation between number of parameters and dimensions of $W_\pm^{cs,cu}$:

$$p = m_+^u - m_-^u - m_-^c + 1. \tag{2.5}$$

We refer to the original work of Beyn in [5] for a precise persistence result for the connecting orbit γ. Here, we simply recall the essence of the key result of Beyn about the well posedness of (1.4): "Suppose that $\bar{z} = (\bar{x}, \bar{\lambda})$ is a connecting orbit between $M_-(\bar{\lambda})$ and $M_+(\bar{\lambda})$, and that the phase condition ψ in (1.4) satisfies a (mild) nondegeneracy requirement.Then the connecting orbit problem (1.4) is well posed if and only if the manifolds W_-^{cu} and W_+^{cs} intersect transversally along $\bar{z}$ in the strong sense of (2.3)-(2.4)".

Remark 2.1. As pointed out in [5], if p does not satisfy (2.5) (and hence (2.3) or (2.4) is violated), then we should either add parameters to the system or add conditions for the parametrization of a manifold of connecting orbits. More precisely, if $p < m_+^u - m_-^u - m_-^c + 1$, we should add $(m_+^u - m_-^u - m_-^c + 1) - p$ parameters in order to have a well-posed problem, while if $p > m_+^u - m_-^u - m_-^c + 1$, then we can add $p - (m_+^u - m_-^u - m_-^c + 1)$ constraints to select a unique connecting orbit.

2.1. **Truncated Problem.** To justify our algorithm, we need to establish the solvability of (1.5) and estimate the error resulting from truncating the original interval of integration $(-\infty, \infty)$ to a finite interval $J := [T_-, T_+]$. In other words, we need to study the error resulting from applying the *projection boundary conditions* to the original problem. The results which follow, in particular Theorems 2.3 and 2.5, generalize the corresponding statements in [4] from the point-to-point case, to the point-to-periodic and periodic-to-periodic cases. For completeness, we outline the key differences and generalizations of Beyn's proofs, following [20]. The following lemma is fundamental.

Lemma 2.2. *Let $F : B_\delta(w_0) \to Z$ be a C^1 mapping from some ball of radius δ in a Banach space W into some Banach space Z. Assume that $F'(w_0)$ is an homeomorphism and that for some constants c_1, c_2 we have*

$$\| F'(w) - F'(w_0) \| \le c_2 < c_1 \le \| F'(w_0)^{-1} \|^{-1}, \quad \forall w \in B_\delta(w_0)\,, \tag{2.6}$$

$$\| F(w_0) \| \le (c_1 - c_2)\, \delta. \tag{2.7}$$

Then F has a unique zero w_c in $B_\delta(w_0)$ and

$$\| w_0 - w_c \| \le (c_1 - c_2)^{-1} \| F(w_0) \|\,, \tag{2.8}$$

$$\| w_1 - w_2 \| \le (c_1 - c_2)^{-1} \| F(w_1) - F(w_2) \|\,, \qquad \forall\ w_1, w_2 \in B_\delta(w_0). \tag{2.9}$$

In the theorems below, we use the spaces

$$W := C^1(J, \mathbb{R}^m) \times \mathbb{R}^p \quad \text{and} \quad Z := C(J, \mathbb{R}^m) \times \mathbb{R}^{m_-^c + m_-^s} \times \mathbb{R}^{m_+^u + 1}.$$

For appropriate $\alpha, \beta > 0$, the norms are defined as

$$\begin{aligned}
\|(x,\lambda)\|_W &= \sup_{t\in J_-} \|x(t)\|e^{\alpha t} + \sup_{t\in J_+} \|x(t)\|e^{-\beta t} + \|\lambda\|, \\
\|(y, r_-, r_+)\|_Z &= \|(y, r_-)\|_{Z_1} + \|(y, r_+)\|_{Z_2}, \quad \text{where} \\
\|(y, r_-)\|_{Z_1} &= \sup_{t\in J_-} \|y(t)\|e^{\alpha t} + \|r_-\|, \quad \text{and} \\
\|(y, r_+)\|_{Z_2} &= \sup_{t\in J_+} \|y(t)\|e^{-\beta t} + \|r_+\|, \quad \|\cdot\| = \|\cdot\|_\infty,
\end{aligned}$$

and where $J_- = [T_-, 0]$ and $J_+ = [0, T_+]$. With these norms, W and Z become Banach spaces. Anticipating the asymptotic convergence of $x(t)$ to $y(t)$ with rate $\epsilon > 0$, we impose the condition that, for some constant C, $\|x(t) - y_\pm(t)\| \le Ce^{-\epsilon|t|}$ as $t \to \pm\infty$.

Theorem 2.3. *Let* (2.3), (2.4) *hold, and let* $(\bar{x}, \bar{\lambda})$ *be an orbit connecting either a hyperbolic equilibrium point* $y_-(\bar{\lambda})$ *or a hyperbolic periodic orbit* $\gamma_-(\bar{\lambda})$*, to a hyperbolic periodic orbit* $\gamma_+(\bar{\lambda})$*. Consider* (1.5) *and assume that* $f \in C^2(\mathbb{R}^{m+p}, \mathbb{R}^m)$*, and that* $L_\pm$ *are* C^1 *(in* λ*).*

Then, there exists $\delta > 0$ *sufficiently small and* $C > 0$*, such that, for sufficiently large interval of integration* $J = [\,T_-,\, T_+\,]$*, the boundary-value problem* (1.5) *has a unique solution* $(x_J,\, \lambda_J)$ *in a ball of radius* δ *in* W*. Moreover, the following estimate holds*

(2.10)
$$\|(\bar{x}|_J, \bar{\lambda}) - (x_J, \lambda)\|_W \le C\left(\|\, L_-(\bar{\lambda})(\bar{x}(T_-) - y_-(s(T_-)))\,\| + \|\, L_+(\bar{\lambda})(x(T_+) - y_+(s(T_+)))\,\|\right).$$

Proof. The idea is to apply Lemma 2.2 to $w_0 = (\bar{x}|_J, \bar{\lambda})$ and

$$F(x,\lambda) = \left(\dot{x} - f(x,\lambda),\ L_-(\lambda)(x(T_-) - y_-(s(T_-))),\ L_+(\lambda)(x(T_+) - y_+(s(T_+)))\right).$$

The steps on the proof in [20, Theorem 4] apply here to obtain a bound $\|F'(w_0)^{-1}\| \le c_1^{-1}$ and to find a $\delta > 0$ such that (2.6) holds with $c_2 = \frac{1}{2}c_1$. Finally, one gets:

(2.11)
$$\|F(\bar{x}|_J, \bar{\lambda})\|_Z = \|L_-(\bar{\lambda})(\bar{x}(T_-) - y_-(s(T_-)))\| + \|L_+(\bar{\lambda})(x(T_+) - y_+(s(T_+)))\| \to 0$$

as $T_\pm \to \pm\infty$, which in turn implies (2.7). Then, by Lemma 2.2, F has a unique zero $w_c = (x_J, \lambda_J)$ in a ball of radius δ in W, and combining (2.8) and (2.11) one obtains the sought result with $C = (c_1 - c_2)^{-1}$. ∎

Remark 2.4. The statement of Theorem 2.3 is true up to a certain time shift, for which the phase condition vanishes, and the error depends solely on the boundary conditions. More precisely, any solution of (1.5) approximates some suitably shifted connecting orbit with an error which is dominated by the error in satisfying the projection boundary conditions. The next result pins down this error.

Theorem 2.5. *Under the assumptions of Theorem 2.3, for $J = [T_-, T_+]$ sufficiently large, we have*[3]*:*

$$\|(\bar{x}|_J, \bar{\lambda}) - (x_J, \lambda)\|_W \le C\, e^{-2\min(\mu_-|T_-|,\, \mu_+T_+)}\,. \tag{2.12}$$

In (2.12), $0 < \mu_- <$ Re μ, for all unstable eigenvalues μ of the Jacobian $f_x(y_-(\bar{\lambda}))$ (if $M_-(\bar{\lambda}) = y_-(\bar{\lambda})$), or all unstable Floquet exponents of the monodromy relative to $\gamma_-(\bar{\lambda})$ (if $M_-(\bar{\lambda}) = \gamma_-(\bar{\lambda})$). Also, $0 < \mu_+ <$ -Re μ, for all stable Floquet exponents μ associated to the periodic orbit $\gamma_+(\bar{\lambda})$.

Proof. The key tools to use are corollaries from the Stable Manifold Theorems for equilibria and periodic orbits (see [16]), which state that solutions starting in the corresponding unstable or stable manifold, sufficiently near the equilibrium or the periodic orbit, approach them exponentially fast, as $t \to -\infty$ or $t \to \infty$ respectively. Moreover, in the case of a periodic orbit, the motion along the connecting orbit is synchronized with that on the periodic orbit (convergence in asymptotic phase). For the boundary condition at an equilibrium the exponential decay of the error is proved in [4]: there exists $T_1 < 0$, such that

$$L_-(\bar{\lambda})(\bar{x}(t) - y_-(s(t))) = O(e^{-2\mu_- t})\,, \quad \text{for} \quad t \le T_1\,. \tag{2.13}$$

Next we give the proof for the exponential decay of the error for the boundary condition at the periodic orbit γ_+. If $0 < \mu_+ <$ -Re μ, for all characteristic exponents μ with negative real part of the periodic orbit $\gamma_+(\bar{\lambda})$, then there exists $T_2 > 0$ such that for all $t \ge T_2$,

$$\bar{x}(t) - \gamma_+(\bar{\lambda}) = O(e^{-\mu_+ t})\,. \tag{2.14}$$

Then, for any $t \ge T_2$ there is always a time shift $s(t):\ 0 \le s(t) \le \tau_+$, such that $\bar{x}(t) - y_+(s(t)) = O(e^{-\mu_+ t})$. With this, by a Taylor expansion, we get

$$\begin{aligned} L_+(\bar{\lambda})(\bar{x}(t) - y_+(s(t))) \;&=\; L_+(\bar{\lambda})(y_+(s(t)) - y_+(s(t))) + L_+(\bar{\lambda})(y_+(s(t)) - y_+(s(t))) \\ &\quad (\bar{x}(t) - y_+(s(t))) + O(\,\|\bar{x}(t) - \bar{y}(s(t))\|^2\,\|). \end{aligned}$$

Therefore,

$$L_+(\bar{\lambda})(\bar{x}(t) - y_+(s(t))) = O(\,\|\bar{x}(t) - y_+(s(t))\|^2\,) \tag{2.15}$$

and by (2.14),

$$L_+(\bar{\lambda})(\bar{x}(t) - y_+(s(t))) = O(e^{-2\mu_+ t}). \tag{2.16}$$

As for the periodic-to-periodic case, if $0 < \mu_- <$ -Re μ for all unstable Floquet exponents of the periodic orbit $\gamma_-(\bar{\lambda})$, then there exists a $T_1 < 0$ such that for all $t \le T_1$,

$$\bar{x}(t) - \gamma_-(\bar{\lambda}) = O(e^{-\mu_+ t}), \text{ and} \tag{2.17}$$

[3] again, (2.12) is true up to a time shift; see Remark 2.4

proceeding in a similar way as we did for the periodic orbit γ_+, we can obtain (for a time shift $s(t)$))

$$L_-(\bar{\lambda})(\bar{x}(t) - \bar{y}_-(s(t))) = O(\, e^{-2\mu_- t}\,). \tag{2.18}$$

Combining (2.13), or (2.18), and (2.16) with inequality (2.10) from Theorem 2.3, we get the sought result. ■

3. Algorithms

We begin writing in details the problems we will solve. All intervals will be normalized to $[0, 1]$.

3.1. **Point-to-Periodic.** We need to find equilibrium $y_-(\lambda)$, periodic orbit $\gamma_+(\lambda)$ and the connection $x(t, \lambda)$ solutions of

$$\begin{cases} \dot{x} \;=\; (T_+ - T_-)f(x,\lambda)\,, \quad 0 \le t \le 1\,, \\ f(y_-(\lambda)) \;=\; 0\,, \\ \dot{y}_+ \;=\; \tau_+ f(y_+,\lambda)\,, \quad 0 \le t \le 1\,, \\ y_+(0) \;=\; y_+(1)\,, \\ L_-(\lambda)\big(x(0,\lambda) - y_-(\lambda)\big) \;=\; 0\,, \\ L_+(\lambda)\big(x(1,\lambda) - y_+(s(1),\lambda)\big) \;=\; 0\,, \\ \psi(x, y_+, \lambda) \;=\; 0\,. \end{cases} \tag{3.1}$$

Recall that in (3.1), $L_-(\lambda) \in \mathbb{R}^{m_-^s,m}$ and $L_+(\lambda) \in \mathbb{R}^{m_+^u+1,m}$, and that we have p free parameters, with p satisfying (2.5) (if not, recall Remark 2.1).

Computationally, it is more convenient to **split** (3.1) in the following form, which is the one we eventually implemented:

$$\begin{cases} \dot{x} \;=\; (T_+ - T_-)f(x,\lambda)\,, \quad 0 \le t \le 1\,, \\ L_-(\lambda)\big(x(0,\lambda) - y_-(\lambda)\big) \;=\; 0\,, \\ L_+(\lambda)\big(x(1,\lambda) - y_+(0,\lambda)\big) \;=\; 0\,, \end{cases} \tag{3.2}$$

and

$$\begin{cases} f(y_-(\widehat{\lambda})) \;=\; 0\,, \\ \dot{y}_+ \;=\; \tau_+ f(y_+,\widehat{\lambda})\,, \quad 0 \le t \le 1\,, \\ y_+(0) \;=\; y_+(1)\,, \\ \sigma(y_+,\widehat{\lambda}) \;=\; 0\,, \end{cases} \tag{3.3}$$

where $\sigma = 0$ is a phase condition for the periodic orbit, serving the role of $\psi = 0$ in (1.5).

Remarks 3.1. Several observations are in order.

(i) In (3.1), the term $y_+(s(1),\lambda)$ in the boundary conditions means that there exists a choice of phase for the periodic orbit making the problem (3.1) well posed. In (3.2), instead, the term $y_+(0,\lambda)$ is the very given point of the periodic solution of (3.3). The value 0 in $y_+(0,\lambda)$ is not the same as setting $t=0$ in (3.2), but it refers to the value $t=0$ in (3.3).

(ii) Notice that the equilibrium and the periodic orbit enter in (3.2) (only) through the boundary conditions.

(iii) Computationally, we will solve (3.2), and solve (3.3) to properly set up the boundary conditions. In other words, we will always need to solve (3.3) with a **fixed** value of λ, which will be determined by the solution process of (3.2). This is the reason we wrote $\widehat{\lambda}$ in (3.3). There are no free parameters in (3.3).

3.2. **Periodic-to-Periodic.** We separate the cases of heteroclinic and homoclinic periodic-to-periodic connections.

3.2.1. *Heteroclinic.* We need to find the two periodic orbits $\gamma_\pm(\lambda)$ and the connection $x(t,\lambda)$ by solving

$$(3.4)\qquad \begin{cases} \dot{x} \;=\; (T_+ - T_-)f(x,\lambda)\,, \quad 0\le t\le 1\,,\\ \dot{y}_\pm \;=\; \tau_\pm f(y_\pm,\lambda)\,, \quad 0\le t\le 1\,,\\ y_\pm(0) \;=\; y_\pm(1)\,,\\ L_-(\lambda)\big(x(0,\lambda)-y_-(s(0),\lambda)\big) \;=\; 0\,,\\ L_+(\lambda)\big(x(1,\lambda)-y_+(s(1),\lambda)\big) \;=\; 0\,,\\ \psi(x,y_+,y_-,\lambda) \;=\; 0\,. \end{cases}$$

In (3.4), $L_-(\lambda)\in\mathbb{R}^{m_-^s+1,m}$ and $L_+(\lambda)\in\mathbb{R}^{m_+^u+1,m}$.

We would like to split (3.4) similarly to what we did for (3.1). For clarity, we proceed in two steps. First, we separate the solution process for one periodic orbit, say $\gamma_-(\lambda)$:

$$(3.5)\qquad \begin{cases} \dot{x} \;=\; (T_+ - T_-)f(x,\lambda)\,, \quad 0\le t\le 1\,,\\ \dot{y}_+ \;=\; \tau_+ f(y_+,\lambda)\,, \quad 0\le t\le 1\,,\\ y_+(0) \;=\; y_+(1)\,,\\ L_-(\lambda)\big(x(0,\lambda)-y_-(0,\lambda)\big) \;=\; 0\,,\\ L_+(\lambda)\big(x(1,\lambda)-y_+(s(1),\lambda)\big) \;=\; 0\,, \end{cases}$$

and

$$(3.6)\qquad \begin{cases} \dot{y}_- \;=\; \tau_- f(y_-,\widehat{\lambda})\,, \quad 0\le t\le 1\,,\\ y_-(0) \;=\; y_-(1)\,,\\ \sigma_-(y_-,\widehat{\lambda}) \;=\; 0\,, \end{cases}$$

where $\sigma_- = 0$ is a phase condition for the periodic orbit γ_-. Again, we remark that, in (3.4), the writing $y_-(s(0), \lambda)$ and $y_+(s(1), \lambda)$ in the boundary conditions means that there exist phase shifts for the periodic trajectories $y_\pm(\cdot, \lambda)$ for which the problem is well posed. In (3.5), instead, $y_-(0, \lambda)$ is now the very given value of the periodic solution of (3.6), while $y_+(s(1), \lambda)$ in (3.5) is still some point on the periodic orbit (that is, there exists a phase shift for the periodic orbit $\gamma_+(\lambda)$ for which (3.5) is well posed).

Now, for computational convenience, we want to separate the task of finding also the other periodic orbit, y_+, from finding the connection. However, if we do so, and fix say $y_+(0, \lambda)$, we will no longer be guaranteed that the boundary condition at the given T_+ is satisfied. Nonetheless, since the connection, solution of (3.5), enters the periodic orbit at $+\infty$ in asymptotic phase, we are guaranteed that there is some possibly different time T_+ at which the boundary condition at T_+ is satisfied. [Geometrically, the connection may have to go around the periodic orbit a bit in order to satisfy the boundary condition at T_+]. Therefore, we propose the following split systems for the heteroclinic periodic-to-periodic connection:

$$
\left\{
\begin{array}{rcl}
\dot{x} & = & (T_+ - T_-) f(x, \lambda), \quad 0 \le t \le 1, \\
L_-(\lambda)\big(x(0, \lambda) - y_-(0, \lambda)\big) & = & 0, \\
L_+(\lambda)\big(x(1, \lambda) - y_+(0, \lambda)\big) & = & 0,
\end{array}
\right.
\tag{3.7}
$$

where T_+ (or T_-) is now a free parameter, and

$$
\left\{
\begin{array}{rcl}
\dot{y}_\pm & = & \tau_\pm f(y_\pm, \widehat{\lambda}), \quad 0 \le t \le 1, \\
y_\pm(0) & = & y_\pm(1), \\
\sigma_-(y_\pm, \widehat{\lambda}) & = & 0,
\end{array}
\right.
\tag{3.8}
$$

with $\sigma_\pm = 0$ phase conditions for the two periodic orbits $\gamma_\pm$. Similarly to before, we solve (3.7), and need to solve (3.8) to set up the boundary conditions. There are no free parameters in (3.8): We always solve (3.8) with a fixed value of λ, hence the notation $\widehat{\lambda}$.

Remark 3.2. In general, appropriate values of T_+ and T_-, ought to be found trying to balance the error due to truncation to a finite interval; see Theorem 2.5. Although it is possible to adapt to the present setting the strategy that Beyn proposed for point-to-point connections in [4] in order to choose T_- and T_+, for the experiments of the next section we have simply taken $T_+ = -T_- \equiv T$. As a consequence, e.g., in (3.7), we end up solving $\dot{x} = 2Tf(x, \lambda)$, $0 < t < 1$, with the parameter T free.

3.2.2. *Homoclinic.* In the case of $\gamma_-(\lambda) = \gamma_+(\lambda)$, the relation (2.5) would give us $p = m_+^u - m_-^u - 1 + 1 = 0$. That is, a homoclinic to a periodic orbit is a codimension 0 phenomenon: It persists with no free parameters. For this reason, we propose the following adaptation of (3.7) (there is no λ dependency in general, and recall

Remark 2.1):

$$
\text{(3.9)} \qquad \begin{cases} \dot{x} \ = \ 2Tf(x)\,, \quad 0 \le t \le 1\,, \\ L_-\big(x(0) - y(0)\big) \ = \ 0\,, \\ L_+\big(x(1) - y(0)\big) \ = \ 0\,, \end{cases}
$$

where the value of T also needs to be found, and

$$
\text{(3.10)} \qquad \begin{cases} \dot{y} \ = \ \tau f(y)\,, \quad 0 \le t \le 1\,, \\ y(0) \ = \ y(1)\,, \\ \sigma(y) \ = \ 0\,. \end{cases}
$$

Again, $y(0)$ is the value at $t = 0$ of the solution of (3.10).

Remark 3.3. At times, one has to find connections for a differential system with extra symmetries and some modifications to the above setups are needed. For example, suppose we have a Hamiltonian system $\dot{x} = J\nabla H(x, \lambda)$, where H is the Hamiltonian and J is the symplectic identity $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, and we need to find a homoclinic to periodic connection. (Similar modifications would be needed for the other cases considered in this work). The system has even dimension $m = 2n$, and the periodic orbit now has two multipliers equal to 1. The regularization technique in [4] (also adopted in [7]) can be used, the idea being to embed the original Hamiltonian system into a larger, generic, system, by introducing an artificial parameter (which at the exact solution will need to be 0). To illustrate, the homoclinic to periodic connection system (3.9) will become

$$
\text{(3.11)} \qquad \begin{cases} \dot{x} \ = \ 2T\big(J\nabla H(x) + \mu \nabla H(x)\big)\,, \quad 0 \le t \le 1\,, \\ L_-\big(x(0) - y(0)\big) \ = \ 0\,, \\ L_+\big(x(1) - y(0)\big) \ = \ 0\,, \end{cases}
$$

where T and μ are free parameters, and L_- and L_+ both are in $\mathbb{R}^{n+1,2n}$. The periodic orbit can be found as solution of

$$
\text{(3.12)} \qquad \begin{cases} \dot{y} \ = \ \tau\big(J\nabla H(y) + \nu \nabla H(y)\big)\,, \quad 0 \le t \le 1\,, \\ y(0) \ = \ y(1)\,, \\ H(y) = \text{constant}, \\ \sigma(y) \ = \ 0\,, \end{cases}
$$

where τ and ν are free and the constraint "$H(y) = \text{constant}$" is required for well posedness of (3.12). For a recent, comprehensive, discussion of techniques for continuing periodic orbits in Hamiltonian systems we refer to [19].

3.3. **Boundary Conditions.** In general, it is important to define the boundary conditions, i.e., L_- and L_+ in (1.5), in such a way that they depend smoothly on λ. This is more than a theoretical restriction imposed by Theorem 2.3, it is also crucial for the success of the numerical methods. Indeed, typically (1.5) is solved by a discretization method for boundary value problems (say, multiple shooting or collocation, see [2, 14]) coupled with Newton's method for solving the resulting nonlinear system: Failure to have smooth functions $L_\pm$ will then give at best reduced convergence rate and may altogether preclude convergence of the Newton iteration.

We adapted to the present setting the approach of [9] whereby smooth orthonormal representations for $L_\pm$ are computed using smooth continuation of block Schur factorizations; for an alternative, one may adapt the approach in [4]. We believe that the way we compute $L_\pm$ and hence set up the boundary conditions is a major contribution of this paper and improvement over the approaches of [5, 7, 20].

The procedure relative to an equilibrium (i.e., to find $L_-(\lambda)$ in (3.2)) is the same as explained in [9], to which we refer for details; in essence, we perform a smooth block Schur factorization of the function $f_x\big(y_-(\lambda)\big)$. For the case of a periodic orbit, the idea is to perform a smooth block Schur factorization of the matrix valued function given by the monodromy matrix (function of λ) associated to the linearized problem. [This is a point where the splitting technique of Section 3.1 proves very convenient]. To clarify, consider defining $L_+(\lambda)$ in (3.2), for λ in a compact set D within which (3.3) has unique hyperbolic equilibrium and periodic solution for each given λ in D. For each given $\lambda \in D$, let $Y_+(1, \lambda)$ be the monodromy matrix associated to the linearization of the periodic trajectory in (3.3). That is,

$$\dot{Y}_+ = \tau_+ f_x\big(y_+(t, \lambda)\big)Y_+ \ , \quad Y_+(0) = I \, .$$

Then, we seek a smooth (in λ) orthogonal function $Q(\lambda)$ such that for all λ

$$Q^T(\lambda)Y_+(1, \lambda)Q(\lambda) = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} , \tag{3.13}$$

where $R_{11} \in \mathbb{R}^{m_+^u+1, m_+^u+1}$, $R_{22} \in \mathbb{R}^{m_+^s, m_+^s}$, and if $\mu \in \sigma(R_{11})$ then $|\mu| \geq 1$, and if $\mu \in \sigma(R_{22})$ then $|\mu| < 1$. Partitioning $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$, we will then take $L_+(\lambda) = Q_1^T(\lambda)$. To make sure that the function L_+ is a smooth function of λ, we use the same technique used for equilibria in [9, 10], to which we again refer for details.

Remark 3.4. The first time, i.e., for the very first value of λ, to find the decomposition (3.13), we compute an ordered Schur form of Y_+ using routines from `Lapack`. At later stages, say during the Newton iteration for solving (a discretized version of) (3.2) (or, similarly, of (3.7), etc.), we use the smooth Continuation of Invariant Subspaces (`CIS`) algorithm of [10]. Furthermore, it is quite commonly the case (see our Examples in the next Section), that one needs to continue a branch of connections with respect to one parameter. In this case, we also use the `CIS` algorithm in order to update the functions $L_\pm$.

Remark 3.5. In [5, 20], the authors do not directly rely on the monodromy matrix and moreover use fixed matrices to define $L_\pm$ in (1.5). In [7], the authors rely on the monodromy matrix to build the matrices $L_\pm$: However, they use generalized eigenvectors to build $L_\pm$ (potentially an unstable procedure), do not update $L_\pm$ during the Newton's process, but keep them frozen, and furthermore do not smoothly update $L_\pm$ during continuation.

3.4. **Implementation.** A major advantage of having split the computation of the connection from that of the periodic orbit(s), is that it is now possible to use sophisticated software for finding the connection, i.e., for solving the boundary value problem (3.2) (or (3.7), or (3.9)). In fact, to find the connection, we have eventually used the spline collocation code `Colsys` from the `Netlib` collection (see also [2]). Since `Colsys` requires explicitly the boundary conditions and their derivatives, we need to solve repeatedly the problems defining $L_\pm$. Now, to find the equilibria in (3.3) is a routine matter, e.g., using Newton's method. To find the periodic solutions in (3.3) (or (3.8) or (3.10)) one may also use some canned software, say `AUTO` (see [14]), and at first we did just this. However, to reduce overhead and to avoid some cumbersome programming details in forcing communication between the periodic solution problem and the connection problem, we ended up also writing our own multiple shooting code to find the periodic orbits; since its performance, on all problems we solved, was excellent, we will refer to the version of our algorithms obtained when using this multiple shooting code for the periodic solutions.

Multiple shooting is a well known approach, and we refer to [2] for generalities on the method. Here, we use it for finding periodic solutions. Thus, we solve

$$\begin{cases} \dot{y} = \tau f(y)\ , \quad 0 < t < 1\,, \\ y(0) = y(1)\ , \\ \sigma(y) = 0\,, \end{cases} \tag{3.14}$$

where we take the standard phase condition

$$\sigma(y) = (y(0) - y_r(0))^T f\big(y_r(0)\big)\ , \tag{3.15}$$

with $y_r(0)$ a given reference vector (possibly, the solution at a previous continuation step). We solve (3.14)-(3.15) coupled with Newton's method. This way, the monodromy matrix can be easily extracted from the Jacobian we have at convergence of Newton's method. Indeed, we need the solution at $t = 1$, $Y(1)$, of the linearized problem about the solution $y(t)$: $\dot{Y} = \tau\, f_y(y(t))\, Y$, $Y(0) = I$. If we set $S_0 = I$, and progressively $S_{j+1} = Y((j+1)h, jh)\, S_j$, $j = 0, \ldots, N-1$, clearly $Y(1) = S_N$. Here, the points $t_j = (j+1)h$, $j = 0, \ldots, N-1$, are the multiple shooting points. Letting $Y_{j+1} = Y((j+1)h, jh)$, $j = 0, \ldots, N-1$, this recursion is better rewritten in the matrix form

$$
\begin{bmatrix}
I & & & & & & \\
Y_1 & -I & & \cdots & & & \\
 & Y_2 & -I & & \cdots & & \\
 & & & & & & \\
 & & & & Y_{N-1} & -I & \\
 & & & & & Y_N & -I
\end{bmatrix}
\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N-2} \\ S_{N-1} \\ S_N \end{bmatrix}
=
\begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.16}
$$

We solve (3.16) by Gaussian elimination with row pivoting (thus minimizing fill), and find the (approximate) monodromy matrix S_N.

4. Examples

Here we illustrate the performance of our algorithms on several problems.

4.1. **Point to Periodic.** We consider (3.1), rewritten as (3.2)-(3.3).

Example 4.1 (Lorenz equation)**.** Consider the well-known Lorenz equations:

$$
\begin{cases}
\dot{x}_1 &= \sigma\,(x_2 - x_1) \\
\dot{x}_2 &= \lambda\, x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= x_1 x_2 - b\, x_3.
\end{cases} \tag{4.1}
$$

First, we take $\sigma = 10$, $b = \frac{8}{3}$ and treat λ as a free positive parameter. The bifurcation diagram for this problem is well known. At $\lambda = 1.0$ there is a pitchfork bifurcation from the trivial equilibrium and further Hopf bifurcation point at $\lambda = 24.7368$ along both pitchfork branches. From these Hopf points, one can continue in λ a branch of hyperbolic periodic orbits (in the direction of decreasing λ), which eventually turns into a homoclinic connection to the origin.

In [5], Beyn approximates a connection from the origin to a periodic orbit for $\lambda \approx 24.05$ (see also [20]). At this λ value (and therefore near it), one has $m_-^u = 1$, and $m_-^s = 2$, and the periodic orbit has $m_+^u = m_+^s = 1$. We want to compute this connection and then continue it with respect to b. The balance (2.5) gives $p = 1$, i.e., we have one free parameter, λ. If this connecting orbit exists for a specific value of λ as the result of the transversal intersection of the one-dimensional unstable manifold of the origin with the two-dimensional center-stable manifold of the periodic orbit, then as we vary b there will be a branch of connections.

Remark 4.2. The main difference between our algorithms and Beyn's approach (and also the approach in [20]) is the way we define the boundary conditions in (3.2). In fact, probably a main merit of our algorithms is the way in which we define and compute the boundary conditions by generalizing in a natural way the application of projection boundary conditions from the case of an equilibrium point to the case of a periodic orbit, and adapting the strategy of continuation of invariant subspaces of [9, 10] to the case of periodic orbits.

We solve (3.2)-(3.3) as explained in the previous section. An initial profile for the connection was given by a crude approximation obtained by a single shooting approach, with the initial guess $\lambda = 24.05$. An initial approximation for the periodic orbit in (3.3) was also obtained by single shooting. The method converges to the connecting orbit with $\lambda = 24.057900322267$. The computed periodic orbit has period $\tau = 0.67717179808618$ and Floquet multipliers $\mu_1 = 1$, $\mu_2 = 1.029332933257$, $\mu_3 = 0.000092936681$. In the next two figures, we show the connection and the periodic orbit at this λ value, and then we show each component of this connecting orbit as function of time (scaled to the interval $[0, 1]$). Notice how the connecting orbit leaves the origin and enters the periodic orbit, after some time $t > 0.5$.

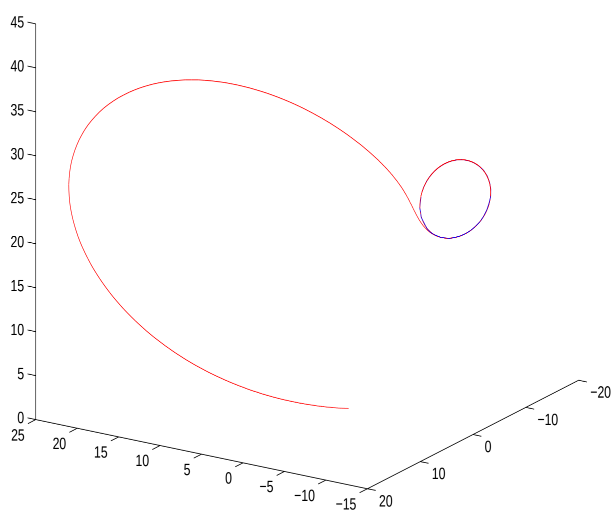

FIGURE 1. (4.1) $b = 8/3$, $\lambda = 24.057900322267$. Connection.

One should compare our Figure 2 with [5, Figure 3]: To account for the difference, recall that solutions are unique only up to a time shift. In Figure 3 we show several other connecting orbits and the corresponding periodic orbits obtained by continuation with respect to the parameter b, and in Figure 4 we plot the third components of these connecting orbits, to illustrate how the family of connecting orbits evolve with respect to time. Continuation in b was done with continuation step equal to 0.005.

In Table 1, we report on the computed values of the stable/unstable Floquet multipliers for several values of b and λ. Observe that these Floquet multipliers

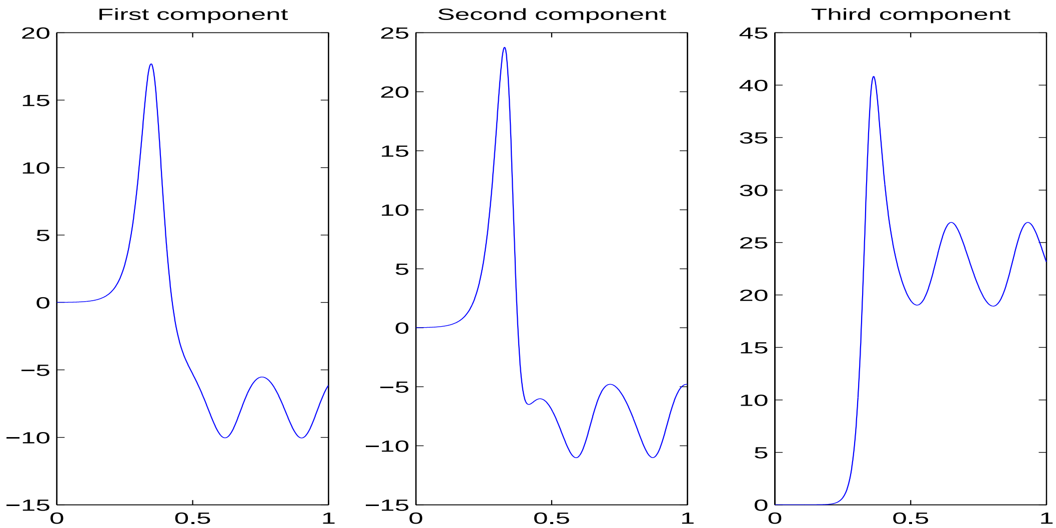

FIGURE 2. (4.1) $b = 8/3$, $\lambda = 24.057900322267$. Solution components.

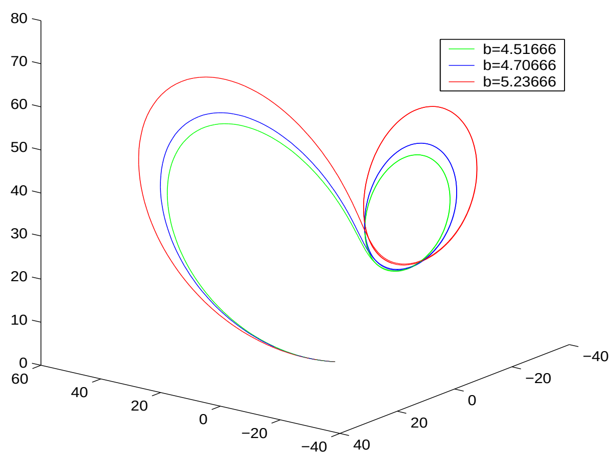

FIGURE 3. (4.1): Branch of Point-to-Periodic connections

change very little during continuation, but the λ's for which there is a connection change rapidly during continuation, thus giving an indirect sign of the robustness of our algorithm. We stopped continuation at $b = 5.23666\ldots$, simply because the computations became quite time-consuming, though we had no indication that the continuation could not be carried further.

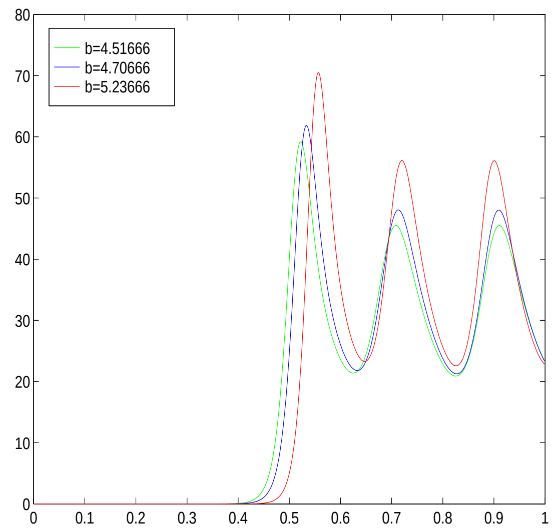

FIGURE 4. (4.1): Third component of connections

b	λ	Period	Floquet multipliers
2.6666666666	24.0579004345	0.677172	0.0000929366 1.0293329332
3.0866666666	26.1026935553	0.61911323893502	0.0001563313 1.0431878965
4.006028666	31.3332910155	0.52695220914778	0.0003390663 1.0852982508
4.5166666666	34.9853184083	0.48557753113517	0.0004812866 1.1101585551
4.7066666666	36.5284173879	0.47041509361629	0.0005458739 1.1193395430
5.2366666666	41.4875170876	0.43295209928827	0.0007732981 1.1447049577

TABLE 1. (4.1): multipliers

4.2. **Periodic-to-periodic.** We give a few examples of heteroclinic connections between periodic orbits.

Example 4.3. We propose the following system as a test problem for heteroclinic connections. We have

$$(4.2)\quad \begin{cases} \dot{x} = (1-w)y + wx(1-x^2) \\ \dot{y} = (1-w)\big(-x + \lambda(1-x^2)y\big) + w(z-\gamma) \\ \dot{z} = (1-w)z\big((z^2-(1+\gamma)^2\big) + w\big[-y+\gamma+\lambda\big(1-(y-\gamma)^2\big)(z-\gamma)\big] \\ \dot{w} = w(1-w) \end{cases}$$

where $\gamma = 3+\lambda$, and λ is a real positive parameter. This system is a homotopy from $w = 0$ to $w = 1$ which in essence takes us to two planar systems living in the (x, y) and (y, z) planes, respectively. In the (x, y) and (y, z) planes the equations reduce to those of van der Pol oscillators. As it is well known, these oscillators have attracting periodic orbits (restricted to their respective planes). There are several heteroclinic connections between the two limit cycles of these van der Pol oscillators, and here we are interested in computing (and continuing) the one from $z = 0$, $w = 0$, call it γ_-, to that with $x = 0$, $w = 1$, call it γ_+. A simple computation shows that associated to both $\gamma_\pm$ there are two multipliers less than 1, one equal to 1, and one greater than 1. Therefore, we have $m^u_\pm = 1$, $m^c_\pm = 1$, and $m^s_\pm$=2. The balance (2.5) will give us $p = 0$, hence there are no free parameters in the problem. In Figure 5 we show the connection for $\lambda = 1/2$, in Figure 6 we show several connections for $\lambda \in [1/2, 1]$, and in Figure 7 we show the second component of these connecting orbits. These computations required an extremely careful initial guess in order to converge to the right connections; once the initial guess was sufficiently close to the solution, our algorithm was able to find the connection and compute a branch of connections with relative ease. On the other hand, we were not able to satisfactorily solve this problem with shooting methods: Usually, we ended up being attracted to the stable values $x = \pm 1$ as $t \to \infty$, and even when apparent convergence to the sought connection was taking place the approximation was instead poor.

Example 4.4. [Coupled Oscillators] Consider the system

$$(4.3)\quad \begin{cases} \dot{u}_1 = u_1 + \beta v_1 - u_1^3 - 3u_1u_2^2 - u_1(v_1^2+v_2^2) - 2v_1v_2u_2 \\ \dot{v}_1 = -\beta u_1 + v_1 - v_1^3 - 3v_1v_2^2 - v_1(u_1^2+u_2^2) - 2u_1u_2v_2 \\ \dot{u}_2 = (1-2\lambda)u_2 + (\beta-2\lambda)v_2 - u_2^3 - 3u_1^2u_2 - u_2(v_1^2+v_2^2) - 2v_1v_2u_1 \\ \dot{v}_2 = -(\beta+2\lambda)u_2 + (1-2\lambda)v_2 - v_2^3 - 3v_1^2v_2 - v_2(u_1^2+u_2^2) - 2u_1u_2v_1 \end{cases}$$

where we fix $\beta = 0.55$ and λ is a free parameter. This model is taken from [1], and it has been often used as a test problem for computation and continuation of invariant tori; e.g., see [11]. Indeed, for $\lambda > 0$, sufficiently small (up to $\lambda \approx 0.26052...$, see [11]), the system has an attracting invariant torus on which there are two periodic orbits γ_- and γ_+ (there is *phase locking* on the torus). The two solutions of (4.3) giving the periodic orbits can be explicitly given (see [1]) as

$$y_-(t) = (\,0,\, 0,\, \rho(t)\cos\theta(t),\, \rho(t)\sin\theta(t)\,), \qquad y_+(t) = (\cos\beta t,\, -\sin\beta t,\, 0,\, 0),$$

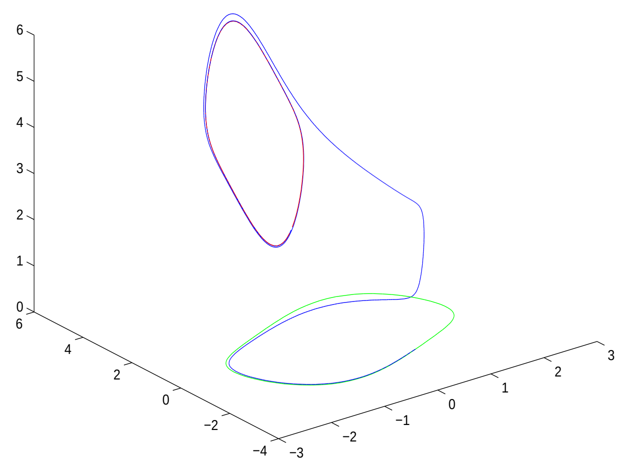

FIGURE 5. (4.2): Periodic-to-Periodic, $\lambda = 0.5$

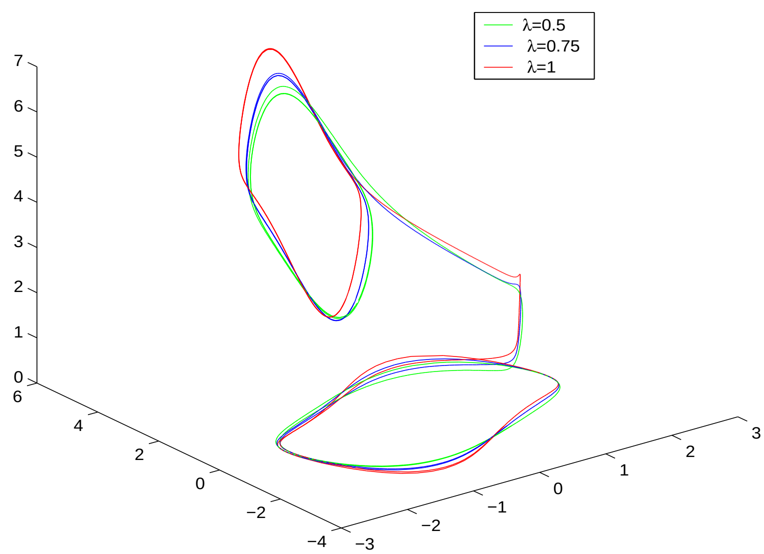

FIGURE 6. (4.2): Branch of Periodic-to-Periodic connections

where

$$\dot{\rho} = (1 - 2\lambda - 2\lambda \sin 2\theta)\rho - \rho^3 \,, \quad \dot{\theta} = -\beta - 2\lambda \cos 2\theta \,.$$

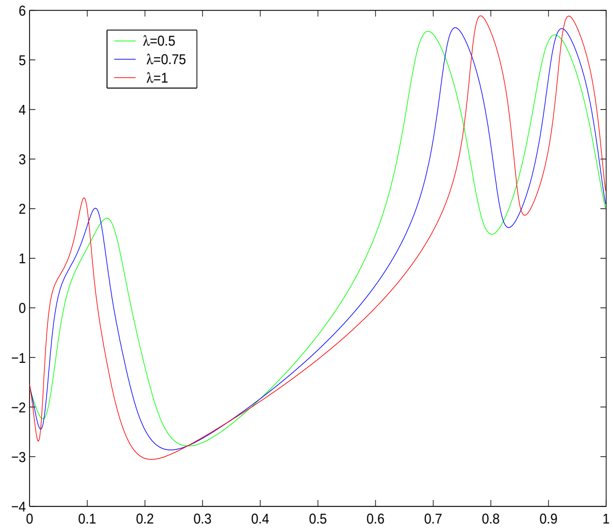

FIGURE 7. (4.2): Second component of connecting orbits

We want to compute a branch of connecting orbits from γ_- to γ_+. We begin at $\lambda = 0.1$, where the Floquet multipliers are

$$\gamma_- : \quad \mu_1 = 1, \quad \mu_2 = 0.000000039, \quad \mu_3 = 0.0000000030, \quad \mu_4 = 10.393836557,$$

$$\gamma_+ : \quad \mu_1 = 1, \quad \mu_2 = 0.0000000001, \quad \mu_3 = 0.00000000001, \quad \mu_4 = 0.11102401100.$$

So, γ_- has a 1-dimensional unstable manifold, and γ_+ has a 3-dimensional stable manifold . Therefore, the relation (2.5) would give $p = -1$. Recalling Remark 2.1, we may add one extra condition to uniquely determine a connecting orbit. Motivated by the splitting technique which we adopted for the algorithms, we thus can directly attempt solving (3.7) and (3.8) with no free parameter (not even the value of T_+ (or T_-) is free). In fact, it turned out to be a rather simple task to compute and continue this heteroclinic connection from $\lambda = 0.1$ to $\lambda = 0.195$. At $\lambda = 0.195$, the computation became demanding, though we had no reason to suspect that it could not be continued farther. The unstable multiplier of γ_- grows quite rapidly as λ increases (e.g., it is already 240.187 at $\lambda = 0.195$) and is responsible for a progressively more rapid approach to γ_+.

In Figure 8, we show several connections for different values of λ. Since the periodic orbits lie in different planes and only have the origin in common, for our 3-dimensional rendition we used the coordinates $(u_2, v_2, \sqrt{u_1^2 + v_1^2})$. In Figure 9, instead, we show the fourth component, v_2, of these connecting orbits: It is apparent that as λ increases the connecting orbit gets close to the periodic orbit γ_+ in a shorter time.

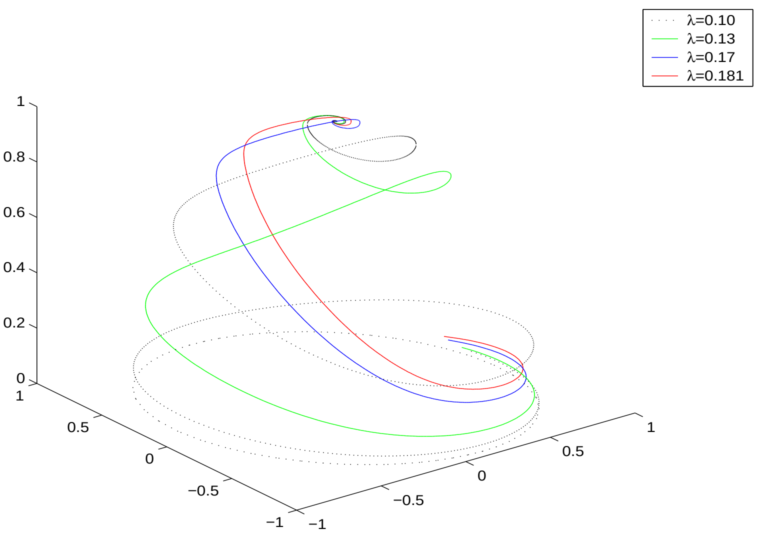

FIGURE 8. (4.3): Connections for several λ

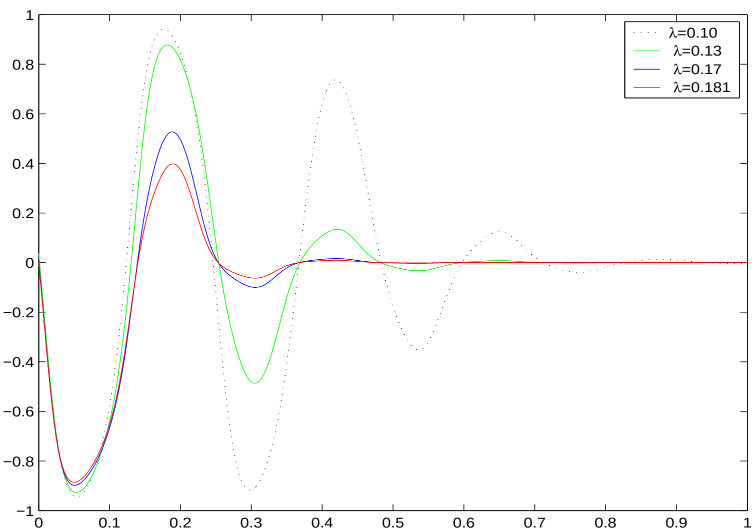

FIGURE 9. (4.3): Fourth component of connections

5. CONCLUSIONS

We have proposed, justified, implemented, and tested, a class of algorithms for finding connections between periodic orbits. The algorithms rest on two natural ideas: (i) to split the computation of the periodic orbits from that of the connection, and (ii) to use the monodromy matrices obtained when computing the periodic orbits to set up projection boundary conditions for the connection. To enforce smoothness in the boundary conditions, we made use of an algorithm which computes smooth orthonormal bases for invariant subspaces. We have solved several examples, and shown that the proposed techniques are quite reliable.

As far as we know, our is the first work where a systematic study of a numerical method for computing connections between periodic orbits has been undertaken. Our implementations can probably be ameliorated in several ways, and we propose to consider some of these in future studies. For example, we anticipate improvements towards the selection of the continuation steps and the approximation of the periodic orbits. We also anticipate comparison of our algorithm with the techniques in [7] and [18, 21] for computation of homoclinic to periodic connections.

References

[1] D.G. Aronson, E.J. Doedel, H.G. Othmer. "An analytical and numerical study of the bifurcations in a system of linearly-coupled oscillators", Physica D **25** (1987), pp. 20-104.

[2] U. Ascher, R. Mattheij, R. Russell. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Prentice Hall (1998).

[3] F. Bai, G. Lord, A. Spence. "Numerical computations of connecting orbits in discrete and continuous dynamical systems", Int. J. of Bif. and Chaos, **6** (1996), pp. 1281-1293.

[4] W.J. Beyn. "The numerical computation of connecting orbits in dynamical systems", IMA J. Numer. Anal. **9** (1990), pp. 379-405.

[5] W.J. Beyn. "On well-posed problems for connecting orbits in dynamical systems", Contemporary Mathematics **172** (1994), pp. 131-168.

[6] A.R. Champneys, Y.A. Kuznetsov, B. Sandstede. "A numerical toolbox for homoclinic bifurcation analysis", Int. J. Bif. and Chaos **6** (1996), pp. 867-887.

[7] A.R. Champneys, G.J. Lord. "Computation of homoclinic solutions to periodic orbits in a reduced water-wave problem", Physica D **102** (1997), pp. 101-124.

[8] S. Coombes. "From periodic traveling waves to traveling fronts in the spike-diffuse-spike model of dendritic waves", Mathematical Biosciences **170** (2001), pp. 155–172.

[9] J. Demmel, L. Dieci, M. Friedman. "Computing connecting orbits via an improved algorithm for continuing invariant subspaces", SIAM J. Scientific Computing **22** (2001), pp. 81-94.

[10] L. Dieci, M. Friedman. "Continuation of invariant subspaces", Numer. Linear Algebra Appl. **8** (2001), pp. 317-327.

[11] L. Dieci, J. Lorenz. "Lyapunov-type numbers and torus breakdown: Numerical aspects and a case study", Numerical Algorithms **14** (1997). pp. 79-102.

[12] E.J. Doedel, M. Friedman. "Numerical computation of heteroclinic orbits", J. Comp. Appl. Math. **26** (1989), pp. 159-170.

[13] E.J. Doedel, M. Friedman, B. Kunin. "Succesive continuation for locating connecting orbits", Num. Algor. **14** (1997), pp. 103-124.

[14] E.J. Doedel, A.R. Champneys, T.F. Fairgrieve, Yu.A. Kuznetsov, B. Sandstede, and X.J. Wang. `AUTO97`*: Continuation and bifurcation software for ordinary differential equations (with* `HomCont`*)*. Technical Report, 1997. (*ftp.cs.concordia.ca*)

[15] E. Freire, J. Rodriguez, E. Gamero, E. Ponce. "A case study for homoclinic chaos in an autonomous electronic circuit", Physica D **62** (1993), pp. 230-243.

[16] Ph. Hartman. *Ordinary Differential Equations*, Wiley & Sons (1964).

[17] B. Katzengruber, M Krupa, and P Szmolyan. "Bifurcation of traveling waves in extrinsic semiconductors", Physica D **144** (2000), pp. 1–19.

[18] W.S. Koon, M.W. Lo, J.E. Marsden, S.D. Ross. "Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics", Chaos **10** (2000), pp. 427–469.
[19] F.J. Munõz-Almaraz, E. Freire, J. Galàn, E. Doedel, A. Vanderbauwhede. "Continuation of periodic orbits in conservative and Hamiltonian systems", Physica D **181** (2003), pp. 1-38.
[20] T. Pampel. "Numerical approximation of connecting orbits with asymptotic rate", Numer. Math. **90** (2001), pp. 309–348.
[21] R. Serban, W.S. Koon, M. Lo, J.E. Marsden, L.R. Petzold, S. D. Ross, R. S. Wilson. "Halo orbit mission correction maneuvers using optimal control", Automatica **38** (2002), pp. 571–583.

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ERRATUM: POINT-TO-PERIODIC AND PERIODIC-TO-PERIODIC CONNECTIONS.

LUCA DIECI AND JORGE REBAZA

Following is a list of some corrections we wanted to make to our work before its publication. Unfortunately, the corrections arrived late to the Publisher and were not accounted for in the paper *Point-to-Periodic and Periodic-to-Periodic Connections*, which appeared in *BIT* **44**, pp.41-62, 2004.

1. P.44, line 8. Replace [5] with [5,20].
2. P.45, line 18. Replace $[T_-, T_+]$ with $J := [T_-, T_+]$.
3. P.45, lines 23-24. Modify as "... of Beyn's proofs, following [20]. The following lemma ... ".
4. P.45, replace last 4 lines with the following between quotes.

"In the theorems below, we use the spaces

$$W := C^1(J, \mathbb{R}^m) \times \mathbb{R}^p \quad \text{and} \quad Z := C(J, \mathbb{R}^m) \times \mathbb{R}^{m_-^c + m_-^s} \times \mathbb{R}^{m_+^u + 1}\,.$$

For appropriate $\alpha, \beta > 0$, the norms are defined as

$$\begin{aligned}
\|(x, \lambda)\|_W &= \sup_{t \in J_-} \|x(t)\| e^{\alpha t} + \sup_{t \in J_+} \|x(t)\| e^{-\beta t} + \|\lambda\|, \\
\|(y, r_-, r_+)\|_Z &= \|(y, r_-)\|_{Z_1} + \|(y, r_+)\|_{Z_2}\,, \quad \text{where} \\
\|(y, r_-)\|_{Z_1} &= \sup_{t \in J_-} \|y(t)\| e^{\alpha t} + \|r_-\|\,, \quad \text{and} \\
\|(y, r_+)\|_{Z_2} &= \sup_{t \in J_+} \|y(t)\| e^{-\beta t} + \|r_+\|\,, \quad \|\cdot\| = \|\cdot\|_\infty\,,
\end{aligned}$$

and where $J_- = [T_-, 0]$ and $J_+ = [0, T_+]$. With these norms, W and Z become Banach spaces. Anticipating the asymptotic convergence of $x(t)$ to $y(t)$ with rate $\epsilon > 0$, we impose the condition that, for some constant C, $\|x(t) - y_\pm(t)\| \le Ce^{-\epsilon|t|}$ as $t \to \pm\infty$."

- As a consequence of the above change, the norms in the statements of Theorems 2.1 and 2.2 need to be changed as well. So, the following changes are needed.

5. P.46, line 6-8. It is now $K_\delta = \{(x, \lambda) \,:\, \|(x, \lambda) - (\bar{x}|_J, \bar{\lambda})\|_W \le \delta\}$.
6. P.46, formula (2.10). The left hand side changes to

$$\|(\bar{x}|_J, \bar{\lambda}) - (x_J, \lambda_J)\|_W$$

7. P.46, line 13. Replace [4, Theorem 3.1] with [20, Theorem 4].

8. P.46, formula (2.11). The left hand side changes to

$$\|F(\bar{x}|_J, \bar{\lambda})\|_Z$$

9. P.46, formula (2.12). The left hand side changes to

$$\|(\bar{x}|_J, \bar{\lambda}) - (x_J, \lambda_J)\|_W$$

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